IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-0661, p-ISSN: 2278-8727, Volume 21, Issue 4, Ser. 1 (July. – August. 2025), PP 44-61
www.iosrjournals.org

Square Summable Coefficients And Integral Means With Boundary Limits Of Dirichlet Series

Ali Mohamed Abu Oam
Faculty Of Pure And Applied Sciences, Department Of Mathematics And Computer, International University Of Africa, Khartoum Sudan

Abstract
We investigate the composition operators on the space of Dirichlet series and to show the relation of Dirichlet series on Hilbert spaces, and to discuss the integral and boundry of Dirichlet series.
Keywords: *Dirichlet series, Hilbert space, Drury Arveson space, polydisk, interpolation theorem, interpolating sequence, full interpolating problem, Carleson embedding, pure isometry.*

Date of Submission: 19-07-2025 Date of Acceptance: 29-07-2025

I. Introduction

We will give an answer to the following :
Let $\mathcal{H}$ be the space of Dirichlet series with square summable coefficients, $f \in \mathcal{H}$ means that the function has the form
$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ (1)
With $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. By the Cauchy–Schwarz inequality, the functions in $\mathcal{H}$ are all holomorphic on the half-plane $\mathbb{C}_{1/2} = \{s \in \mathbb{C}: \Re s > 1/2\}$. The coefficients $\{a_n\}_n$ can be retrieved from the holomorphic function $f(s)$ so that $\|f\|_{\mathcal{H}}^{c} = \sum_{n=1}^{\infty} |a_n|^2$ defines a Hilbert space norm on $\mathcal{H}$.
This chapter deals with the boundary behavior of functions in the Hardy spaces $\mathcal{H}^p$ for ordinary Dirichlet series. The main result, answering a question of $H.$ Hedenmalm, shows that the classical $F.$ Carlson theorem on integral means does not extend to the imaginary axis for functions in $\mathcal{H}^\infty$, i.e., for ordinary Dirichlet series in H^∞ of the right half-plane.

Section(3.1): ***Composition Operators on the Space of Dirichlet Series***
In this section, a complete answer to this question is found. In the process, we encounter the space $\mathcal{D}$ of functions *f,* which in some half-plane :
$\mathbb{C}_\theta = \{c \in \mathbb{C}: \Re s > \theta\}, (\theta \in \mathbb{R})$ admit representation by a convergent Dirichlet series (1). It is, in a sense, a space of germs of holomorphic functions. It is important to note that if a Dirichlet series converges on $\mathbb{C}_\theta$ then it converges ab-solutely and uniformly on $\mathbb{C}_\vartheta$, provided $\vartheta > \theta + 1$ (see e.g. [1]).In terms of the coefficients, $f \in \mathcal{D}$ means that a_n grows at most polynomially in the index variable *n.* We shall use the notation $\mathbb{C}_+$ to denote the right half-plane, $\mathbb{C}_+ = \{ s \in \mathbb{C}: \Re s > 0\}$,although strictly speaking we probably ought to keep the notation consistent and write $\mathbb{C}_0$ instead. Throughout the section, the term *half-plane* will be used in the re-stricted sense of a half-plane of the type $\mathbb{C}_\theta$ for some $\theta \in \mathbb{R}$. It should be men-tioned that, by the closed graph theorem, every composition operator $C_\Phi : \mathcal{H} \to \mathcal{H}$ is automatically bounded.
The first question that arises naturally in connection with this problem is: For what functions Φ analytic in some half-plane $\mathbb{C}_\theta$ and mapping it into $\mathbb{C}_{1/2}$ does the composition operator C_Φ map the space $\mathcal{H}$ into $\mathcal{D}$?
The next theorem answers the original question posed above.

Theorem(3.1.1) *An analytic function* $\Phi: \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ *defines a bounded composition operator* $C_\Phi : \mathcal{H} \to \mathcal{H}$ *if and only if:*
(a) *it is of the form*
$\Phi(s) = c_0 s + \varphi(s)$ *where* $c_0 \in \mathbb{N} \cup \{0\}$ *and* $\varphi \in \mathcal{D}$.
(b) Φ *has an analytic extension to* C_+, *also denoted by* Φ, *such that*
(i) $\Phi(C_+) \subset C_+$ *if* $0 < c_0$, *and*
(ii) $\Phi(C_+) \subset C_+$ *if* $c_0 = 0$.
This Theorem is a Dirichlet series analog of the classical Littlewood subordination

principle [2].Indeed, in case Φ fixes the point $+\infty$, which happens precisely when $0 < c_0$, the composition operator C_Φ is a contraction on $\mathcal{H}$.
The nonegative integer c_0, which appears both in Theorem (3.1.1) and in Theorem (3.1.2), contains much information about the mapping function Φ. We call this c_0 the *characteristic* of Φ.

Theorem(3 .1.2) $(\theta \in \mathbb{R})$ *An analytic function* $\Phi: \mathbb{C}_\theta \to \mathbb{C}_{1/2}$ *generates a composition operator* $C_\Phi : \mathcal{H} \to \mathcal{D}$ *if and only if it has the form*
$\Phi(s) = c_0 s + \varphi(s) \quad where\ c_0 \in \mathbb{N} \cup \{0\} \ \ and \ \ \varphi \in \mathcal{D}.$
Proof: we shall need this simple and well-known lemma.
Lemma(3.1.3) *Let m be a positive integer, and let* $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ *be a Dirichlet series from the class* $\mathcal{D}$*, starting from the index m: Then* $m^s f(s) \to a_m$ *uniformly as* $\Re s \to +\infty$.
We are now able to prove the necessity part of Theorem(3.1.2). Suppose that $fo\Phi \in \mathcal{D}$ for every $f \in \mathcal{H}$, In particular, $k^{-\Phi(s)} \in \mathcal{D}$ for all $k \in \mathbb{N}$. Denote the cor-responding series by
$k^{-\Phi(s)} = \sum_{n=N(k)}^{\infty} b_n^{(k)} n^{-s}$ (2)
Where $N(k) \in \mathbb{N}$ is the index of the first nonzero coefficient. Multiplying the equality (2) by $N(k)^s$ and applying Lemma(3.1.3), we arrive at
$exp(slogN(k) - \Phi(s)logk) \to b_{N(k)}^{(k)}$ as $\Re s \to +\infty$. (3)
With uniform convergence. Here, "log" stands for natural logarithm. Observe that the function of s in the exponent on the left-hand side is holomorphic in $\mathbb{C}_\theta$ (the half-plane appearing in the formulation of Theorem (3.1.2)), so it maps $\mathbb{C}_\theta$ into a connected domain. Moreover, it maps any half-plane $\mathbb{C}_\vartheta$ contained in $\mathbb{C}_\theta$ into a connected domain as well. On the other hand, it follows from (3) that, for s with sufficiently large real part, the values of $slogN(k) - \Phi(s)logk$ are contained in the set $U(k) + 2\pi i\mathbb{Z}$ where $\mathbb{Z}$ is the set of all integers and $U(k)$ is an arbitrarily small open neighborhood of the point $logb_{N(k)}^{(k)}$ (here "log" stands for the principal branch of the logarithm). Hence, there must exist an integer q such that
$slogN(k) - \Phi(s)logk \to logb_{N(k)}^{(k)} + 2\pi iq$ as $\Re s \to +\infty$. (4)
Dividing the both parts of (3) by $s \log k$ (for $k > 1$), we have

$$\lim_{\Re s \to +\infty} \frac{\Phi(s)}{s} = \frac{logN(k)}{logk}$$

with uniform convergence (by Lemma(3.1. 3). It follows that the real number

$$c_0 = \frac{logN(k)}{logk}$$

does not depend on *k:* We can look at this relation from the other side: $N(k) = k^{c_0}$ is an integer for all positive integers *k*.
The following result is indubitably known:
Lemma(3.1.4) *A real number c such that* n^c *is an integer for all positive integers n is itself a nonnegative integer.*
Proof. In the case $c < 0$ the statement is obvious: on the one hand, $n^c \to 0$ as $n \to +\infty$ on the other, it must be an integer for all *n*. Hence $n^c = 0$, for sufficiently large *n,* which is impossible.
The case $c > 0$ can be reduced to a similar situation by means of taking finite differences. We recall the definition of the *first difference* of a sequence $\{x_n\}_{n=1}^{\infty}$ as the sequence $\{\Delta x_n\}_{n=1}^{\infty}$where $\Delta x_n = x_{n+1} - x_n$ The differences of higher orders are then defined inductively.
Let k be the least integer that is $\geq c$. We consider the sequence $\{y_n\}_{n=1}^{\infty}$, $y_n = \Delta^k x_n$, with $x_n = n^c$. We observe that $y_n = O(n^{c-k})$ as $n \to \infty$, and we consider a series of the form $f(t) = \sum_{j=0}^{\infty} a_j t^{c-j}$ that is absolutely convergent for $t > 1$.
The difference operation $\Delta f(t) = f(t+1) - f(t)$ carries it into a series of the same kind, but starting from $j = 1$, as

$$(t+1)^c - t^c = t^c((1+1/t)^c - 1) = ct^{c-1} + \frac{c(c-1)}{2} t^{c-2} + \cdots, \quad t > 1,$$

with absolute convergence on the indicated interval. It follows that *k* applications of the operation Δ to $f(t)$ results in a series starting from $j = k$, which proves the observation.

Hence, $y_n = 0$ as $n \to \infty$ unless *c* equals the integer *k*. Since the numbers y_n are integers, we must then have $y_n = 0$ for sufficiently large *n;* say $n \geq N$. On the other hand, the sequence $\{y_n\}_{n=1}^{\infty}$ is the restriction to the set ℕ of a function *y(z),*
which is holomorphic on $\mathbb{C}(-\infty, 0]$ and grows no faster than a power of $|z|$ as
$|z| \to \infty$. If such a function vanishes on the set $\mathbb{N} \cap [N, +\infty)$, it must be identically
zero. Hence $y_n = 0$, and since the kernel of Δ^k consists of those sequences that are polynomials in *n* of degree k -1 or less, the original sequence $x_n = n^c$ is a polynomial of degree at most k -1This is possible only if $c \leq k$ -1which contradicts the definition of *k*. Hence $c = k \in \mathbb{N}$ as desired.
The case $c = 0$ is trivial.
From the lemma we conclude that $c_0 \in \mathbb{N} \cup \{0\}$. We shall now consider more closely the function $\varphi(s) = \Phi(s) - c_0 s$. We claim that φ belongs to $\mathcal{D}$.
Multiplying (2) by $k^{c_0 s}$ we obtain

$$k^{-\varphi(s)} = \sum_{m=k^{c_0}}^{\infty} b_m^{(k)} \left(\frac{m}{k^{c_0}}\right)^{-s}$$

Dropping the superscript, we can write this relationship as

$$k^{-\varphi(s)} = \tilde{b}_0 + \tilde{b}_1(1 + \tfrac{1}{k^{c_0}})^{-s} + \tilde{b}_2(1 + \tfrac{2}{k^{c_0}})^{-s} + \cdots = \tilde{b}_0 + h(s) \qquad (5)$$

where the notation $\tilde{b}_j$ stands for the shifted coefficients, $\tilde{b}_j = b_{k^{c_0}+j}^{(k)}$. Combining
(5) with (4), we obtain

$$-\varphi(s) log k = log \tilde{b}_0 + log\left(1 + \frac{h(s)}{\tilde{b}_0}\right) + 2\pi i q$$

on a half-plane where the principal branch of the logarithm defines a holomorphic
function, which is assured if $|h(s)| < |\tilde{b}_0|$ there. The Dirichlet series

$$\sum_{m=k^{c_0}}^{\infty} b_m^{(k)} m^{-s}$$

is in $\mathcal{D}$, so that (by Lemma(3.1.3)) the function $h(s)$ defined by (5) tends to 0 unif-ormly as $\Re s \to +\infty$. Expanding $log(1 + z)$ in a Taylor series around $z = 0$ with:
$z = \frac{h(s)}{\tilde{b}_0}$, we have

$$-\varphi(s) log k = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \tilde{b}_0{}^{-n} h(s)^n + log \tilde{b}_0 + 2\pi i q,$$

with convergence for *s* with $|h(s)| < |\tilde{b}_0|$.
Let us open the brackets in every expression $h(s)^n$ and rearrange the terms,
which is allowed in the half-plane of absolute convergence of $h(s)$. It follows that
$\varphi(s)$ is a series of the form

$$\varphi(s) = \sum_{q=0}^{\infty} \sum_{n_1,\ldots,n_q=1}^{\infty} \beta_{n_1,\ldots,n_q} \left(1 + \frac{n_1}{k^{c_0}}\right)^{-s} \ldots \left(1 + \frac{n_q}{k^{c_0}}\right)^{-s},$$

which converges absolutely in some half-plane. In other words, $\varphi(s)$ is a converg-ent.
Dirichlet series over the multiplicative semigroup $\mathfrak{S}(k^{c_0})$ generated by the set $\{1 + j/k^{c_0}\}_{j\in\mathbb{N}}$. Note that $\varphi(s)$ does not depend on *k,* and that it is a Dirichlet series over $\mathfrak{S}(k^{c_0})$ for every $k \in \mathbb{N}$.The following lemma now completes the proof of the assertion that $\varphi(s)$ belongs to the class $\mathcal{D}$.

Lemma(3.1.5) ($c_0 \in \mathbb{N} \cup \{0\}$) *The intersection of* $\mathfrak{S}(k^{c_0})$ *over all* $k \in \mathbb{N}$ *consists only of positive integers.*
Hence a Dirichlet series over the intersection of all $\mathfrak{S}(k^{c_0})$ is an ordinary Dirichlet
series.
Proof . Suppose that a number α lies in the intersection of $\mathfrak{S}(2^{c_0})$, and $\mathfrak{S}(3^{c_0})$. As an element of $\mathfrak{S}(2^{c_0})$, α admits a representation by a fraction with denomina-tor $(2^{c_0})^n$ for some $n \in \mathbb{N}$. Similarly, α is a fraction with denominator $(3^{c_0})^m$ for some $m \in \mathbb{N}$. Since c_0 is a nonnegative integer, this is possible only if α is an integer.
It now follows that Φ has the form $\Phi(s) = c_0 s + \varphi(s)$ where $c_0 \in \mathbb{N} \cup \{0\}$ and
$\varphi \in \mathcal{D}$. This completes the necessity part of Theorem(3.1.2).
We turn to the sufficiency part, and suppose that Φ is a holomorphic mapping
$\mathbb{C}_\theta \to \mathbb{C}_{\frac{1}{2}}$ of the form

$$\Phi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s}$$

Where $c_0 \in \mathbb{N} \cup \{0\}$ and the series $\varphi(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ converges in some half plane.A series in $\mathcal{D}$,the space of convergent Dirichlet series,actually converges
absolutely in the half-plane one unit to the right of the half-plane of convergence, in particular, this applies to φ. We shall show that the composition $f \circ \Phi$ belongs to $\mathcal{D}$ for every function $f \in \mathcal{H}$. For k = 1, 2, 3, ..., we expand

$$k^{-\Phi(s)} = k^{-c_0 s}.k^{-\varphi(s)} = k^{-c_0 s - c_1} exp\left(-(log k)\sum_{n=2}^{\infty} c_n n^{-s}\right)$$

$$= k^{-c_0 s - c_1} \prod_{n=2}^{\infty} exp(-(log k) c_n n^{-s}) \qquad (6)$$

The relationship (6) holds in the half-plane of absolute convergence of the series $\varphi(s)$. Let us take an element $f(s) = \sum_{k=1}^{\infty} a_k k^{-s}$, $f \in \mathcal{H}$. We want to plug the Dirichlet series expansion for every $k^{-\Phi(s)}$obtained by opening the brackets in the product in (6), into $f \circ \Phi(s)$ and so derive a Dirichlet series for the composition $f \circ \Phi$ by rearrangement of the terms. To justify this operation, we need to check that the series formally obtained this way converges absolutely in some half-plane. That is, we need to prove the absolute convergence of the Dirichlet series obtained by expanding

$$\sum_{k=1}^{\infty} a_k k^{-\Phi(s)} = \sum_{k=1}^{\infty} a_k k^{-c_0 s - c_1} \prod_{n=2}^{\infty}\left(1 + \sum_{j=1}^{\infty} \frac{(-c_n log k)^j}{j!} n^{-js}\right) \qquad (7)$$

The absolute convergence of the Dirichlet series expanded from (7) follows from the convergence of

$$\sum_{k=1}^{\infty} |a_k| k^{-\Re(c_0 s + c_1)} \prod_{n=2}^{\infty}\left(1 + \sum_{j=1}^{\infty} \frac{(|c_n| log k)^j}{j!} n^{-j\Re s}\right)$$

$$= \sum_{k=1}^{\infty} |a_k| k^{-\Re(c_0 s + c_1)} \prod_{n=2}^{\infty} k^{|c_n| n^{-\Re s}}$$

$$= \sum_{k=1}^{\infty} |a_k| k^{-\Re(c_0 s + c_1)} exp\left(log k \sum_{n=2}^{\infty} |c_n| n^{-\Re s}\right) \qquad (8)$$

The expression $\sum_{n=2}^{\infty} |c_n| n^{-\Re s}$ is uniformly bounded in some half-plane $s \in \mathbb{C}_\vartheta$ $(\vartheta \in \mathbb{R})$. In the case of characteristic $c_0 = 1, 2, 3, ...$ the absolute converg- -ence of the right-hand side of (8) in $\mathbb{C}_\vartheta$ follows, provided ϑ is positive and suffic- iently large.

In case of characteristic $c_0 = 0$, we need to check that $\Re c_1 > 1/2$. Once this has been done, by Lemma(3.1.3) it follows that

$\sum_{n=2}^{\infty} |c_n| n^{-\Re s} \to 0$ as $\Re s \to +\infty$

with uniform convergence. Hence, in some sufficiently remote half-plane $\mathbb{C}_\vartheta$, the inequality

$$\sum_{k=1}^{\infty} |a_k| k^{-\Re c_1} exp\left(log k \sum_{n=2}^{\infty} |c_n| n^{-\Re s}\right) \le \sum_{k=1}^{\infty} |a_k| k^{-\frac{1}{2}-\varepsilon}$$

holds with some $\varepsilon > 0$, and the convergence of the right-hand part of (8) follows.

We turn to the assertion $\Re c_1 > 1/2$. The function Φ: $\mathbb{C}_\theta \to \mathbb{C}_{\frac{1}{2}}$ has the expansion

$\Phi(s) = \varphi(s) = \sum_{n=1}^{\infty} c_n n^{-s}$, and by Lemma(3.1.3), c_1 equals the limit of $\Phi(s)$ as $\Re s \to +\infty$. Hence $\Re c_1 \ge 1/2$,almost what we want to prove. If Φ is constant, then $\Phi(s) = c_1$ and $\Re c_1 > 1/2$. If Φ is not constant then there is a first index n = 2,3, 4,..., such that the coefficient c_n is different than 0, call this index N. Then, for large positive values of $\Re s$,

$$\Phi(s) = \varphi(s) = c_1 + c_N N^{-s} + O\left((N+1)^{-\Re s}\right) \qquad (9)$$

In a sufficiently remote half-plane $\mathbb{C}_\vartheta$, the error term is negligible compared with the second term $c_N N^{-s}$, so that the image of $\mathbb{C}_\vartheta$ under Φ is a slightly perturbed (punctured) disk centered at c_1. In particular, since Φ maps $\mathbb{C}_\theta$ into $\mathbb{C}_{\frac{1}{2}}$, the point

c_1 must be an interior point in $\mathbb{C}_{\frac{1}{2}}$.
The proof of Theorem (3.1.2) is now completed.
Here, we shall obtain the following partial result.

Proposition(3.1.6) *If the holomorphic function* $\Phi: \mathbb{C}_{\frac{1}{2}} \rightarrow \mathbb{C}_{\frac{1}{2}}$ *has the property that it induces a bounded composition operator* $C_\Phi: \mathcal{H} \rightarrow \mathcal{H}$ *then almost every function* Φ_χ *has an analytic extension to* $\mathbb{C}_+$.

Proof. By Theorem (3.1.2) Φ has the form $\Phi(s) = c_0 s + \varphi(s)$ where $c_0 \in \mathbb{N} \cup \{0\}$ and $\varphi \in \mathcal{D}$. For each $n = 1, 2, 3, \ldots$, n^{-s} is in $\mathcal{H}$, so that $C_\Phi(n^{-s}) = n^{-\Phi(s)}$ is in $\mathcal{H}$, because of the assumption. It follows that $(n^{-\Phi})_\chi$ is holomorphic in $\mathbb{C}_+$ almost surely in χ [1], we have

$$n^{-\Phi_\chi(s)} = \chi(n)^{-c_0}(n^{-\Phi})_\chi(s) \quad (10)$$

in the half-plane of uniform convergence for the Dirichlet series φ .The right-hand side of (10) provides an analytic extension of the function $n^{-\Phi_\chi(s)}$ to $\mathbb{C}_+$ for almost every character χ. Since a countable union of null sets is a null set, it follows that, almost surely in χ, the functions $n^{-\Phi_\chi(s)}(n = 1, 2, 3, \ldots)$ are all analytic in $\mathbb{C}_+$. Fix a character χ with this property and consider the functions $n^{-\Phi_\chi(s)}$ for all $n \in \mathbb{N}$. The only possible singularities in $\mathbb{C}_+$ of the function $\Phi_\chi(s)$ are at the zeros of the funct-ion $n^{-\Phi_\chi} = \chi(n)^{-c_0}(n^{-\Phi})_\chi$. Let $s_0 \in \mathbb{C}_+$, and let $m_n(s_0, \chi)$ stand for the multip-licity of the zero at s_0 that the analytic extension of the function $n^{-\Phi_\chi}$ develops (if $m_n(s_0, \chi) = 0$ then there is no zero). We calculate that, in the half-plane of absol-ute convergence for the Dirichlet series $\varphi(s)$,

$$\frac{\left(n^{-\Phi_\chi(s)}\right)'}{n^{-\Phi_\chi(s)}} = \Phi'_\chi(s) \log n \quad (11)$$

The left-hand part of (11) is a meromorphic function in $\mathbb{C}_+$, with at most simple poles, so the relationship (11) provides such a meromorphic continuation of the fu-nction $\Phi'_\chi(s)$ to $\mathbb{C}_+$. Let $\rho(s_0, \chi) = \lim_{s \to s_0}(s - s_0)\Phi'_\chi(s)$ be the residue of $\Phi'_\chi(s)$ at $s = s_0$. The residue of the left-hand side of (11) at the point $s = s_0$ equ-als the multiplicity $m_n(s_0, \chi)$, an integer. Therefore, for each $n = 2, 3, 4, \ldots$, the number $\rho(s_0, \chi) \log n$ is an integer, which is possible only if $\rho(s_0, \chi) = 0$, in which case $m_n(s_0, \chi) = 0$ for all n. The proof of the proposition is complete.

In this section we shall demonstrate the following :

claim(3.1.7):

If a function $\Phi: \mathbb{C}_{\frac{1}{2}} \rightarrow \mathbb{C}_{\frac{1}{2}}$ *generates a continuous composition operator* $C_\Phi: \mathcal{H} \rightarrow \mathcal{H}$ *, so that* $\Phi(s) = c_0 s + \varphi(s)$ where $c_0 \in \mathbb{N} \cup \{0\}$ *and* $\varphi \in \mathcal{D}$, *then*: (a) *if* $c_0 = 0$ *then* Φ *extends to a holomorphic mapping* $\mathbb{C}_+ \rightarrow \mathbb{C}_{\frac{1}{2}}$ *and* (b) *if* $c_0 > 0$ *then* Φ *extends to a holomorphic mapping* $\mathbb{C}_+ \rightarrow \mathbb{C}_+$.

Proof. We assume that $\Phi: \mathbb{C}_{\frac{1}{2}} \rightarrow \mathbb{C}_{\frac{1}{2}}$ generates a continuous composition operator $C_\Phi: \mathcal{H} \rightarrow \mathcal{H}$ and let $f \in \mathcal{H}$, for every $\chi \in \Xi$ we have that

$$(f o \Phi)_\chi(s) = f_{\chi^{c_0}} o \Phi_\chi(s), \qquad s \in \mathbb{C}_{\frac{1}{2}} \quad (12)$$

Since $f o \Phi \in \mathcal{H}$,[49] shows that, almost surely in χ , $(f o \Phi)_\chi$ extends Holomorph-ically to $\mathbb{C}_+$. Also, by Proposition (3.1.6) Φ_χ extends analytically to $\mathbb{C}_+$ almost surely in χ. Moreover,for characteristic c_0=1,2,3,..., $f_{\chi^{c_0}}$ is almost surely holomor-phically extendable to $\mathbb{C}_+$ because the transformation $\chi \mapsto \chi^{c_0}$ is measure-preserv-ing (the pre-image of a set has the same mass as the set itself). However, for char-acteristic $c_0 = 0$ we have $f_{\chi^{c_0}} = f$, and all we know about this function is that it is holomorphic on $\mathbb{C}_{\frac{1}{2}}$.

We first consider the case of characteristic c_0 =1, 2, 3,..., and let $\chi \in \Xi$ belong to the set of full measure with the properties that $(f o \Phi)_\chi$, Φ_χ and $f_{\chi^{c_0}}$ all extend analytically to $\mathbb{C}_+$. We wish to prove that Φ_χ maps $\mathbb{C}_+$ to $\mathbb{C}_+$ (after all, Φ is a verti-cal limit function of Φ_χ). The image $\Phi_\chi(\mathbb{C}_+)$ of $\mathbb{C}_+$ under Φ_χ is a connected open subset of $\mathbb{C}$, because the holomorphic mapping Φ_χ is nonconstant.

Let Ω consist of all points $s \in \mathbb{C}_+$for which $\Phi_\chi(s) \in \mathbb{C}_+$ it is an open subset of $\mathbb{C}_+$. Since Φ_χ maps $\mathbb{C}_{\frac{1}{2}}$ to $\mathbb{C}_{\frac{1}{2}}$, it follows that Ω contains the half-plane $\mathbb{C}_{\frac{1}{2}}$. Let Ω_0 be the connectivity component of Ω that contains $\mathbb{C}_{\frac{1}{2}}$. Then, by analytic continuation, (12) holds for all $s \in \Omega_0$. If Ω is not all of $\mathbb{C}_+$then the same goes for Ω_0, and we can find a boundary point $s_0 \in \partial\Omega_0$ with $s_0 \in \mathbb{C}_+$. By wiggling the point slightly, we can make sure that $\Phi'_\chi(s) \neq 0$, so that Φ_χ is conformal near s_0. The point $\Phi_\chi(s)$ lies on the imaginary axis $\partial\mathbb{C}_+$, and (12)(which is valid for $s \in \Omega_0$) shows that $f_{\chi^{c_0}}$ has an analytic extension across a small segment of the imaginary axis near $\Phi_\chi(s)$. This extension is given by , $(fo\Phi)_\chi o\ \Phi_\chi{}^{-1}$ where the mapping $\Phi_\chi{}^{-1}$ refers to the inverse to the conformal map that Φ_χ defines from a neighborhood of s_0 to a neig-hborhood of $\Phi_\chi(s_0)$. In conclusion, if Φ_χ does not map $\mathbb{C}_+$ to $\mathbb{C}_+$, then $f_{\chi^{c_0}}$ nec-essarily extends holomorphically across a small segment of the imaginary axis.

We shall see that there is a function $f \in \mathcal{H}$ such that, almost surely in χ, f_χ does not extend analytically to any region larger than $\mathbb{C}_+$ (in other words, the imaginary axis is a natural boundary for the function f_χ), hence, the same can be said for the function $f_{\chi^{c_0}}$. This means that, for many (in fact, almost all) characters χ conside-red here, $f_{\chi^{c_0}}$, has $\partial\mathbb{C}_+$ as a natural boundary, which forces Φ_χ to map $\mathbb{C}_+$ to $\mathbb{C}_+$, as claimed.

We turn to the remaining case of characteristic $c_0 = 0$, where the relation (12) simplifies a bit as follows:

$$(fo\Phi)_\chi(s) = fo\Phi_\chi(s), \qquad s \in \mathbb{C}_{\frac{1}{2}} \tag{13}$$

Let $\chi \in \Xi$ belong to the set of full measure with the properties that $(fo\Phi)_\chi$ and Φ_χ both extend analytically to $\mathbb{C}_+$. We wish to prove that Φ_χ maps $\mathbb{C}_+$ to $\mathbb{C}_{\frac{1}{2}}$ for applying Φ_χ in place of Φ, guarantees that Φ also maps $\mathbb{C}_+$ to $\mathbb{C}_{\frac{1}{2}}$. As before, let Ω be the open set of all points $s \in \mathbb{C}_+$ for which $\Phi_\chi(s) \in \mathbb{C}_{\frac{1}{2}}$. Since Φ_χ maps $\mathbb{C}_{\frac{1}{2}}$ to $\mathbb{C}_{\frac{1}{2}}$, Ω contains the half-plane $\mathbb{C}_{\frac{1}{2}}$. Let Ω_0 be the connectivity component of Ω that contains $\mathbb{C}_{\frac{1}{2}}$. Then, by analytic continuation,(13) holds for all $s \in \Omega_0$. If Ω is not all of $\mathbb{C}_+$ then the same goes for Ω_0, and we can find a boundary point $s_0 \in \partial\Omega_0$ with $s_0 \in \mathbb{C}_{\frac{1}{2}}$. By wiggling the point slightly, we can make sure that $\Phi'_\chi(s) \neq 0$, so that Φ_χ is conformal near s_0. The point $\Phi_\chi(s_0)$ lies on the vertical line $\partial\mathbb{C}_{\frac{1}{2}}$, and (13), valid for $s \in \Omega_0$, shows that f has an analytic extension across a small segment of the line $\partial\mathbb{C}_{\frac{1}{2}}$. In conclusion, if Φ_χ does not map $\mathbb{C}_+$ to $\mathbb{C}_{\frac{1}{2}}$, then f necessarily exte-nds holomorphically across a small segment of the vertical line $\partial\mathbb{C}_{\frac{1}{2}}$.

We shall see that there is a function $f \in \mathcal{H}$ that does not extend holomorphically to any region larger than $\mathbb{C}_{\frac{1}{2}}$.This forces Φ_χ to map $\mathbb{C}_+$ to $\mathbb{C}_{\frac{1}{2}}$, as claimed.

Let us consider the function

$$f(s) = \sum_p a_p\, p^{-s}$$

where the summation runs over the primes p and

$$a_p = \frac{1}{\sqrt{p}logp}$$

Clearly, $f \in \mathcal{H}$ .The vertical limit functions of f are

$f_\chi = \sum_p a_p\, \chi(p) p^{-s}$,

Where $\chi(p)$, $p = 2, 3, 5, 7, 11,\ldots$, are to be thought of as independent uniformly distributed stochastic variables on $\mathbb{T}$, so they have mean value 0 and variance 1. By H. Helson (see[1]),the Dirichlet series $f_\chi(s)$ converges on $\mathbb{C}_+$, so that $f_\chi(s)$ is hol-omorphic on $\mathbb{C}_+$ almost surely in χ. The stochastic variable $f_\chi(s)$ has variance $\sum_p |a_p|^2 p^{-2\Re s}$, which diverges for $\Re s < 0$ hence, by the central limit [4] (applic-able because of the regular behavior of each term $|a_p|^2 p^{-2\Re s}$), the quantity

$$\frac{\sum_{p:p\leq N} a_p \chi(p) p^{-s}}{\sum_{p:p\leq N} |a_p|^2 p^{-2\Re s}}$$

tends to the unit Gaussian distribution in the complex plane as $N \to \infty$ for $\Re s < 0$ so that $a_p\chi(s)p^{-s}$ diverges almost surely. It follows that the abscissa of converg-ence for f_χ is almost surely the line $\Re s < 0$. The derivative of the function f_χ is

$$f_\chi{}' = -\sum_p \chi(p) p^{-s-1/2}$$

Proposition(3.1.8) *Let g_χ be the Dirichlet series*

$$g_\chi(s) = \sum_p \chi(p) p^{-s-1/2}$$

(a) *For a dense set of characters χ, the line $\Re s = 1/2$ is both abscissa of conver-gence and natural boundary for the series g_χ.*
(b) *For almost all characters χ, the line $\Re s = 0$ is both abscissa of convergence and natural boundary for the series g_χ.*
We now show that *f* does not extend beyond $\mathbb{C}_{\frac{1}{2}}$. It follows from the relation $f_\chi{}' = -g_\chi$ that the function f_χ has the same two properties (a) and (b) of proposit-ion (3.1.8) as does the function g_χ. The final touches of the proof run as follows. For a dense set of χ, f_χ has $\mathbb{C}_{\frac{1}{2}}$ as its maximal domain of holomorphy (i.e., it has $\partial\mathbb{C}_{\frac{1}{2}}$ as natural boundary), so this is true in particular for a single character χ_0. We then let the function f_{χ_0} play the role of f in the argument treating the case $c_0 = 0$. Moreover, for almost all χ, f_χ has $\mathbb{C}_+$ as its maximal domain of holomo-rphy. The claim is proved.

Section(3.2): Dirichlet Series on Hilbert spaces

Let $\mathcal{W} = \|\omega_n\|_{n=n_0}^{\infty}$ be a sequence of positive numbers. In this section we are concerned with Hilbert spaces of functions representable by Dirichlet series:

$$\mathcal{H}_\omega = \{f(s) = \textstyle\sum_{n=n_0}^{\infty} a_n n^{-s} \,|\, \|f\|_{\mathcal{H}_\omega}^2 = \sum_{n=n_0}^{\infty} |a_n|^2\, \omega_n < \infty\} \qquad (14)$$

The prototypical case,where $\omega_n \equiv 1$, $n \geq 1$,was first studied By H. Hedenmalm, P. Lindqvist and K. Seip [1]. Among other results, they characterized the multipliers of the space .
One purpose of this section is to consider the scale of spaces obtained from the weight sequences ω^α, defined for $n \geq 2$ by

$$\omega_n^\alpha = (log n)^\alpha \qquad (15)$$

For brevity, we shall write $\mathcal{H}_\alpha$ for the space $\mathcal{H}_{\omega^\alpha}$, specifically

$$\mathcal{H}_\alpha = \left\{ f(s) = \sum_{n=n_0}^{\infty} a_n n^{-s} \,|\, \sum_{n=n_0}^{\infty} |a_n|^2\, (log n)^\alpha < \infty \right\}$$

(When $\alpha = 0$, it is more natural to let $n_0 = 1$ and to include the constant functions in $\mathcal{H}_0$. It is not essential to any of the issues we discuss here).
Before going further, let us remind some basic facts about Dirichlet series. A nice treatment can be found in Titchmarsh [5].
We shall follow the convention of writing the complex variable $s = \sigma + it$. A
Dirichlet series is a series of the form

$$\sum_{n=1}^{\infty} a_n n^{-s} \qquad (16)$$

Such a series may converge for no values of s, if it converges for any particular s_0, then it converges for all s with $\sigma > \Re(s_0)$. Therefore the largest open set in which a series (16) converges is a half-plane (at what points on the boundary of the half-plane the series converges is, in general, a delicate question). Let us adopt the notation, for ρ a real number Ω_ρ is the half-plane

$$\Omega_\rho = \{s \in \mathbb{C} | \sigma > \rho\}$$

Let $\sigma_c = inf\{\Re(s) : \sum_{n=1}^{\infty} a_n n^{-s}$ converges$\}$
this is called the *abscissa of convergence* of the series. The largest domain of convergence of the series is Ω_{σ_c}.
There are three other abscissae assoc-iated with the series (16) which we shall need. The first is the *abscissa of absolute convergence*, σ_a, defined by

$$\sigma_a = inf\left\{\Re(s) : \sum_{n=1}^{\infty} a_n n^{-s} \quad \text{converges absolutely}\right\}$$

Obviously $\sigma_a \geq \sigma_c$, it is straightforward that $\sigma_a \leq \sigma_c + 1$, because if $\sum_{n=1}^{\infty} a_n n^{-s}$ converges, then $|a_n| n^{-\sigma} = o(1)$. The second is the *abscissa of boundednes* σ_b, defined by

$$\sigma_b = inf\left\{\rho\colon \sum_{n=1}^{\infty} a_n n^{-s} \quad \text{converges to a bounded function in } \Omega_\rho\right\}$$

The third abscissa is the *abscissa of uniform convergence* σ_u, defined by

$$\sigma_u = inf\left\{\rho\colon \sum_{n=1}^{\infty} a_n n^{-s} \quad \text{converges uniformly in } \Omega_\rho\right\}$$

Clearly $\sigma_u \geq \sigma_b$, H. Bohr proved that $\sigma_u = \sigma_b$, and $\sigma_a \leq \sigma_b + 1/2$ [6]. Note that the series does not necessarily define a bounded function in Ω_{σ_b}, but the function it represents is bounded in all *strictly smaller* half-planes. If all the coefficients a_n are positive, then all three of $\sigma_c, \sigma_b, \sigma_a$ coincide.

We shall let $\mathcal{D}$ denote the set of functions that can be represented in some half- plane by a Dirichlet series.

Let $f(s)$ be holomorphic in the half-plane Ω_ρ. Let $\varepsilon > 0$. A real number $\mathcal{T}$ is called an ε *translation number of* f if

$$sup_{s\in\Omega_\rho}|f(s+i\mathcal{T}) - f(s)| \leq \varepsilon$$

The function $f(s)$ is called *uniformly almost periodic* in the half-plane Ω_ρ if, for every $\varepsilon > 0$, there exists a positive real number M such that every interval in $\mathbb{R}$ of length M contains at least one ε translation number of f.

We shall need the following theorem. The proof can be found in [7].

Theorem(3.2.99 *Suppose that $f(s)$ is represented by a Dirichlet series that converges uniformly in the half-plane Ω_ρ. Then f is uniformly almost periodic in Ω_ρ.*

Returning to the spaces $\mathcal{H}_\alpha$, it follows from the Cauchy-Schwarz inequality that any function in any space $\mathcal{H}_\alpha$ has $\sigma_a \leq 1/2$. Moreover, for all $\varepsilon > 0$,the function

$$\zeta\left(\frac{1}{2} + \varepsilon + s\right) - 1 = \sum_{n=2}^{\infty} \frac{1}{n^{\frac{1}{2}+\varepsilon}} n^{-s}$$

is in every $\mathcal{H}_\alpha$ and has a pole at $\frac{1}{2} - \varepsilon$, so the largest common domain of analyticity of the functions in any $\mathcal{H}_\alpha$ is $\Omega_{1/2}$. The reproducing kernel for $\mathcal{H}_\omega$ is

$k(s,u) = \sum_{n=n_0}^{\infty} \frac{1}{\omega_n} n^{-s-u}$ (17)

for the spaces $\mathcal{H}_\alpha$ this is essentially a fractional derivative or integral of the ζ function at $s + \bar{u}$.

We claim that the scale of spaces $\mathcal{H}_\alpha$ is in many ways analogous to the scale of spaces of holomorphic functions in the unit disk defined by

$K_\alpha = \{g(z) = \sum_{n=0}^{\infty} a_n z^n \mid \sum_{n=0}^{\infty} |a_n|^2 (n+1)^\alpha < \infty\}$ (18)

To discuss the Bergman-like Spaces, throughout this discussion, μ will be a positive Radon measure on [0,∞) for which

$\int_0^\infty n_0^{-2\sigma}\, d\mu(\sigma) = \int_0^\infty e^{-2(log n_0)\sigma}\, d\mu(\sigma) < \infty$ (19)

for some positive integer n_0. We also assume that

0 is in the support of μ. (20)

We define ω_n, for $n \geq n_0$, by

$\omega_n = \int_0^\infty n^{-2\sigma}\, d\mu(\sigma).$ (21)

Letting μ_α be the measure

$$d\mu_\alpha(\sigma) = \frac{2^{-\alpha}}{\Gamma(-\alpha)} \sigma^{-1-\alpha}$$

gives the weights from (15) for $\alpha < 0$:

$\int_0^\infty n^{-2\sigma}\, d\mu_\alpha(\sigma) = (log n)^\alpha, \qquad n \geq= 2, \ \ \alpha < 0.$

We let μ_0 be the unit point mass at 0,which has all of its moments equal to1.For any measure satisfying(19) and(20),the moments ω_n are a decreasing sequence that decays more slowly than any negative power of n:for all ε>0, there exists $c > 0$ so that

$\omega_n > cn^{-\varepsilon}$ (22)

Therefore, every space $\mathcal{H}_\omega$ consists of functions analytic in $\Omega_{1/2}$, and contains functions that are not analytically extendable to any larger domain.

Nonetheless there is a dense subspace of functions in $\mathcal{H}_\omega$ whose norms can be obtained by evaluating suitable integrals over the larger half-plane Ω_0. For the case of $\mathcal{H}_0$, the following theorem is due to F.Carlson [9]. Note that we do not assume that either side of (23) is finite.

Theorem(3.2.10) *Let $f(s) = \sum_{n=n_0}^{\infty} a_n n^{-s}$ be a function in $\mathcal{D}$ that has $\sigma_b = 0$ Let μ satisfy (19) and (20), and let ω_n be given by (21). Then*

$$\sum_{n=n_0}^{\infty} |a_n|^2 \omega_n = \lim_{c\to 0+} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt \int_0^{\infty} d\mu(\sigma) |f(s+c)|^2 \qquad (23)$$

Proof: Fix $0 < c < 1$, and let $\varepsilon > 0$. Let δ be given by

$$\delta = \frac{\varepsilon}{\left(1 + \mu\left[c, \frac{1}{c} + c\right]\right)\left(1 + \|f\|_{\Omega_0}\right)}.$$

By Bohr's theorem, the Dirichlet series of f converges uniformly in $\overline{\Omega_c}$, so there exists N so that

$$\left|\sum_{n=n_0}^{N'} a_n n^{-s} - f(s)\right| < \delta \quad \forall s \in \overline{\Omega_c},\ \forall N' \geq N.$$

Then

$$\operatorname*{Lim}_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt \int_0^{\frac{1}{c}} d\mu(\sigma) |f(s+c)|^2$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt \int_0^{1/c} d\mu(\sigma) \left|\sum_{n=n_0}^{N'} a_n n^{-(s+c)}\right|^2 + O(\varepsilon) \qquad (24)$$

As

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, n^{-(\sigma+it)} m^{-(\sigma-it)} = \delta_{mn} n^{-2\sigma},$$

We get from (24) that

$$\operatorname*{Lim}_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt \int_0^{\frac{1}{c}} d\mu(\sigma) |f(s+c)|^2 =$$

$$\sum_{n=n_0}^{N'} |a_n|^2 \int_0^{\frac{1}{c}} d\mu(\sigma) n^{-2\sigma-2c} + O(\varepsilon) \qquad (25)$$

for all $N' \geq N$.Taking the limit in (25) as c decreases to 0, and noting that ε can be made arbitrarily small for N large enough, we get

$$\sum_{n=n_0}^{\infty} |a_n|^2 \omega_n = \lim_{c\to 0+} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt \int_0^{1/c} d\mu(\sigma) |f(s+c)|^2.$$

But this is the same limit as (23).
Note that the integrals

$$\frac{1}{2T} \int_{-T}^{T} dt |f(\sigma + it)|^2$$

Are monotonically decreasing as a function of σ, so if $\mu(\{0\}) = 0$, the monotone convergence theorem yields:

Corollary(3.2.11) *Assume the hypotheses of Theorem(3.2.10), and also that $\mu(\{0\}) = 0$. Then*

$$\sum_{n=n_0}^{\infty} |a_n|^2 \omega_n = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt \int_0^{\infty} d\mu(\sigma) |f(s)|^2 \qquad (26)$$

By a *multiplier* of $\mathcal{H}_\omega$ we mean a function ϕ with the property that ϕf is in $\mathcal{H}_\omega$ for every f in $\mathcal{H}_\omega$. It follows from the closed graph theorem that for any multiplier ϕ, the operator of multiplication by ϕ, which we denote M_ϕ, is bounded. It is somewhat surprising that, although the spaces $\mathcal{H}_\omega$ consist of functions analytic in $\Omega_{1/2}$, the multipliers are somehow forced to extend to be analytic on all of Ω_0. For the case $\mu = \mu_0$, the following theorem is due to Hedenmalm, Lindqvist and Seip [1].

Theorem(3.2.12) *Let μ satisfy (19) and (20), and let* ω_n *be given by (21). Then the multiplier algebra of* $\mathcal{H}_\omega$ *is isometrically isomorphic to* $H^\infty(\Omega_0)\cap\mathcal{D}$*, where the norm on* $H^\infty(\Omega_0)\cap\mathcal{D}$ *is the supremum of the absolute value on* Ω_0*.*

Proof: It is clear that any multiplier ϕ must be in $\mathcal{D}$, just by considering $\phi(s)\cdot {n_0}^{-s}$. We shall prove the theorem in two parts:

(A) Show that if $\phi \in \mathcal{D}$ has $\sigma_b = 0$, then

$$\|M_\phi\| = \|\phi\|_{\Omega_0} \qquad (27)$$

(B) Show that if ϕ is a multiplier of $\mathcal{H}_\omega$,then ϕ is analytic and bounded in Ω_0.

Proof of (A): Suppose $\phi(s) = \sum_{n=1}^{\infty} b_n\, n^{-s}$ is bounded in all half-planes strictly smaller than Ω_0. Let $f(s) = \sum_{n=n_0}^{N}\sum_{n=1}^{\infty} a_n\, n^{-s}$ be a finite sum in $\mathcal{H}_\omega$.
Then ϕf has $\sigma_b = 0$, so by Theorem(3.2.10):

$$\|M_\phi f\|^2 = \lim_{c\to 0+}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dt \int_0^{\infty} d\mu(\sigma)|\phi(\sigma+c)f(\sigma+c)|^2 \le \|\phi\|_{\Omega_0}^2\|f\|^2$$

So if $\|\phi\|_{\Omega_0}$ is finite, then M_ϕ is bounded on the dense subspace of $\mathcal{H}_\omega$, consisting of finite sums, so extends by continuity to be a multiplier of the whole space.
We must show that $\|M_\phi\|$ equals $\|\phi\|_{\Omega_0}$. So let us assume tha t $\|M_\phi\|$ = 1 and $\|\phi\|_{\Omega_0}$ > 1, and derive a contradiction. (We are not assuming that $\|\phi\|_{\Omega_0}$ is necessarily finite). For each σ > 0, let

$$N_\sigma = sup_t|\phi(\sigma+it)|$$

By the Phragmen-Lindelӧf Theorem, N_σ is a strictly decreasing function of σ. Indeed, for σ very large, N_σ tends to $|b_1|$ which is less than or equal to $\|M_\phi\|$ < $\|\phi\|_{\Omega_0}$, so the conclusion follows by applying the Phragmen- Lindelӧf Theorem to the function $e^{\varepsilon s}\phi(s)$ for an appropriate choice of ε on a vertical strip.
Moreover, in each half-plane Ω_c for c > 0, the Dirichlet series of ϕ converges uniformly to ϕ by Bohr's Theorem, so by Theorem(3.2.9), ϕ is uniformly almost periodic in Ω_c. Therefore there exists $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ > 0 so that, for large enough T,

$$|\{t: |\phi(\sigma+it)| > 1+\varepsilon_1, -T\le t\le T\}| \ge \varepsilon_2(2T) \quad \forall \varepsilon_3 \le \sigma \le \varepsilon_3+\varepsilon_4 \qquad (28)$$

Since multiplication by ϕ is a contraction, so is multiplication by ϕ^j for any positive integer j. Therefore

$$\|\phi^j(s){n_0}^{-s}\|^2 \le \omega_{n_0} \qquad \forall j\in\mathbb{N}$$

So by Theorem(3.2.10), we conclude that

$$\omega_{n_0} \ge \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dt\int_0^{\varepsilon_4} d\mu(\sigma)\left|\phi^j(\sigma+\varepsilon_3)\right|^2 {n_0}^{-2(\sigma+\varepsilon_3)}$$

$$\ge \varepsilon_2(1+\varepsilon_1)^{2j}{n_0}^{-4\varepsilon_3}\mu([0,\varepsilon_4]) \qquad (29)$$

As the right-hand side of (29) tends to infinity with j, we get a contradiction.

Proof of (B): Let p_j denote the j^{th} prime, and let $\mathbb{N}_N$ denote the set of positive integers all of whose prime factors are in the set $\{p_1, \ldots, p_N\}$:

$$\mathbb{N}_N = \{p_1^{\upsilon_1}\ldots p_N^{\upsilon_N}: \upsilon_1,\ldots,\upsilon_N \in \mathbb{N}\}$$

For every positive integer N, let Q_N denote orthogonal projection from $\mathcal{H}_\omega$ onto the closed linear span of the functions

$$\{n^{-s}: n\in\mathbb{N}_N, n\ge n_0\}$$

Suppose ϕ is a multiplier of $\mathcal{H}_\omega$. Then we have

$$Q_N M_\phi Q_N = M_{Q_N(\phi)}Q_N = Q_N M_\phi \qquad (30)$$

Moreover, by a truncated version of the Euler product formula, we have that $f(s) = \sum a_n\, n^{-s}$ is any function in $\mathcal{H}_\omega$, then

$$|Q_N(f)(s)| = \left|\sum_{n\in\mathbb{N}_N} a_n n^{-s}\right| \le (sup|a_n|)\prod_{j=1}^{N}\left(1-{p_j}^{-\sigma}\right)^{-1}.$$

So if the coefficients of f are bounded, then $Q_N(f)$ is a bounded function in Ω_c for every c > 0.As $(Q_N f)(s+\varepsilon) = Q_N(f(s+\varepsilon))$, if the coefficients of f are $O(n^{\varepsilon})$ for every $\varepsilon > 0$, we have that $\sigma_b(Q_N f) \le \varepsilon$ for every $\varepsilon > 0$, *i.e.* $\sigma_b(Q_N f) \le 0$. By (22), as the weights decay more slowly than any negative power of n, it follows that for every f in $\mathcal{H}_\omega$, the coefficients of f are indeed $O(n^{\varepsilon})$ for every $\varepsilon > 0$, and *a fortiori* this hypothesis holds for every ϕ in the multiplier algebra of $\mathcal{H}_\omega$ (since $\phi(s)\, {n_0}^{-s}$ is in $\mathcal{H}_\omega$).
Therefore we can conclude that
$\sigma_b(Q_N\phi) \le 0, \quad \forall N > 1.$
Moreover, by (30), multiplication by $Q_N\phi$ on $Q_N\mathcal{H}_\omega$ is a compression of M_ϕ, so

$$\|M_{Q_N\phi}\|_{Q_N\mathcal{H}_\omega} \le \|M_\phi\|_{\mathcal{H}_\omega} < \infty$$

By repeating the argument in Part (A) and estimating

$$\|(Q_N\phi)^j 2^{-\nu s}\|$$

For $2^\nu \ge n_0$, we therefore conclude that
$\|Q_N\phi\|_{\Omega_0} \le \|M_\phi\|_{\mathcal{H}_\omega} \quad \forall N.$
By a normal families argument, some subsequence of $Q_N\phi$ converges uniformly on compact subsets of Ω_0 to some $H^\infty(\Omega_0)$ function, ψ say. On compact subsets of Ω_1 where the Dirichlet series for ϕ converges absolutely $Q_N\phi$ converges unifor-mly to ϕ. Therefore $\phi = \psi$,and so φ must be bounded and analytic in all of Ω_0.
It is a theorem of Khintchine and Kolmogorov that if the series $\sum |c_n|^2$ is finite, then almost every series $\sum \pm c_n$ converges (see [10]). It follows that if $\sum a_n n^{-s}$ is in $\mathcal{H}_\omega$, then for almost every choice of signs, $\sum \pm a_n n^{-s}$ will converge in Ω_0 (and in $cl(\Omega_0)$ for $\mathcal{H}_\alpha$ with $\alpha \ge 0$). This may help explain why the multipliers of $\mathcal{H}_\omega$ extend analytically to Ω_0.
To know about Dirichlet-like Spaces , let μ be a measure satisfying conditions (19) and (20) as above, and let ω_n be defined by (21) for $n \ge 2$. Define another weight sequence $\omega^{\flat}$ by

$$\omega_n^{\flat} = (\log n)^2 \omega_n.$$

The space $\mathcal{H}_{\omega^\flat}$ is exactly the set of functions whose derivatives are in $\mathcal{H}_\omega$, and is analogous to the Dirichlet space. We shall prove that the multipliers of $\mathcal{H}_{\omega^\flat}$ are contained in the multipliers of $\mathcal{H}_\omega$.One can prove a similar result for higher order derivatives, but for simplicity we stick to the case of a single derivative.

Theorem(3.2.13) *With notation as above, the multipliers of $\mathcal{H}_{\omega^\flat}$ are contractively contained in the multipliers of $\mathcal{H}_\omega$.*
Proof: We shall boot-strap from the following claim.
Claim(3.2.14) *There is a constant $K < \infty$ such that, if ϕ is a multiplier of $\mathcal{H}_{\omega^\flat}$ of norm one, and both ϕ and ϕ' have $\sigma_b \le 0$, then ϕ is a multiplier of $\mathcal{H}_\omega$ of norm at most K.*
Suppose the claim were proved. Let ψ be any multiplier of $\mathcal{H}_{\omega^\flat}$ of norm one. Then for every N, $Q_N\psi$ satisfies the hypotheses of the claim, so is a multiplier of $\mathcal{H}_\omega$ of norm at most K. By taking the weak-star limit of a subsequence of $Q_N\psi$, we can conclude that ψ is a multiplier of $\mathcal{H}_\omega$ of norm at most K.
To show K must be 1, assume it were greater. Then there would be a multiplier ϕ of $\mathcal{H}_{\omega^\flat}$ of norm one, which has norm greater than $\sqrt{K}$ as a multiplier of $\mathcal{H}_\omega$. By Theorem(3.2.12),

$$\|M_{\phi^2}\|_{\mathcal{H}_\omega} = \|M_\phi\|_{\mathcal{H}_\omega}^2$$

Then ϕ^2 would be a multiplier of norm one of $\mathcal{H}_{\omega^\flat}$, and have norm greater than K as a multiplier of $\mathcal{H}_\omega$, a contradiction.
We shall prove the claim with $K = \sqrt{2}$. Suppose the claim is false. Then there is some finite Dirichlet series

$$f(s) = \sum_{n=n_0}^{N} a_n n^s$$

in $\mathcal{H}_\omega$ of norm 1 such that $\|\phi f\| > K$. Let

$$g(s) = \sum_{n=n_0}^{N} a_n \frac{1}{\log n} n^s$$

be the primitive of f, which is of norm one in $\mathcal{H}_{\omega^\flat}$. Let

$$B = \{s \in \Omega_0 : |\phi(s)| > 1\}$$

By Theorem(3.2.10), there exists $c > 0$ such that

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt \int_0^\infty d\mu(\sigma) |\phi(s+c) f(s+c)|^2 \chi_B(s+c) > K^2 - 1, \qquad (31)$$

As ϕ^j is a multiplier of $\mathcal{H}_{\omega^\flat}$ of norm at most one for every positive integer j, we have

$$1 \geq \|\phi^j g\|_{\mathcal{H}_{\omega^{\mathcal{b}}}} = \|\phi^j f + j\phi^{j-1}\phi' g\|_{\mathcal{H}_\omega}$$

Therefore

$$1 \geq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt \int_0^\infty d\mu(\sigma) |\phi(s+c)|^{2(j-1)} \left|\phi(s+c)f(s+c) + j\phi^j(s+c)g(s+c)\right|^2 \chi_B(s+c)$$

$\geq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt \int_0^\infty d\mu(\sigma) \left|\phi(s+c)f(s+c) + j\phi^j(s+c)g(s+c)\right|^2 \chi_B(s+c)$
(32)

By subtracting ϕf + $\phi' g$ from ϕf +j $\phi' g$ and using Minkowski's inequality on (32), we get

$$4 \geq (j-1)^2 \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt \int_0^\infty d\mu(\sigma) |\phi'(s+c)g(s+c)|^2 \chi_B(s+c)$$

for all j, and so the limit is zero.Therefore by Cauchy-Schwarz,(32) becomes

$$1 \geq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt \int_0^\infty d\mu(\sigma) |\phi(s+c)f(s+c)|^2 \chi_B(s+c)$$

This contradicts (31) if $K \geq \sqrt{2}$.

Let us say a Radon measure ν supported in $cl(\Omega_0)$ is an α-Carleson measure if there exists some constant C such that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{|\Im s|\leq T} |f(s)|^2 d\nu(s) \leq C\|f\|^2_{\mathcal{H}_\alpha}$$

for every finite Dirichlet series f(s) $=\sum_{n=2}^N a_n n^{-s}$. Then we have

Corollary(3.2.15) *For* $0 < \alpha < 2$, *the function* ϕ *is a multiplier of* $\mathcal{H}_\alpha$ *if and only if*
(i) ϕ *is in* $\mathcal{D} \cap H^\infty(\Omega_0)$, *and*
(ii) The measure $|\phi'(s)|^2 \, d\mu_{\alpha-2}(\sigma)dt$ *is* α*-Carleson.*

Proof: The necessity of Condition (i) follows from Theorem (3.2.13). For condition (ii), observe that by Cauchy's theorem, if ϕ is in $\mathcal{D} \cap H^\infty(\Omega_0)$, then $\sigma_b(\phi') \leq 0$. The function ϕ is a multiplier if and only if

$$\|\phi f' + \phi' f\|_{\mathcal{H}_{\alpha-2}} \leq C\|f\|_{\mathcal{H}_\alpha}$$

for every finite Dirichlet series f. If ϕ satisfies Condition (i), then

$$\|\phi f'\|_{\mathcal{H}_{\alpha-2}} \leq \|\phi\|_{\Omega_0} \|f\|_{\mathcal{H}_\alpha}$$

So such a ϕ is a multiplier if and only if

$$\|\phi' f\|^2_{\mathcal{H}_{\alpha-2}} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_0^\infty |\phi(s)|^2 \, |f(s)|^2 d\mu_{\alpha-2}(\sigma) dt$$

is controlled by $\|f\|^2_{\mathcal{H}_\alpha}$. if and only if Condition (ii) holds.

What is a Space of Dirichlet series with the Pick property? Let $\mathcal{H}$ be a Hilbert function space on a set X with reproducing kernel k.We say $\mathcal{H}$ has the *Pick prope-rty* if, given any distinct points $\lambda_1, \ldots, \lambda_N$ in X and any complex numbers z_1,..., z_n, then a necessary and sufficient condition for the existence of a function ϕ in the closed unit ball of the multiplier algebra of $\mathcal{H}$ that has the value z_i at each λ_i is that the matrix

$$\left[k(\lambda_i, \lambda_j)(1 - z_i \bar{z}_j)\right]_{i.j=1}^N$$

be positive semi-definite. We say $\mathcal{H}$ has the *complete Pick property* if, for any positive integer s, any distinct points λ_1,...,λ_N in X and any s -by- s matrices Z_1,...,Z_n,then anecessary and sufficient condition for the existence of a function ϕ in the closed unit ball of the multiplier algebra of $\mathcal{H} \otimes \mathbb{C}^s$ that has the value Z_i at each λ_i is that the Ns -by- Ns matrix

$$\left[k(\lambda_i, \lambda_j)(1 - Z_i Z_j^*)\right]_{i.j=1}^N$$

Be positive semi-definite. See [42] for a treatment of complete Pick kernels. For every integer ≥ 2, let $F(n)$ be the number of ways n can be factored, where the order matters. Let $F(1) = 1$. Then the following identity holds [5,8,12]

$$\sum_{n=1}^{\infty} \frac{F(n)}{n^s} = \frac{1}{2-\zeta(s)} \quad (33)$$

For the rest of this section, we shall fix

$$\omega_n = \frac{1}{F(n)} \quad (34)$$

and consider the space $\mathcal{H}_\omega$ (with $n_0 = 1$). The kernel function for $\mathcal{H}_\omega$ is then

$$k(s,u) = \frac{1}{2-\zeta(s+\bar{u})} \quad (35)$$

As the reciprocal of k has only one positive square,it follows from the McCullough - Quiggin theorem (see [13], [14]), that k is a complete Pick kernel.

Section(3.3): INTEGRAL AND BOUNDARY OF DIRICHLET SERIES:

A classical theorem of F.Carlson [9]says that if an ordinary Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad (36)$$

converges in the right half-plane $\Re s > 0$ and is bounded in every half-plane $\Re s \geq \delta > 0$, then for each $\sigma > 0$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |f(\sigma+it)|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma} \quad (37)$$

From a modern viewpoint, Carlson's theorem is a special case of the general ergodic theorem, as will be explained below.

A natural question, first raised by H. Hedenmalm [15], is whether the identity (37) remains valid when $\sigma = 0$, provided $f(s)$ is a bounded function in $\Re s > 0$. The problem makes sense because we may replace $f(\sigma + it)$ by the non-tangential limit $f(it)$, which in this case exists for almost every t. We note that the general ergodic theorem is of no help for this problem.

We denote by $\mathcal{H}^\infty$ the class of functions $f(s)$ that are bounded in $\Re s > 0$ with f represented by an ordinary Dirichlet series (36) in some half-plane. We will use the notation

$$\|f\|_\infty = sup_{\sigma>0}|f(\sigma+it)| \quad \text{and} \quad \|f\|_2^2 = \sum_{n=1}^{\infty}|a_n|^2.$$

The result is that there is no "boundary version" of Carlson's Theorem:

To see how to obtain Carlson's theorem as a special case of the general ergodic theorem,we resort to a fundamental observation of Bohr [6]. We put

$$z_1 = 2^{-s}, z_2 = 3^{-s}, \ldots, z_j = p_j^{-s}, \ldots,$$

Where p_j denotes the *j*-th prime, then, in view of the fundamental theorem of arithmetic, the Dirichlet series(36) can be considered as a power series in infinitely many variables. For a given Dirichlet series f we denote by F the corresponding extension to the infinite polydisc $\mathbb{D}^\infty$, then if F happens to be a function of only n variables, it is immediate from Kronecker's theorem and the maximum principle that

$$\|f\|_\infty = \|F\|_\infty \quad (38)$$

where the norm on the right-hand side is the $H^\infty(\mathbb{D}^\infty)$ norm. The result is the same in the infinite-dimensional case, but some care has to be taken when defining the norm in the polydisc. (See[1]) We can now think of any vertical line $t \mapsto \sigma + it$ as an ergodic flow on the infinite-dimensional torus $\mathbb{T}^\infty$:

$$(\mathcal{T}_1, \mathcal{T}_2, \ldots) \mapsto (p_1^{-it}\mathcal{T}_1, p_2^{-it}\mathcal{T}_2, \ldots) \; for \; (\mathcal{T}_1, \mathcal{T}_2, \ldots) \in \mathbb{T}^\infty$$

If $F(p_1^{-\sigma}z_1, p_2^{-\sigma}z_2, \ldots)$ is continuous on $\mathbb{T}^\infty$, then the general ergodic theorem yields (37). A similar problem concerning integral means of nontangential limits can be stated for the closely related space $\mathcal{H}^2$ which consists of those Dirichlet series of the form (36) for which $\|f\|_2 < \infty$. In this case, $f(s)/s$ belongs to the Hardy space H^2 of the half-plane $\sigma > 1/2$, (see [1, 3]):

$$\int_\theta^{\theta+1} \left|f\left(\frac{1}{2}+it\right)\right|^2 dt \leq C\|f\|_2^2 \quad (39)$$

with C an absolute constant independent of θ. It follows immediately that we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \left|f\left(\frac{1}{2}+it\right)\right|^2 dt = \sum_{n=1}^{\infty}|a_n|^2 n^{-1} \quad (40)$$

for every function f in $\mathcal{H}^2$, since the space of Dirichlet polynomials is dense in $\mathcal{H}^2$ and the identity holds trivially when f is a Dirichlet polynomial.

Now, we make some simple observations in order to clarify what the problem is really about. We note that another way of phrasing Hedenmalm's question is to ask whether we have:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T|f(it)|^2dt=\lim_{\sigma\to 0+}\lim_{T\to\infty}\frac{1}{T}\int_0^T|f(\sigma+it)|^2dt$$

for every f in $\mathcal{H}^\infty$. We observe that for a finite interval, say for $t_1 < t_2$, we have indeed

$$\int_{t_1}^{t_2}|f(it)|^2dt=\lim_{\sigma\to 0+}\int_{t_1}^{t_2}|f(\sigma+it)|^2dt,$$

As follows by Lebesgue's dominated convergence theorem. Similarly , by applying Cauchy's integral theorem and again Lebesgue's dominated convergence theorem, we get

$$a_n=\lim_{T\to\infty}\frac{1}{T}\int_0^T|f(it)|^2\,n^{it}dt,$$

for every positive integer n. Let us also note that the upper estimate

$$\liminf_{T\to\infty}\frac{1}{T}\int_0^T|f(it)|^2\,dt\geq\lim_{\sigma\to 0+}\lim_{T\to\infty}\frac{1}{T}\int_0^T|f(\sigma+it)|^2\,dt=\|f\|_2^2$$

may be obtained from the Poisson integral representation of $[f(\sigma+it)]^2$, i.e.,

$$[f(\sigma+it)]^2=\frac{1}{\pi}\int_{-\infty}^{\infty}[f(i\mathcal{T})]^2\frac{\sigma}{(t-\mathcal{T})^2+\sigma^2}dt$$

We conclude from these observations that the counterexamples of the Theorem should be functions whose nontangential limits have increasing oscillations when the argument t tends to ∞.

We begin by recalling some terminology and briefly reviewing Rudin's method for constructing real parts of analytic functions in the polydisc $\mathbb{D}^n$ with given boundary values almost everywhere on the distinguished boundary $\mathbb{T}^n$.Rudin treats $\mathbb{D}^n$ with arbitrary $n\geq 1$,but we shall need only the case $n=2$. We refer to [16].

We employ the complex notation for points on the distinguished boundary $\mathbb{T}^2$ of the bidisc $\mathbb{D}^2$. The normalized Lebesgue measure on $\mathbb{T}^2$ is denoted by m_2. The distance between $\mathcal{T}=(\mathcal{T}_1,\mathcal{T}_2)$ and $\mathcal{T}'=(\mathcal{T}_1',\mathcal{T}_2')$ is

$$d(\mathcal{T},\mathcal{T}')=max(|\mathcal{T}_1-\mathcal{T}_1'|,|\mathcal{T}_2-\mathcal{T}_2'|),$$

and $B(\mathcal{T},r)$ stands for the ball with center $\mathcal{T}$ and radius r. We set

$$P_r(\mathcal{T})=\frac{(1-r^2)^2}{|1-r\mathcal{T}_1|^2|1-r\mathcal{T}_2|^2}\,,o<r<1,$$

Where $\mathcal{T}=(\mathcal{T}_1,\mathcal{T}_2)$ is a point in $\mathbb{T}^2$.In particular,the Poisson integral of a measure μ on $\mathbb{T}^2$ can then be expressed in the form:

$$P\mu(r\mathcal{T})=\int_{\mathbb{T}^2}P_r(\mathcal{T}\overline{\omega})\mu(d\omega),$$

Where $\mathcal{T}\overline{\omega}=(\mathcal{T}_1\overline{\omega}_1,\mathcal{T}_2\overline{\omega}_2)$ for every finite Borel measure μ and every $\mathcal{T}\in\mathbb{T}^2$, the Poisson maximal operator is defined by setting : $P_*|\mu|(\mathcal{T})=sup_{r\in(0,1)}P_r|\mu|(\mathcal{T})$.

The following estimate is immediate.

Lemma(3.3.16) *We have* $P_r(\mathcal{T})\leq 16\left(d\left(\mathcal{T},(1,1)\right)\right)^{-2}$ *for* $r\in(0,1)$. *In particular, if* $s=d(\mathcal{T},supp(\mu))>0$, *then* $P_*\mu(\mathcal{T})\leq 16s^{-2}\|\mu\|$.

Let $g:\mathbb{T}^2\to\mathbb{R}$ be a strictly positive, integrable, and lower semicontinuous function, we may express it as

$$g=\sum_{j=1}^{\infty}p_j$$

where the p_j are non-negative trigonometric polynomials on $\mathbb{T}^2$. For each $j\geq 1$, Rudin shows that one may choose a positive singular measure μ_j with $\mu_j(\mathbb{T}^2)=\int_{\mathbb{T}^2}p_j dm_2$ and so that $P(p_j-\mu_j)$ is the real part of an analytic function on $\mathbb{D}^2$. More specifically, μ_j is chosen to be of the form $p_j\lambda_{kj}$, where $kj\geq deg(p_j)$ and for any positive integer k the measure λ_k has the Fourier series expansion

$$\lambda_k=\sum_{j=-\infty}^{\infty}exp\left(ikj(\theta_1,\theta_2)\right)\qquad(41)$$

On $\mathbb{T}^2$, where (θ_1, θ_2) corresponds to the point $(e^{i\theta_1}, e^{i\theta_2})$ on $\mathbb{T}^2$. This measure is positive, has mass one, and with respect to the standard Euclidean identification $\mathbb{T}^2 = [0,2\pi)^2$ of the 2-torus, it is just the normalized 1-measure sup-ported on $2k = 1$ line segments of $\mathbb{T}^2 = [0,2\pi)^2$ parallel to the direction (1,-1). On the torus, its support consists of k equally spaced closed "rings".
For s in the right half-plane $\mathbb{C}^+ = \{z \in \mathbb{C}: \Re z > 0\}$, we set $\phi(s) = (2^{-s}, 3^{-s})$.
The induced boundary map takes the form : $\phi(it) = \big(exp(-ilog(2)t), \exp(-ilog(3)t)\big)$.
We denote the image of the boundary by L. Thought of as a subset of $[0,2\pi)^2$, L consists of a dense set of segments that have common direction vector $v_0 = (log(2), log(3))$.

Lemma(3.3.17) *Let a summable sequence of nonnegative numbers $a_k (k = 1,2, \dots)$ be given. If the measure μ satisfies*

$$0 \le \mu \le \sum_{k=1}^{\infty} a_k \lambda_k$$

Then $\lim_{r\to 1-} P\mu(\mathcal{T}) = 0$ *for almost every* $\mathcal{T} \in L$.

Proof. It is enough to prove the claim for $\mu = \sum_{k=1}^{\infty} a_k \lambda_k$. By [16], we know that $\lim_{r\to 1-} P\mu(\mathcal{T}) = 0$for m_2-a.e. $\mathcal{T} \in \mathbb{T}^2$. Pick any segment $J \in L$ of length 1/2, say. By Fubini's Theorem we see that for almost every $s \in [0, 1/2]$ the claim holds for almost every $J \in L + s(1,-1)$. However, since the measure μ is invariant with respect to the translation $\mathcal{T} \to \mathcal{T} + s(1,-1)$,we see that the statem-ent is true for every $s \in [0, 1/2]$. In particular, we have : $\lim_{r\to 1-} P\mu(\mathcal{T}) = 0$ for almost every $\mathcal{T} \in J$, by expressing L as a countable union of such segments, we obtain the conclusion of the lemma.

Lemma(3.3.18) *Given ε > 0, there is an open set $U \subset \mathbb{T}^2$ with $m_2(U) < \varepsilon/2$ and a probability measure μ on $\mathbb{T}^2$ such that the function*

$$h = P(\chi_U + (1/2)\chi_{U^c}) - P\mu$$

is the real part of a function in the unit ball of $H^\infty(\mathbb{D}^2)$ Moreover, $\lim_{r\to 1-} h(r\mathcal{T}) = 1$
for almost every $\mathcal{T} \in L$ with respect to the Hausdorff 1-measure on L.

Proof. We begin by covering L with a thin open strip U that becomes thinner and thinner so that $m_2(U) < \varepsilon/2$. For example, we may take

$$U = \cup_{t\in\mathbb{R}} B\left(\phi(it), \frac{\varepsilon}{100(1+|t|)^2}\right).$$

The next step is to run Rudin's construction with respect to the positive and lower semicontinuous function $\chi_U + (1/2)\chi_{U^c}$. Thus we choose strictly positive trigonometric polynomials $p_1, p_2, \dots$ on $\mathbb{T}^2$ in such a way that
$\sum_{j=1}^{\infty} p_j = \chi_U + (\varepsilon/2)\chi_{U^c}$ at every point of $\mathbb{T}^2$. Moreover, by a compactness argument, we observe that we may perform the selection in such a way that

$$0 < p_j(\mathcal{T}) \le j^{-2} \quad if \quad d(\mathcal{T}, \partial U) \ge j^{-1} \qquad (42)$$

We may also require that $\int_{\mathbb{T}^2} p_j dm_2 \le j^{-2}$ We set $\mu_j = p_j \lambda_{k(j)}$ and observe that

$$\|\mu_j\| = \int_{\mathbb{T}^2} p_j dm_2 \le j^{-2} \qquad (43)$$

Write $\lambda_0 = \sum_{j=1}^{\infty} j^{-2} \lambda_{k(j)}$
Then, according to Lemma(3.3.17), we have

$$\lim_{r\to 1-} P\lambda_0(r\mathcal{T}) = 0 \quad for \quad \mathcal{T} \in L\backslash E \qquad (44)$$

where E has linear measure zero. A fortiori, we have in particular that

$$\lim_{r\to 1-} P\mu_j(r\mathcal{T}) = 0 \quad for \quad \mathcal{T} \in L\backslash E \qquad (45)$$

We now set $\mu = \sum_{j=1}^{\infty} \mu_j$. The fact that

$$h = P(\chi_U + (\varepsilon/2)\chi_{U^c}) - P\mu$$

is the real part of an analytic function in the unit ball of $H^\infty(\mathbb{D}^2)$ is immediate from Rudin's theorem [16]. Since U is open and the mass of the two-dimensional Poisson kernel concentrates on any neighborhood of the origin as $r \to 1-$, we see that $\lim_{r\to 1-} P\left(\chi_U + \left(\frac{\varepsilon}{2}\right)\chi_{U^c}\right)(r\omega) = 1$ for every $\omega \in U$. Hence it remains to verify that $\lim_{r\to 1-} P\mu(r\mathcal{T}) \to 0$ for almost every $\mathcal{T} \in L$ with respect to Hausdorff 1-measure on L. In fact, we will show that

$$\lim_{r\to 1-} P\mu(r\mathcal{T}) = 0 \quad if \quad \mathcal{T} \in L\backslash E \qquad (46)$$

which is clearly sufficient.
Fix an arbitrary $\mathcal{T} \in L\backslash E$. Write $s = d(\mathcal{T}, \partial U) > 0, B = B\left(\mathcal{T}, \frac{s}{2}\right)$, and set
$\mu_k^a = \chi_B \mu_k$ and $\mu_k^b = \mu_k - \mu_k^a$.
Pick $k_0 \geq (s/2)^{-1}$. We clearly have
$\sum_{k=k_0}^{\infty} \mu_k^a \leq \lambda_0$ (47)
so that (44) implies that
$\lim_{r\to 1-} P\left(\sum_{k=k_0}^{\infty} \mu_k^a\right)(r\mathcal{T}) = 0$ (48)
On the other hand, we have $d\left(\mathcal{T}, supp\left(\mu_k^b\right)\right) \geq s/2$ and $\|\mu_k^b\| \leq \|\mu_k\| \leq k^{-2}$
Hence Lemma(3.3.17) yields
$P_*\left(\sum_{k=k_0}^{\infty} \mu_k^b\right)(r\mathcal{T}) \leq 64s^{-2} \sum_{k=k_0}^{\infty} k^{-2} \leq C(\mathcal{T})k_0^{-1}$ (49)
By (45), we have $\lim_{r\to 1-} P\left(\sum_{k=1}^{k_0-1} \mu_k\right)(r\mathcal{T}) = 0$ (50)
As k_0 can be chosen arbitrarily large, we obtain the desired conclusion by combining this fact with (48) and (49).

Theorem(3.3.19) *The following two statements hold:*
(i) *There exists a function f in $\mathcal{H}^\infty$ such that*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |f(it)|^2 dt$$

does not exist.
(ii) *Given $\varepsilon > 0$, there exists a singular inner function g in $\mathcal{H}^\infty$ such that $\|g\|_2 \leq \varepsilon$.*

Proof. We begin by proving part (ii) of the theorem. Let h be the function given in Lemma(3.3.18), and assume that it is the real part of the analytic function H on $\mathbb{D}^2$.When k is large enough,the function $R = exp(k(H-1))$ satisfies $\|R\|_{H^\infty(\mathbb{D}^2)} = 1$ and $\|R\|_{H^2(\mathbb{D}^2)} \leq \varepsilon$. Moreover, its modulus has radial boundary values1 at almost every point of the set L with respect to linear measure. It is almost immediate from this that the function

$$g(s) = R(\phi(s)) = R(2^{-s}, 3^{-s})$$

is, by construction, a singular inner function in $\mathbb{C}^+$ with $\|g\|_{\mathcal{H}^2} < \varepsilon$. The only matter that requires a little attention, is how we conclude that $|g|$ has unimodular boundary values almost everywhere. The point is that horizontal boundary approach in $\mathbb{C}^+$ does not transfer exactly via ϕ to radial approach, but instead to what we will call quasi-radial approach.
This means that $(r_1\omega_1, r_2\omega_2) \to (\omega_1, \omega_2)$where $r_1 \to 1^-$ and $r_2 \to 1^-$in such a way that the ratio $(1-r_1)$=$(1-r_2)$ stays uniformly bounded from above and below. However, apart from a change of non-essential constants, the proof of Lemma (3.3.18) remains valid for quasi-radial approach.This is easily verified for Lemma (3.3.16), and it remains true for the basic theorem [16], on radial limits of singular measures [21].These remarks conclude the proof of part (ii) of Theorem(3.3.19).
We now turn to the proof of part (i) of Theorem (3.3.19). The basic construction is similar to the one in the proof of part (ii), so we only indicate the required changes. To simplify the notation, we identify the imaginary axis with L. Lebesgue measure on the imaginary axis is denoted by ν. This time we cover only part of the image of the imaginary axis L by an open set U. To this end, given $\varepsilon > 0$, we first construct by induction a sequence of open subsets $U_1, U_2, \ldots \subset T^2$ with the following properties for each $n \geq 1$:

(i) There is $t_n \geq n$ so that $\nu(U_n \cap [0, it_n]) > \left(1 - \frac{\varepsilon}{2}\right) t_n$.
(ii) The closures $\overline{U_1}, \overline{U_2}, \ldots, \overline{U_n}$ are disjoint.
(iii) The set U_n is a finite union of open dyadic squares and

$$\sum_{j=1}^{n} m_2(U_j) < \frac{\varepsilon}{2}$$

In the first step, we set $t_1 = 1$ and, apart from a finite number of points, we cover $[0, it_1]$ by a finite union of dyadic open cubes U_1 with $m_2(U_1) = m_2(\overline{U_1}) < \frac{\varepsilon}{2}$. Assume then that sets $U_1, \ldots, U_n$ with the right properties have been found. Since we are dealing with finite unions of open squares,it holds that $m_2\left(\overline{\bigcup_{j=1}^n U_j}\right) \leq \sum_{j=1}^n m_2(U_j) < \frac{\varepsilon}{2}$, and hence we may apply the continuous version of Weyl's equi-distribution theorem for Kronecker flows in order to select $t_{n+1} \geq n+1$ with

$$\nu\left(\left(\overline{\bigcup_{j=1}^n U_j}\right) \cap [0, it_{n+1}]\right) < \frac{\varepsilon}{2}$$

Then U_{n+1} is obtained by covering a sufficiently large portion of the set $[0, it_{n+1}]\setminus\bigcup_{j=1}^{n}U_j$ by a union of open dyadic squares that has a positive distance to $\overline{\bigcup_{j=1}^{n}U_j}$ and satisfies $m_2(U_{n+1}) < \varepsilon - \sum_{j=1}^{n} m_2\left(U_j\right)$. This completes the induction.

Set $U = \bigcup_{k=1}^{\infty}U_{2k-1}$ and $V = \bigcup_{k=1}^{\infty}U_{2k}$. We run the Rudin construction exactly as in the proof of part (ii) corresponding to the lower semicontinuous boundary function $\chi_U + \left(\frac{\varepsilon}{2}\right)\chi_{U^c}$. Hence, we obtain a polyharmonic function h on $\mathbb{D}^2$ with (quasi-)radial boundary values 1 at almost every point of $U\cap L$ (respectively $\frac{\varepsilon}{2}$ at almost every point of V), and such that h is the real part of the analytic function H on $\mathbb{D}^2$. By property (i) of the sets $U_1, U_2, \ldots,$ it is then evident that with sufficiently large k the function

$f(s) = exp\left(k(1 + (2^{-s}, 3^{-s}) - 1)\right)$ satisfies $f \in \mathcal{H}^\infty$ and

$$\liminf_{T\to\infty}\frac{1}{T}\int_0^T |f(it)|^2\, dt \le \varepsilon$$

as well as

$$\liminf_{T\to\infty}\frac{1}{T}\int_0^T |f(it)|^2\, dt = 1$$

Now, to know and prove Fatou theorems for $\mathcal{H}^p$,we will now in some sense return to what appeared as a difficulty in the proof of Theorem(3.3.19), namely that the imaginary axis has measure zero when viewed as a subset of $\mathbb{T}^2$. Thus, a priori, it makes no sense to speak about the restriction to the imaginary axis of a function in $L^p(\mathbb{T}^\infty)$. We will now show that, for functions in $H^p(\mathbb{D}^\infty)$, we can find a meaningful connection to the boundary limits of the corresponding Dirichlet series.

We consider a special type of boundary approach by setting for each $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \ldots) \in \mathbb{T}^\infty$ and $\theta \ge 0$

$$b_\theta(\mathcal{T}) = \left(p_1^{-\theta}\mathcal{T}_1, p_2^{-\theta}\mathcal{T}_2, \ldots\right)$$

We also recall that the Kronecker flow on $\overline{\mathbb{D}}^\infty$ is defined by setting

$$T_t(z_1, z_2, \ldots) = \left(p_1^{-it}z_1, p_2^{-it}z_2, \ldots\right).$$

For an arbitrary $z \in \overline{\mathbb{D}}^\infty$, we denote by $T(z)$ the image of z under this flow, i.e., $T(z)$ is the one-dimensional complex variety $T(z)= \{ T_t(z) \colon t \in \mathbb{R} \}$. We equip $T(z)$ with the natural linear measure, which is just Lebesgue measure on the real *t*-line. Moreover, for $\sigma> 0$, we set $\mathbb{T}_\sigma^\infty = b_\theta(\mathbb{T}^\infty)$. The natural Haar measure $m_{\infty,\sigma}$ on $\mathbb{T}_\sigma^\infty$ is obtained as the pushforward of m_∞ under the map b_θ. The set $\mathbb{T}_{1/2}^\infty$ is of special interest,since in a sense it serves as a natural boundary for the set $\mathbb{D}^\infty \cap \ell^2$, where point evaluations are bounded for the space $H^p(\mathbb{D}^\infty)$ with $p \in (0, \infty)$.

The version of Fatou's theorem for H^∞ reads as follows.

Theorem(3.3.20) *Let F be a function in $H^\infty(\mathbb{D}^\infty)$. Then we may pick a representative $\tilde{F}$ for the boundary function of F on the distinguished boundary $\mathbb{T}^\infty$ such that $\tilde{F}(\mathcal{T}) = \lim_{\theta\to 0^+} F\left(b_\theta(\mathcal{T})\right)$ for almost every $\mathcal{T} \in \mathbb{T}^\infty$. In fact, for every $\mathcal{T} \in \mathbb{T}^\infty$, we have $\tilde{F}(\mathcal{T}') = \lim_{\theta\to 0^+} F\left(b_\theta(\mathcal{T}')\right)$ for almost every $\mathcal{T}' \in T(\mathcal{T})$.*

Proof. Recall that by [69]the values of $H^\infty(\mathbb{D}^\infty)$-functions are well-defined in $\mathbb{D}^\infty$ at points z with coordinates tending to zero, i.e. for $z \in c_0$. We simply define the desired representative $\tilde{F}$ for the boundary values by setting $\tilde{F}(\mathcal{T}) = \lim_{\theta\to 0^+} F(\theta o \mathcal{T})$ whenever this limit exists and otherwise $\tilde{F}(\mathcal{T}) = 0$. The Borel measurability of $\tilde{F}$ is clear. The second statement follows immediately by considering for each $\mathcal{T} \in \mathbb{T}^\infty$ the analytic function $f_{\mathcal{T}} \colon f_{\mathcal{T}}(\theta + it) = F\left(T_t b_\theta(\mathcal{T})\right)$ and observing that for each $\mathcal{T} \in \mathbb{T}^\infty$ we have $f_{\mathcal{T}} \in \mathcal{H}^\infty$. Now the classical Fatou theorem applies to $f_{\mathcal{T}}$. The fact that the set $\left\{\mathcal{T} \in \mathbb{T}^\infty : = \lim_{\theta\to 0^+} F\left(b_\theta(\mathcal{T})\right) \text{ exists}\right\}$ has full measure is an immediate consequence of the ergodicity of the Kronecker flow $\{T_t\}_{t\ge 0}$ and the second statement. Finally, we observe that it is easy to check the formula

$\hat{F}(\beta) = p_1^{\beta_{1\sigma}} \ldots p_k^{\beta_{k\sigma}} \int_{\mathbb{T}^\infty} F\left(b_\theta(\mathcal{T})\right)\bar{\mathcal{T}}^\beta m_\infty(d\mathcal{T})$

For the Fourier coefficients of an $H^\infty(\mathbb{D}^\infty)$-function. Lebesgue's dominated convergence theorem now yields $\hat{\tilde{F}} = \hat{F}$, whence $\tilde{F} = F$ almost surely, and this finishes the proof of the first statement.

To arrive at a similar result for $\mathcal{H}^p$,we need to make sense of the restriction $F \longmapsto F|_{\mathbb{T}_{1/2}^\infty}$ as a map from $H^p(\mathbb{D}^\infty)$ to $L^p\left(\mathbb{T}_{1/2}^\infty, m_{\infty,1/2}\right)$.When F is a polynomial, we must have

$$F|_{\mathbb{T}^{\infty}_{\frac{1}{2}}}(\mathcal{T}) = F\left(b_{1/2}(\mathcal{T})\right)$$

Since this formula can be written as a Poisson integral and the polynomials are dense in $H^p(\mathbb{D}^\infty)$, this leads to a definition of $F|_{\mathbb{T}^{\infty}_{\frac{1}{2}}}$ for general F.Indeed, by using elementary properties of Poisson kernels for finite polydiscs, we get that $F \longmapsto F|_{\mathbb{T}^{\infty}_{1/2}}$ is a contraction from $H^p(\mathbb{D}^\infty)$ to $L^p\left(\mathbb{T}^{\infty}_{1/2}, m_{\infty,1/2}\right)$.

Theorem(3.3.21) *Let F be a function $H^p(\mathbb{D}^\infty)$ fo r $p \geq 2$.Then we may pick a representative $\tilde{F}_{1/2}$ for the restriction $F|_{\mathbb{T}^{\infty}_{1/2}}$ on the distinguished boundary $\mathbb{T}^\infty$ such that $\tilde{F}(\mathcal{T}) = \lim_{\theta \to \frac{1}{2}+} F\left(b_\theta(\mathcal{T})\right)$ for almost* every *$\mathcal{T} \in \mathbb{T}^\infty$. In fact, for every $\mathcal{T} \in \mathbb{T}^\infty$, we have $\tilde{F}_{1/2}(\mathcal{T}') = \lim_{\theta \to \frac{1}{2}+} F\left(b_\theta(\mathcal{T}')\right)$ 0) for almost* every *$\mathcal{T}' \in T(\mathcal{T})$.*

Proof. The existence of the boundary values is obtained just as in the proof of Theorem (3.3.20). This time one applies the known embedding for $p = 2$ to define $\tilde{F}_{1/2}$.
We may now observe that if F is in $H^p(\mathbb{D}^\infty)$($p \geq 2$), then we have for every $\mathcal{T} \in \mathbb{T}^{\infty}_{1/2}$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left|\tilde{F}(T_t\mathcal{T})\right|^p dt = \left\|\tilde{F}_{1/2}\right\|^p_{L^p(\mathbb{T}^\infty)} \qquad (51)$$

Indeed, (51) holds for polynomials. Hence, the fact that polynomials are dense in $H^p(\mathbb{D}^\infty)$, we obtain (51). It is rather puzzling that (51), which may be understood as a strengthened variant of the Birkhoff-Khinchin ergodic theorem for functions in $H^p(\mathbb{D}^\infty)$, is known to hold only when $p = 2,4,6, ...$

References

[1] H. Hedenmalm, P. Lindqvist And K. Seip, A Hilbert Space Of Dirichlet Series And Systems Of Dilated Functions In $L^2(0,1)$, Duke Math. J. 86, 1-37 (1997).
[2] J. E. Littlewood, On Inequalities In The Theory Of Functions Proc. London Math. Soc. 23 (1925), 481–519.
[3] H. L. Montgomery, Ten Lectures On The Interface Between Analytic Number Theory And Harmonic Analysis, Volume 84 Of Cbms Regional Conference Series In Mathematics, Ams, 1994.
[4] B. V. Gnedenko, The Theory Of Probability, 4th Ed., Chelsea, New York, 1967.
[5] E.C. Titchmarsh. The Theory Of Functions. Oxford University Press, London, 1932.
[6] H. Bohr, Äuber Die Bedeutung Der Potenzreihen Unendlich Vieler Variabeln In Der Theorie Der Dirichletschen Reihen $\sum A_N/N^S$, Nachr. Akad. Wiss. Gäottingen Math.-Phys. Kl. (1913), 441-488.
[7] A.S. Besicovitch. Almost Periodic Functions. Cambridge University Press, London, 1932
[8] Jim Agler, John E. Mccarthy, Pick Interpolation And Hilbert Function Spaces, Grad. Stud. Math., Vol. 44, American Mathematical Society, Providence, Ri, 2002.
[9] F. Carlson, Contributions µa La Thffeorie Des S¶Eries De Dirichlet, Note I, Ark.Mat.16,No.18(1922),1-19.
[10] P. L´Evy. Sur Les S´Eries Dont Les Termes Sont Des Variables ´Eventuelles Ind´Ependentes. Studia Math., 3:119–155, 1931.
[11] W. Cohn, Interpolation And Multipliers On Besov And Sobolev Spaces, Complex Variables Theory Appl. 22, 35_45 (1993).
[12] E.C. Titchmarsh. The Theory Of The Riemann Zeta-Function, Second Edition. Oxford University Press, Oxford, 1986.
[13] S.A. Mccullough. Carath´Eodory Interpolation Kernels. Integral Equations And Operator Theory, 15(1):43–71, 1992.
[14] P. Quiggin. For Which Reproducing Kernel Hilbert Spaces Is Pick's Theorem True? Integral Equations And Operator Theory, 16(2):244–266, 1993.
[15] H. Hedenmalm, Dirichlet Series And Functional Analysis, In: The Legacy Of Niels Henrikabel, Springer (2004), Pp. 673-684.
[16] W.Rudin, Function Theory In Polydiscs, W.A.Benjamin Inc.,New York,1969.
[17] J. E. Mccarthy, Hilbert Spaces Of Dirichlet Series And Their Multipliers,Trans.Amer. Math. Soc. 356(3), 881_893 (2004).
[18] H. Hedenmalm And E. Saksman, Carleson's Convergence Theorem For Dirichlet Series,Pacific J. Math., 208 (2003), 85-109.
[19] F. Bayart, S. V. Konyagin And H. Queffelec, Convergence Almost Everywhere And Divergence Everywhere Of Taylor And Dirichlet Series, Real Anal. Exchange, 29 (2003/04), 557-586.
[20] F. Bayart, Hardy Spaces Of Dirichlet Series And Their Composition Operators, Monatsh.Math., 136 (2002), 203-236.
[21] K. Seip, Beurling Type Density Theorems In The Unit Disk, Invent. Math.113, 21_39 (1993).
[22] Brian J. Cole, T.W. Gamelin, Representing Measures And Hardy Spaces For The Infinite Polydisk Algebra, Proc. Lond.Math. Soc. 53 (3) (1986) 112–142.
[23] L. Carleson,An Interpolation Problem For Bounded Analytic Functions, Amer. J.Math 80, 921_930 (1958).