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Effect And Comparison Between Fractional And Ordinary Derivatives For Mathematical Models

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***Abstract*: This study presents the effect and comparison between fractional derivatives and ordinary derivatives for mathematical models. The study focus fractional heat- wave equations, the homotopy perturbation method (HPM) applied in this paper. In this method, the solution considered as the sum of an infinite series. The HPM is no need to use Adomian's polynomials to calculate the nonlinear terms. To show the efficiency and accuracy of this method, we compared the results of the fractional derivatives orders with ordinary derivative order for nonlinear fractional reaction diffusion systems. Approximate solutions for different values of fractional derivatives together with non-fractional derivative and absolute errors are represented graphically in two and three dimensions.**

***Keywords:* Fractional Calculus, partial Differential equations, Homotopy Perturbution Method.**

I. INTRODUCTION

Recently, it has turned out that many phenomena in engineering and other sciences can be described by models using mathematical tools from fractional calculus (FC). Fractional calculus owes its origin to a question of whether the meaning of a derivative to an integer order could be extended to still be valid when n is not an integer. This question was first raised by L'Hopital on September 30th, 1695. On that day, in a letter to Leibniz, he posed a question [1]. Leibniz's notation for the n^{th} derivative of the linear function, L'Hopital curiously asked what the result would be if $n = 0.5$, Leibniz responded that it would be "an apparent paradox, from which one day useful consequences will be drawn. Following this unprecedented discussion, the subject of fractional calculus caught the attention of other great mathematicians, many of whom directly or indirectly contributed to its development. They included Euler, Laplace, Fourier, Lacroix, Abel, Riemann and Liouville. In 1819, Lacroix became the first mathematician to publish a paper that mentioned a fractional derivative. [1]
The objective of this work establish the effect of mathematical models results between fractional derivative and ordinary derivative

Definition (1): Gamma Function

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}dt = (n-1)!, \quad n \in IN \qquad (1)$$

which is convergent for $n > 0$. A recurrence formula for gamma function are [2]

$$\Gamma(n+1) = n\Gamma(n) \quad for \quad n \in IR^+ \qquad (2)$$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \quad for \quad n \in IR^- \qquad (3)$$

Definition (2): Riemann- Liouville Fractional Integral Operator

Suppose that $\alpha > 0$, $n-1 < \alpha \leq n$, the Riemann-Lioville fractional integral define as [3]

$${}_a^{RL}D_t^{-\alpha}\big(f(t)\big) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-u)^{\alpha-1}f(u)du \qquad (4)$$

Note: Riemann- Liouville fractional differential operator define as ${}^{RL}D^{\alpha}f(t) = D^n D^{\alpha-n}f(t),\ \alpha < n$

Definition (3): Caputo Fractional Differential Operator

Suppose that $\alpha > 0$, $n-1 < \alpha \leq n$, the Caputo fractional differential define as [2, 3]

$${}_a^{C}D_t^{\alpha}\big(f(t)\big) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_a^t \frac{f^n(u)}{(t-u)^{\alpha-n+1}}du, & n-1 < \alpha < n \\ \dfrac{d^n}{dt^n}f(t) & \alpha = n \in N \end{cases} \qquad (5)$$

Riemann-Lioville and Caputo fractional integral for polynomial is

$${}_0^{RL}D_t^{-\alpha}(t^n) = {}_0^{C}D_t^{-\alpha}\big(f(t)\big) = \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}t^{\alpha+n} \qquad (6)$$

Caputo fractional integral for derivative is

$$ {}_0^C D_t^{-\alpha}\left(D^\alpha f(t)\right) = f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0), n-1 < \alpha \leq n \quad (7)$$

Definition (4): The Mittag- Leffler Function

Suppose $\alpha > 0,\ \beta > 0$, then the Mittag-Leffler function define by

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)} \quad (8)$$

II. Mathematical Model

In this section, we discuss the fractional heat- wave equation

$$D_t^\alpha u = \frac{1}{c^2} u_{xx}, \qquad 0 < \alpha \leq 2 \quad (9)$$

If:

i. $0 < \alpha < 1$, is called fractional heat equation
ii. $\alpha = 1$, is called non-fractional heat equation
iii. $1 < \alpha < 2$, is called fractional wave equation
iv. $\alpha = 2$, is called non-fractional wave equation

III. Homotopy Perturbation Method

The Homotopy Perturbation Method (HPM), which provides an analytical approximate solution. In this section, we extened HPM to Eq. (9) according to this method we construct the following simple homotopy [5]

$$D_t^\alpha u = p \frac{1}{c^2} u_{xx}, \qquad 0 < \alpha \leq 2 \quad (10)$$

where $p \in [0,1]$ is an embedding parameter. In case $p = 0$, Eq. (10) is fractional differential equations, which is easy to solve; when $p = 1$, Eq. (10) turns out to be the original system in Eq. (9). The basic assumption is that the solution can be written as a power series in p [4, 7]

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \quad (11)$$

If $p \to 1$, we obtain the analytical approximate solution of Eq. (10), then

$$u = \lim_{p\to 1} \sum_{n=0}^{\infty} p^n u_n(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0 + u_1 + \cdots (12)$$

Homotopy Perturbation Method for Fractional Heat- Wave Equations

$$D_t^\alpha u = \frac{1}{c^2} u_{xx}, \qquad 0 < \alpha \leq 2, \quad t \geq 0$$

BCs: $\quad u(x,0) = e^x \ \&\ u_t(x,0) = x,\ 0 \leq x \leq L$

$$D_t^\alpha u = p \frac{1}{c^2} u_{xx}$$

Integration both sides, we obtain

$$D_t^{-\alpha} D_t^\alpha u = p \frac{1}{c^2} D_t^{-\alpha}(u_{xx})$$

$$u(x,t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \frac{\partial^k u(x,0)}{\partial t^k} = p \frac{1}{c^2} D_t^{-\alpha}(u_{xx}), \qquad n-1 < \alpha \leq n$$

Now define $u(x,t) = \sum_{m=0}^{\infty} p^m u_m$ and consider $c = 1$

$$\sum_{m=0}^{\infty} p^m u_m = \sum_{k=0}^{n-1} \frac{t^k}{k!} \frac{\partial^k u(x,0)}{\partial t^k} + p\, D_t^{-\alpha}[(\sum_{m=0}^{\infty} p^m u_m)_{xx}]$$

Coefficients of p

p^0: $\quad u_0 = \sum_{k=0}^{n-1} \frac{t^k}{k!} \frac{\partial^k u(x,0)}{\partial t^k}$

p^1: $\quad u_1 = D_t^{-\alpha}[(u_0)_{xx}]$

p^2: $\quad u_2 = D_t^{-\alpha}[(u_1)_{xx}]$

p^3: $\quad u_3 = D_t^{-\alpha}[(u_2)_{xx}]$

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IV. Results and Discussion

In this section, we discuss the solution for difference derivative values

Case 1: suppose $0 < \alpha \leq 1$

$$\sum_{m=0}^{\infty} p^m u_m = u(x,0) + p\, D_t^{-\alpha}[(\sum_{m=0}^{\infty} p^m u_m)_{xx}]$$

Coefficients of p

p^0: $\quad u_0 = u(x,0) = e^x$

p^1: $\quad u_1 = D_t^{-\alpha}[(u_0)_{xx}] = D_t^{-\alpha}[e^x] = e^x \frac{t^\alpha}{\Gamma(\alpha+1)}$

p^2: $\quad u_2 = D_t^{-\alpha}[(u_1)_{xx}] = e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$

p^3: $\quad u_3 = D_t^{-\alpha}[(u_2)_{xx}] = e^x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$

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$$u(x,t) = \lim_{p\to 1} \sum_{m=0}^{\infty} p^m u_m = u_0 + u_1 + u_2 + u_3 + \cdots$$

$$u(x,t) = e^x + e^x \frac{t^\alpha}{\Gamma(\alpha+1)} + e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + e^x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots$$

$$u(x,t) = e^x \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} = e^x E_{\alpha,1}(t^\alpha)$$

The solution of non- fractional heat equation is

$$u(x,t) = e^x \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k+1)} = e^x E_{1,1}(t) = e^{x+t}$$

The solution of fractional heat equation when $\alpha = 0.5$, is

$$u(x,t) = e^x \sum_{k=0}^{\infty} \frac{(\sqrt{t})^k}{\Gamma\left(\frac{1}{2}k+1\right)} = e^x E_{0.5,1}(\sqrt{t})$$

TABLE I. THE NUMERICAL RESULTS OF HEAT EQUATION

x	Time=0.4 hr.			
	$\alpha = 1$	$\alpha = 0.8$	$\alpha = 0.5$	$\alpha = 0.2$
0.0000	1.4918	1.0000	1.0000	1.0000
1.0000	4.0552	2.7183	2.7183	2.7183
2.0000	11.0232	7.3891	7.3891	7.3891
3.0000	29.9641	20.0855	20.0855	20.0855
4.0000	81.4509	54.5982	54.5982	54.5982
5.0000	221.4064	148.4132	148.4132	148.4132
6.0000	601.8450	403.4288	403.4288	403.4288
7.0000	1.6360e+003	1.0966e+003	1.0966e+003	1.0966e+003
8.0000	4.4471e+003	2.9810e+003	2.9810e+003	2.9810e+003
9.0000	1.2088e+004	8.1031e+003	8.1031e+003	8.1031e+003
10.0000	3.2860e+004	2.2026e+004	2.2026e+004	2.2026e+004

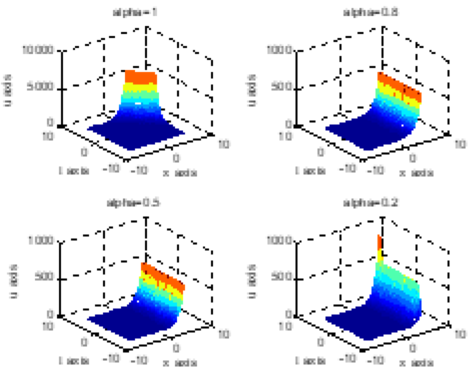

Fig. 1. Graphical presentation of fractional heat equation.

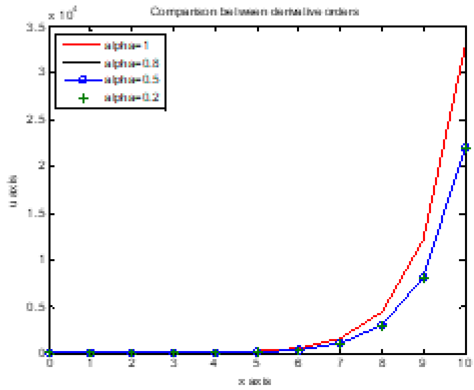

Fig. 2. Comparison between derivative orders of heat equation.

Table 1: shows the approximate solution of fractional diffusion equation with difference orders. Fig. (1): compares the surface of diffusion equation between fractional order and ordinary order, in Fig. (2): we get difference between ordinary order with multiple fractional orders.

Case 2: suppose $1 < \alpha \le 2$

$$\sum_{m=0}^{\infty} p^m u_m = u(x,0) + u_t(x,0)t + p\, D_t^{-\alpha}[(\sum_{m=0}^{\infty} p^m u_m)_{xx}]$$

Coefficients of p

$$p^0: \qquad u_0 = u(x,0) = e^x + x^2 t$$

$$p^1: \quad u_1 = D_t^{-\alpha}[(u_0)_{xx}] = e^x \frac{t^\alpha}{\Gamma(\alpha+1)} + 2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$

$$p^2: \qquad u_2 = D_t^{-\alpha}[(u_1)_{xx}] = e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$p^3: \qquad u_3 = D_t^{-\alpha}[(u_2)_{xx}] = e^x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$\vdots$$

$$u(x,t) = \lim_{p\to 1} \sum_{m=0}^{\infty} p^m u_m = u_0 + u_1 + u_2 + u_3 + \cdots$$

$$u(x,t) = e^x\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right) + x^2 t + 2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$

$$u(x,t) = e^x \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} + x^2 t + 2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$

$$= e^x E_{\alpha,1}(t^\alpha) + x^2 t + 2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$

The solution of non- fractional wave equation is

$$u(x,t) = e^x \sum_{k=0}^{\infty} \frac{t^{2k}}{\Gamma(2k+1)} + x^2 t + \frac{t^3}{3}$$
$$= e^x E_{2,1}(t^2) + x^2 t + \frac{t^3}{3}$$
$$= x^2 t + \frac{t^3}{3} + e^x \cosh(t)$$

The solution of fractional wave equation when $\alpha = 1.5$, is

$$u(x,t) = e^x \sum_{k=0}^{\infty} \frac{(\sqrt{t^3})^k}{\Gamma\left(\frac{3}{2}k+1\right)} + x^2 t + \frac{4\sqrt{t^5}}{15\sqrt{\pi}}$$
$$= e^x E_{1.5,1}\left(\sqrt{t^3}\right) + x^2 t + \frac{4\sqrt{t^5}}{15\sqrt{\pi}}$$

TABLE II. THE NUMERICAL RESULTS OF WAVE EQUATION

x	Time=0.4 hr.			
	$\alpha = 2$	$\alpha = 1.8$	$\alpha = 1.5$	$\alpha = 1.2$
0.0000	1.1024	1.0328	1.0609	1.1099
1.0000	3.3600	3.1510	3.1792	3.2282
2.0000	9.6094	9.0218	9.0500	9.0990
3.0000	25.3353	23.7183	23.7464	23.7954
4.0000	65.4459	61.0309	61.0590	61.1081
5.0000	170.4667	158.4459	158.4741	158.5231
6.0000	450.5571	417.8615	417.8897	417.9387
7.0000	1.2052e+003	1.1163e+003	1.1163e+003	1.1163e+003
8.0000	3.2483e+003	3.0066e+003	3.0066e+003	3.0067e+003
9.0000	8.7924e+003	8.1355e+003	8.1355e+003	8.1356e+003
10.0000	2.3852e+004	2.2066e+004	2.2067e+004	2.2067e+004

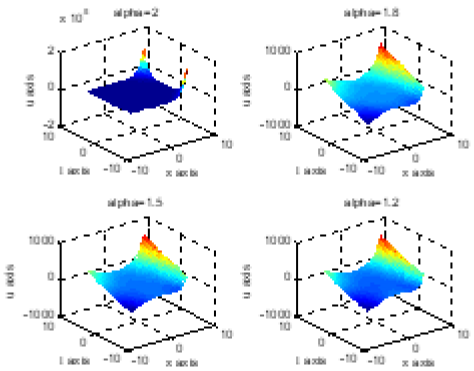

Fig. 3. Graphical presentation of non-fractional wave equation

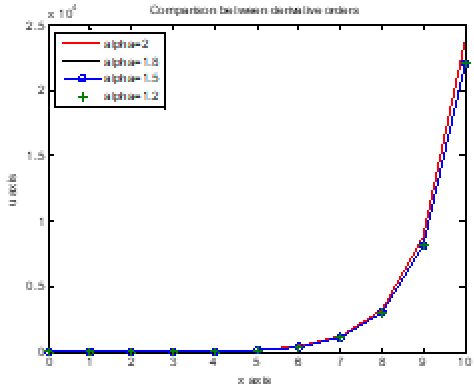

Fig. 4. Comparison between derivative orders of wave equation.

Table 2: shows the approximate solution of fractional wave equation with difference orders. Fig. (3): compares the surface of wave equation between fractional order and ordinary order, in Fig. (4): shows converges between ordinary order with multiple fractional orders.

CONCLUSION

The mathematical models is very important step for physical problem, we have concluded that the fractional derivative of heat equation is more accurate than ordinary derivative order, but in the wave equation we get small difference between fractional derivative and ordinary derivative order, we recommended researchers would use fractional derivatives when derivation the mathematical models for some phenomena.

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