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A Novel Exponential Fitted Differential-Difference Method ForSingularly Perturbed Delay Differential Equations Type With Small Delays

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Abstract
A new exponentially fitted three term method is developed for the numerical treatment of linear second order singularly perturbed delay differential equations which involves the small delay in first- differentiated term. The solution of such equations with the interval and boundary conditions exhibits left layer or right layer behaviour. The method uses the Taylor's series expansion for constructing an equivalent valid version of the original problem first and then, to derive a new three term finite difference recurrence relationship/scheme. The non-uniformity in the solution is resolved by the introduction of a suitable fitting parameter in the derived new scheme using the theory of singular perturbations. Finally, the resulting system of algebraic equations is solved by the well-known "discrete invariant algorithm". Method is analyzed for the convergence, and the theory is illustrated by solving several test example problems. Computational results are tabulated and compared to show the applicability, accuracy and efficiency of the method. Theory and computation show that the method is able to approximate the solution very well with second order convergence rate.

Keywords:*Differential-difference equation, singular perturbation problem, boundary layer, Stability and convergence*

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1 Introduction:

A singularly perturbed delay differential equation (SPDE) is an ordinary differential equation with a small parameter multiplying the highest derivative and including a delay parameter. Differential-difference equations (DDEs) are equations in which the evolution of a state variable depends on its history in an inconsistent manner, i.e., the rate of change of a physical system is influenced by both its present state and its past history. These equations have been extensively used in the fields of population dynamics [1], nonlinear delay differential equations related to physiological control systems [2], the red blood cell system [3], predator-prey models [4], nerve pulse propagation [5], mathematical modelling of neuron variability, reaction diffusion equations and control systems, neuronal variability problems connected to patterns of nerve action potentials formed by unit quintal inputs occurring at random are studied [6], fluid dynamics, fluid mechanics, micro scale heat transfer, diffusion in polymers, fluid flow in high Reynolds number, advection dominated heat and mass transfer, magneto hydro-dynamics flow, chemical reactor theory, control of chaotic system, environmental science, hydro dynamics of liquid helium, evolutionary biology, depolarization in Stein's model to describe the human pulpit light reflex and second-sound theory, theory of plates and shell, semiconductor mass device models, biomechanics, etc.[7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

Authors such as Bellman and Cooke [22] Doolan et al. [23] and Driver [24] have recently published papers and books that elucidate several approaches to solving differential-difference equations with perturbation. Lange and Miura[19] accomplished a study on the solution of SPDEs. Subsequently, Kadalbajoo et al. began investigating the numerical solution of SPDEs using various methods such as the finite difference method [10], hybrid method [11], uniformly convergent fitting approach, invariant embedding technique, and control upwind finite difference discretization [12,13,14]. Kumar and Kadalbajoo employed the fitted mesh B-spline method and the parametric uniform method to address the SPDEs [18] Andargie proposes a difference scheme on a uniform mesh that is fitted exponentially. The use of numerical techniques with an exponential integrating factor can be observed in references [25, 14, 16, 17, 18]. Chawla et. al. [26, 27] and Geng et. al. [28] utilized the tridiagonal finite difference approach to propose an enhanced kernel method for solving SPDEs. Numerical solutions to SPDEs are produced using several methods, including the finite difference technique, finite element method, Haar wavelet method, spline method, tension spline method, and Liouville Green transform method. These methods are discussed in references [29, 30, 7, 11, 15, 21] correspondingly.

In this paper, we present a uniform mesh based new exponentially fitted differential-difference method for the numerical approximation to the solution of singularly perturbed boundary value problems for the second order delay differential equations which have the boundary layer at left end or right end of the underlying interval. The considered singularly perturbed equation with the interval and boundary conditions can be addressed as two parameter problem. When the delay δ, the equation reduces to the singularly perturbed ordinary differential equation (one parameter problem) which with small ε exhibits layer behavior depending upon the sign of the coefficient functions. We have obtained the proposed scheme by introducing of a constant fitting factor in a new three term recurrence relationship derived. Accuracy and efficiency of the proposed scheme is analysed on uniform mesh and proved that the scheme is capable of producing second order accurate uniformly convergent solution. The effect of delay on the boundary layer behaviour of the solution is examined by plotting the graphs of the solution for varying δ with fixed ε.

The advantages of the proposed method are as follows:

- very easy to understand with less problem preparation.

- easy for computing the solution.

- more accurate than other similar methods.

Objective Based on the information provided, our primary goal for this work is to create a reliable, efficient, and precise numerical approach to solve a class of SPDEs that display a boundary layer (BL) at one end (left or right) of the specified interval.

The paper is organized as follows: Problem formulation and properties of solution is given in Section: 2. The numerical scheme of solution for the considered problem having boundary layer at left end points and right end points of the underlying interval is described in the sections: 3 and section: 4. Convergence analysis of the proposed scheme is presented in the Section: 5. Numerical experiments with the computational results in terms of maximum absolute point-wise errors and numerical rate of convergence are presented in the Section: 6 under the heading 'Numerical Illustrations'. Conclusion is presented in the Section: 7. Paper ends with the references.

2 Problem formulation and properties of solution:

Consider the following. singularly perturbed delay differential equation (SPDE) of second order with negative shift in the first derivative term:

$$\varepsilon u''(w) + f(w)\, u'(w-\delta) + g(w)\, u(w) = r(w) \text{ on } 0 < w < 1, \tag{2.1}$$

under the interval and the conditions:

$u\,(w) = \eta(w)\ on - \delta\ \leq w \leq 0\ and\ u\,(w) =\ \zeta,$ (2.2)

where δ is the small positive delay (negative shift) parameters to be order of $0 < \delta = o(\varepsilon) \ll 1$, $\varepsilon(0 < \varepsilon \ll 1)$is small singular perturbation parameter. The functions f(w), g(w), r(w) and $\eta(w)$ are sufficiently smooth in for all $w \in (0,1) = \Omega$and ζ is fixed. The solution of equation (2.1) with (2.2) exhibits the layer behavior at the left end of the interval if f(w)>0 and at the right end of the interval if f(w)<0 The solution of the problem displays oscillatory or twin-layer behavior, depending on whether g(w) is positive or negative, for f(w)=0.

Applying Taylor series expansion, we can derive the term u'(w-δ) in a retarded manner as

$u'((w - \delta) \approx u'(w) - \delta u''(w) + O(\delta^2)$ (2.3)

Substituting equation (2.3) into equation (2.1), we get an asymptotically equivalent singularly perturbed two-point boundary value problem (BVP) of the following form:

$\mu u''(w) + f(w)u(w) + g(w)u(w) = r(w), for\ all\ w \in [0,1]$

$\mu u''(w) + A(w,u,u') = 0; 0 \leq w \leq 1,$ (2.4)

where $A(w,u,u') = f(w)u'(w) + g(w)u(w) - r(w)$

with conditions

$u(0) = \eta(0), u(1) = \eta$ (2.5)

with $0 < \mu = \varepsilon - \delta\xi \ll 1$ such that $\xi = \min_{0\leq w\leq 1} f(w)$, where η_0 and ζ are finite constants. Equation (2.1) is transition able to equation (2.4) if δ is sufficiently small, improving computational efficiency. Further details on the validity of this transition can be found in the articles [12, 31]. Thus, the solution of equation (2.4) will provide a good approximation to the solution of equation (2.1).

For the differential equation given in (2.4) let us represent the differential operator L as $Lu(w) = \mu u''(w) + f(w)u'(w) + g(w)u(w)$ [32].

Lemma 1 *(Maximum Principle) Let $\varrho(w)$ be a differentiable function such that $\varrho(w)$ is greater than or equal to 0 and $\varrho(1)$ is greater than or equal to 0. Given $L_\tau\varrho(w) \leq 0$ for every w in the interval (0,1), it follows that $\varrho(w)$ is greater than or equal to 0 for all w in the closed interval [0,1].*

Proof. See on [31]

Lemma 2 *(Stability Principle) Let $u(w)$ denote the solution to the problem described in equation (2.4). By combining it with equation (2.5), we obtain*

$\|u(w)\| \leq \chi^{-1}\|r\| + max(|\eta_0|, |\zeta|).$ *(2.6)*

where $\|\cdot\|$ is the L_∞ norm given by $\|u(w)\| = \min_{0\leq w\leq 1}|u(w)|$ and χ^{-1} is lower bound of g(w) on [0,1].

Proof. Assume that κ^+ and κ^- are distinct barrier functions, and that $\kappa^\pm(w) = \chi^{-1}\|r\| + max(|\eta_0|, |\zeta|) \pm u(w)$. Applying Lemma-1 to the comparison function $\kappa(w) \pm u(w)$ yields the desired estimate without delay. For more information, please refer to [31].

Lemma 3 *The problem represented by equations (2.4) – (2.5) has a solution denoted as u(w) that meets*

$$|u(w)| \leq C\left[1 + e^{\left(\frac{-f^* w}{\mu}\right)}\right]$$

and

$$\left|u^{(m)}(w)\right| \le C\left[1+\mu^{-m}e^{\left(\frac{-f^{*}w}{\mu}\right)}\right], \forall\, m \ge 1,$$

where f^* is lower bound of $f(w)$.

Proof. See on [33] or [34].

3 The numerical scheme for left-end boundary layer problems:

This section describes the approach for solving the boundary value problem (2.4) with (2.5), which is identical to the boundary value problem (2.1) with (2.2), under the assumptions that $f(w) > 0$ and $g(w) < 0$ in [0,1]. These assumptions implies that the boundary layer will be close to w = 0.

The solution of (2.4) with (2.5) is of the following form (cf. [35,12], pp.22-26):

$$u(w) = u_0(w) + \frac{f(0)}{f(w)}\big(\eta(0) - u_0(0)\big)e^{-\int_0^w\left(\frac{f(w)}{\mu}\right)dw} + o(\mu) \tag{3.1}$$

where $u_0(w)$ is denoted by the solution of the following problem:

$$f(w)u'_0(w) + g(w)u_0(w) = r(w);\; u_0(0) = \zeta \tag{3.2}$$

Considering Taylor's series expansions for f(w) and g(w) about the point $'w = 0'$ up to their first terms, the equation (3.1) becomes:

$$u(w) = u_0(w) + \big(\eta(0) - u_0(0)\big)e^{-\left(\frac{f(0)}{\mu}\right)w} + o(\mu) \tag{3.3}$$

Furthermore, considering equation (3.3) at the point $w = w_t = th, t = 0,1,2,3,\dots,M$ and taking the limit as$h \to 0$, we obtain

$$\lim_{h\to 0} u(th) = u_0(0) + \big(\eta(0) - u_0(0)\big)e^{-f(0)t\rho} + o(\mu) \tag{3.4}$$

where $\rho = h/\mu$.

Now, applying Taylor's series expansion procedure, we have

$$u(w_{t+1}) = u_{t+1} = u_t + hu_t' + \frac{h^2}{2!}u_t'' + \frac{h^3}{3!}u_t''' + \frac{h^4}{4!}u_t^4 + \frac{h^5}{5!}u_t^5 + \frac{h^6}{6!}u_t^6 + u_t^7 + o(h^8) \tag{3.5}$$

$$u(w_{t-1}) = u_{t-1} = u_t - hu_t' + \frac{h^2}{2!}u_t'' - \frac{h^3}{3!}u_t''' + \frac{h^4}{4!}u_t^4 - \frac{h^5}{5!}u_t^5 + \frac{h^6}{6!}u_t^6 - \frac{h^7}{7!}u_t^7 + o(h^8) \tag{3.6}$$

From finite differences using equations (3.5) and (3.6), we get

$$u_{t-1} - 2u_t + u_{t+1} = \frac{2h^2}{2!}u_t'' + \frac{2h^4}{4!}u_t^4 + \frac{2h^6}{6!}u_t^6 + o(h^8) \tag{3.7}$$

Adding both sides $\frac{h^2}{12\mu}\left[A\big(w_{t+1}, u_{t+1}, u_{t+1}^{'+}\big) + 4A\big(w_t, u_t, u_t^{'0}\big) + A\big(w_{t-1}, u_{t-1}, u_{t-1}^{'-}\big)\right]$ from the above equation in (3.7), we get

$$u_{t-1} - 2u_t + u_{t+1} + \frac{h^2}{12\mu}\left[A\left(w_{t+1}, u_{t+1}, u_{t+1}^{'+}\right) + 4A\left(w_t, u_t, u_t^{'0}\right) + A\left(w_{t-1}, u_{t-1}, u_{t-1}^{'-}\right)\right] = \frac{2h^2}{2!}u_t^{''} + \frac{2h^4}{4!}u_t^4 + \frac{2h^6}{6!}u_t^6 + o\left(h^8\right) + \frac{h^2}{12\mu}\left[A\left(w_{t+1}, u_{t+1}, u_{t+1}^{'+}\right) + 4A\left(w_t, u_t, u_t^{'0}\right) + A\left(w_{t-1}, u_{t-1}, u_{t-1}^{'-}\right)\right] \quad (3.8)$$

above equation in (2.4), we get

$$\mu u_{t+1}^{''} + A\left(w_{t+1}, u_{t+1}, u_{t+1}^{'+}\right) = 0 \quad (3.9)$$

$$\mu u_t^{''} + A\left(w_t, u_t, u_t^{'0}\right) = 0 \quad (3.10)$$

and

$$\mu u_{t-1}^{''} + A\left(w_{t-1}, u_{t-1}, u_{t-1}^{'-}\right) = 0 \quad (3.11)$$

Substituting (3.9), (3.10) and (3.11) in right side of (3.8), and simplifying the equation (3.8), we get

$$u_{t-1} - 2u_t + u_{t+1} + \frac{h^2}{12\mu}\left[A\left(w_{t+1}, u_{t+1}, u_{t+1}^{'+}\right) + 4A\left(w_t, u_t, u_t^{'0}\right) + A\left(w_{t-1}, u_{t-1}, u_{t-1}^{'-}\right)\right] = \frac{-h^2}{2\mu}A\left(w_t, u_t, u_t^{'}\right) + O\left(h^6\right) \quad (3.12)$$

Now the equation (3.12) converts into special form:

$$u_{t-1} - 2u_t + u_{t+1} + \frac{h^2}{12\mu}\left[A\left(w_{t+1}, u_{t+1}, u_{t+1}^{'+}\right) + 4A\left(w_t, u_t, u_t^{'0}\right) + A\left(w_{t-1}, u_{t-1}, u_{t-1}^{'-}\right)\right] + \frac{h^2}{2\mu}A\left(w_t, u_t, u_t^{'0}\right) + \frac{h^2}{12\mu}\left[A\left(w_{t+1}, u_{t+1}, u_{t+1}^{'+}\right) - A\left(w_{t-1}, u_{t-1}, u_{t-1}^{'}\right)\right] + o\left(h^6\right) \quad (3.13)$$

where we approximate$u_{t+1}^{'} = u_{t+1}^{'+}$ and $u_{t-1}^{'} = u_{t-1}^{'-}$ using non symmetric finite differences

$$u_t^{'} = \frac{u_{t+1} - u_{t-1}}{2h} \quad (3.14)$$

$$u_{t+1}^{'+} = \frac{3u_{t+1} - 4u_t + u_{t-1}}{2h} \quad (3.15)$$

$$u_{t-1}^{'-} = \frac{-u_{t+1} + 4u_t - 3u_{t-1}}{2h} \quad (3.16)$$

Substituting $F\left(w, u, u^{'}\right)$ (equations (2.4)), equations (3.4), (3.15) and (3.16) in equation (3.13) and simplifying the equation (3.13), we get

$$\frac{\mu}{h^2}\left(u_{t-1} - u_t + u_{t+1}\right) + f_{t+1}\left(\frac{u_{t-1} - 4u_t + 3u_{t+1}}{24h}\right) + \frac{g_{t+1}}{12}u_{t+1} - \frac{r_{t+1}}{12} + f_t\left(\frac{u_{t+1} - u_{t-1}}{6h}\right) + \frac{g_t}{3}u_t - \frac{r_t}{3} + f_{t-1}\left(\frac{3u_{t-1} + 4u_t - u_{t+1}}{24h}\right) + \frac{g_{t-1}}{12}u_{t-1} - \frac{r_{t-1}}{12} + u_{t+1}\left\{\frac{f_t}{4h} + \frac{3f_t f_{t+1}}{48\mu} + \frac{hf_t g_{t+1}}{24\mu} + \frac{f_{t-1}f_t}{48\mu}\right\} - u_t\left\{\frac{f_t}{12\mu}\left(f_{t-1} + f_{t+1}\right) - \frac{g_t}{2}\right\} + u_{t-1}\left\{\frac{f_t f_{t+1}}{48\mu} - \frac{f_t}{4h} + \frac{3f_{t-1}f_t}{48\mu} - \frac{hf_t g_{t-1}}{48\mu}\right\} + \frac{hf_t}{24h}\left(r_{t-1} - r_{t+1}\right) - \frac{r_t}{2} = 0 \quad (3.17)$$

By introducing a constant fitting factor $\sigma(\rho)$in the above scheme (3.17), we get

$$\frac{\sigma(\rho)\mu}{h^2}\left(u_{t-1} - u_t + u_{t+1}\right) + f_{t+1}\left(\frac{u_{t-1} - 4u_t + 3u_{t+1}}{24h}\right) + \frac{g_{t+1}}{12}u_{t+1} - \frac{r_{t+1}}{12} + f_t\left(\frac{u_{t+1} - u_{t-1}}{6h}\right) + \frac{g_t}{3}u_t - \frac{r_t}{3} + f_{t-1}\left(\frac{3u_{t-1} + 4u_t - u_{t+1}}{24h}\right) + \frac{g_{t-1}}{12}u_{t-1} - \frac{r_{t-1}}{12} + u_{t+1}\left\{\frac{f_t}{4h} + \frac{3f_t f_{t+1}}{48\mu} + \frac{hf_t g_{t+1}}{24\mu} + \frac{f_{t-1}f_t}{48\mu}\right\} - u_t\left\{\frac{f_t}{12\mu}\left(f_{t-1} + f_{t+1}\right) - \frac{g_t}{2}\right\} + u_{t-1}\left\{\frac{f_t f_{t+1}}{48\mu} - \frac{f_t}{4h} + \frac{3f_{t-1}f_t}{48\mu} - \frac{hf_t g_{t-1}}{48\mu}\right\} + \frac{hf_t}{24h}\left(r_{t-1} - r_{t+1}\right) - \frac{r_t}{2} = 0 \quad (3.18)$$

Multiplying (3.18) by h and taking the limit as $h \to 0$ we get

$$\lim_{h\to 0}\sigma(\rho)\mu\left(\frac{u_{t-1} - 2u_t + u_{t+1}}{2h}\right) + f(0)\lim_{h\to 0}\left(u_{t+1} - u_{t-1}\right) + \frac{hf(0)^2}{12} \quad (3.19)$$

Now from the equation (3.4) in equation (3.19) and simplifying the equation (3.19) we get:

$$\sigma(\rho) = \rho f(0) coth\left(\frac{f(0)\rho}{2}\right) - \frac{(\rho f(0))^2}{12} \tag{3.20}$$

which is required constant fitting factor $\sigma(\rho)$.

Finally, using equation (3.18) and the value of $\sigma(\rho)$ obtained by equation (3.12), We get the proposed new exponentially fitted three-term scheme/recurrence relationship:

$$P_t u_{t-1} - Q_t u_t + R_t u_{t+1} = H_t, (t = 1, 2, 3, \dots, M-1) \tag{3.21}$$

where

$$P_t = \frac{\mu\sigma(\rho)}{h^2} + \frac{f_{t+1}}{24h} - \frac{10f_t}{24h} - \frac{3f_{t-1}}{24h} + \frac{g_{t-1}}{12} + \frac{f_t}{48\mu}(f_{t+1} + 3f_{t-1}) - \frac{hg_{t-1}f_t}{24\mu}$$

$$Q_t = \frac{\mu\sigma(\rho)}{h^2} + \frac{f_{t+1}}{6h} - \frac{5g_t}{6} - \frac{f_{t-1}}{6h} + \frac{f_t}{12\mu}(f_{t+1} + f_{t-1})$$

$$R_t = \frac{\mu\sigma(\rho)}{h^2} + \frac{3f_{t+1}}{24h} + \frac{10f_t}{24h} - \frac{f_{t-1}}{24h} + \frac{g_{t+1}}{12} + \frac{f_t}{48\mu}(3f_{t+1} + f_{t-1}) + \frac{hg_{t+1}f_t}{24\mu}$$

$$H_t = \frac{1}{12}(r_{t+1} + 10r_t + r_{t-1}) - \frac{hf_t}{24\mu}(r_{t-1} - r_{t+1})$$

The relation (3.21) represents a system of $(M-1)$ equations with $(M+1)$ unknowns u_o to u_M. These $(M-1)$ equations together with the boundary conditions $u = \mathrm{u}(0) = \eta(0) = \eta_0$, $u(0) = u_M = \zeta$ given by (2.5) are sufficient to solve for the unknowns u_o to u_M. The resulting matrix problem associated with (3.21) is a tridiagonal system of linear equations. The coefficient matrix of such a system of equations is non-singular, if it is either strictly diagonally dominant or irreducibly diagonally dominant [36]. Moreover, if these conditions hold, the Thomas Algorithm (or Discrete Invariant Imbedding algorithm) described in [37] provides a numerically stable and convergent algorithm for solving the system.

Assuming $f(w) = F$ and $g(w) = G$ as constant functions in [0,1], where F and G are finite constants, we can easily find that the inequalities:$P_t > 0, Q_t > 0, R_t > 0, (P_t + R_t)$and $|P_t| \le |R_t|$satisfies the assumptions $f(w) = F > 0$ and $g(w) = G < 0$.The coefficient matrix of the tridiagonal system of equations (3.21) with boundary conditions (2.5) is irreducibly diagonally dominating and therefore non-singular.

We have solved the resulting tri-diagonal system of equations using the Discrete Invariant Imbedding algorithm [37, 38].

4 Description of the method for Right-End Boundary Layer Problems:

This section is devoted to the description of the method for solving the boundary value problem (2.4) with (2.5) which is equivalent to the boundary value problem (2.1) with (2.2) under the assumptions that $f(w) = F > 0$ and $g(w) = G < 0$ in [0, 1]. These assumptions merely implies that the boundary layer will be in the neighbourhood of $w = 1$.

The solution of (2.4) with (2.5) is of the following form (cf. [35], pp.22-26):

$$u(w) = u_0(w) + \frac{f(1)}{f(w)}(\zeta - u_0(0))e^{-\int_w^1\left(\frac{f(w)}{\mu}\right)dw} + o(\mu) \tag{4.1}$$

where $u_0(w)$represents the solution of the reduced problem:

$$f(w)u'_0(w) + g(w)u_0(w) = r(w);\ u_0(0) = \eta \tag{4.2}$$

Expanding $f(w)$ and $g(w)$in (4.1) with the help of the Taylor's series about the point $'w = 1'$. and restricting to their first terms, we obtain:

$$(w) = u_0(w) + \big(\zeta - u_0(1)\big)e^{\left(\frac{f(1)}{\mu}\right)(1-w)} + o(\mu) \tag{4.3}$$

Moreover, considering equation (4.3) at the point $w = w_t = th, t = 0,1,2,3, \dots, M$ and taking the limit as $h \to 0$ we obtain

$$\lim_{h\to 0} u(th) = u_0(0) + \big(\zeta - u_0(1)\big)e^{f(1)(\frac{1}{\mu} - t\rho)} + o(\mu) \tag{4.4}$$

where $\rho = h/\mu$.

Now, from the equation (4.4) in equation (3.19) and simplifying the equation (3.19) we get:

$$\sigma(\rho) = \rho f(0) coth\left(\frac{f(1)\rho}{2}\right) - \frac{(\rho f(0))^2}{12} \tag{4.5}$$

which is a required constant fitting factor $\sigma(\rho)$ in this right end boundary layer problem case.

Finally, using equation (3.18) and the value of $\sigma(\rho)$given by equation (4.5), we obtain the proposed new exponentially fitted three-term scheme/ recurrence relationship:

$$P_t u_{t-1} - Q_t u_t + R_t u_{t+1} = H_t, (t = 1, 2, 3, \dots, M-1) \tag{4.6}$$

where

$$P_t = \frac{\mu\sigma(\rho)}{h^2} + \frac{f_{t+1}}{24h} - \frac{10f_t}{24h} - \frac{3f_{t-1}}{24h} + \frac{g_{t-1}}{12} + \frac{f_t}{48\mu}(f_{t+1} + 3f_{t-1}) - \frac{hg_{t-1}f_t}{24\mu}$$

$$Q_t = \frac{2\mu\sigma(\rho)}{h^2} + \frac{f_{t+1}}{6h} - \frac{5g_t}{6} - \frac{f_{t-1}}{6h} + \frac{f_t}{12\mu}(f_{t+1} + f_{t-1})$$

$$R_t = \frac{\mu\sigma(\rho)}{h^2} + \frac{3f_{t+1}}{24h} + \frac{10f_t}{24h} - \frac{f_{t-1}}{24h} + \frac{g_{t+1}}{12} + \frac{f_t}{48\mu}(3f_{t+1} + f_{t-1}) + \frac{hg_{t+1}f_t}{24\mu}$$

$$H_t = \frac{1}{12}(r_{t+1} + 10r_t + r_{t-1}) - \frac{hf_t}{24\mu}(r_{t-1} - r_{t+1})$$

The equation (4.6) produces a system of $(M-1)$ equations with $(M-1)$ unknowns u_t to u_{M-1}. These $(M-1)$ equations together with the boundary conditions equation (2.5) are sufficient to solve the obtained tri-diagonal system with the help of an efficient solver called Thomas Algorithm which is also known as 'Discrete Invariant Imbedding algorithm' [37, 38]

Remark: When $f(0) = f(1)$ and $g(0) = g(1)$, both the fitting factorsbecome equal and the constant fitting factor is

$$\sigma(\rho) = \rho f(0) coth\left(\frac{f(0)\rho}{2}\right) - \frac{(\rho f(0))^2}{12} \tag{4.7}$$

5 Convergence analysis :

In this section, we discuss the convergence analysis of the proposed scheme. Writing the tridiagonal system equation (3.21) in matrix-vector form [38], we get

$$AU = C \tag{5.1}$$

in which $A = (m_{tj}), 1 \le t, j \le M-1$ is a tridiagonal matrix of order $(M-1)$, with

$$m_{t,t-1} = \frac{\mu\sigma(\rho)}{h^2} + \frac{f_{t+1}}{24h} - \frac{10f_t}{24h} - \frac{3f_{t-1}}{24h} + \frac{g_{t-1}}{12} + \frac{f_t}{48\mu}(f_{t+1} + 3f_{t-1}) - \frac{hg_{t-1}f_t}{24\mu}$$

$$m_{t,t} = -\frac{2\mu\sigma(\rho)}{h^2} - \frac{f_{t+1}}{6h} + \frac{5g_t}{6} + \frac{f_{t-1}}{6h} - \frac{f_t}{12\mu}(f_{t+1} + f_{t-1})$$

$$m_{t,t-1} = \frac{\mu\sigma(\rho)}{h^2} + \frac{3f_{t+1}}{24h} + \frac{10f_t}{24h} - \frac{f_{t-1}}{24h} + \frac{g_{t+1}}{12} + \frac{f_t}{48\mu}(3f_{t+1} + f_{t-1}) + \frac{hg_{t+1}f_t}{24\mu}$$

And $C = (d_{ij})$ is a column vector with $d_i = \frac{1}{12}(r_{t+1} + r_t + r_{t-1}) - \frac{hf_t}{24\mu}(r_{t-1} - r_{t+1})$ for $t = 1(1)M-1$, with local truncation error

$$T_t(h) = \frac{h^2}{12}\left[\frac{f_t r_r}{\mu}\right] + O(h^3) \tag{5.2}$$

We also have

$$A\bar{U} - T(h) = C \tag{5.3}$$

Where $U = (\bar{u}_0, \bar{u}_1, \bar{u}_2, \dots, \bar{u}_M)$ denotes the actual and $T(h) = (T_0(h), T_1(h), T_2(h), \dots, T_M(h))$ is the local truncation error.

From equation (5.1) and equation (5.3), we get

$$A(\bar{U} - U) = T(h) \tag{5.4}$$

Thus, the error equation is

$$AE = T(h) \tag{5.5}$$

Where $E = (U - \bar{U}) = (e_0, e_1, e_2, \dots, e_M)'$.

Let S_t be the sum of elements of t^{th} row of A then we have

$$S_1 = \sum_{j-1}^{M-1} m_{1,j} = \frac{\mu\sigma}{h^2} - \frac{f_2}{24h} + \frac{5g_1}{6} + \frac{g_2}{12} + \frac{10f_1}{24h} + \frac{f_1 f_0}{48\mu} + \frac{3f_1 f_0}{48\mu} + \frac{hg_2 f_1}{24\mu}$$

$$S_{M-1} = \sum_{j-1}^{M-1} m_{M-1,j} = \frac{-\mu\sigma}{h^2} - \frac{3f_M}{24h} + \frac{5g_{M-2}}{12} + \frac{5g_{M-1}}{6} - \frac{10f_{M-1}}{24h} + \frac{f_{M-2}}{24h} - \frac{3f_{M-1}f_M}{48\mu} - \frac{f_{M-1}f_{M-2}}{48\mu} - \frac{hg_{M-2}f_{M-1}}{24\mu}$$

$$S_t = \sum_{j-1}^{M-1} m_{1,j} = \frac{1}{12}(g_{t-1} + 10g_t + g_{t-1}) + \frac{hg_{t+1}f_t}{24\mu}, t = 2,3,4, \dots, M-2$$

$$= B_{t_0} + O(h)$$

Where $B_{t_0} = \frac{1}{12}(g_{t-1} + 10g_t + g_{t-1})$. Since $0 < \mu \ll 1$, for sufficiently small h the matrix A is irreducible and monotone. Hence, we can conclude A^{-1} exists and it has non-negative elements.

Hence from equation (5.5) we have

$$E = A^{-1}T(h) \tag{5.6}$$

and

$$\|E\| = \|A^{-1}\| \cdot \|T(h)\| \tag{5.7}$$

Let $\bar{m}_{kt}$be the kt^{th}element of A^{-1}Since$\bar{m}_{kt} \geq 0$, from the theory of matrices we have

$$\sum_{t=1}^{M-1} \bar{m}_{k,t} S_t = 1, k = 1, 2, 3, \dots, M-1 \tag{5.8}$$

Therefore, it follows that

$$\sum_{j-1}^{M-1} \bar{m}_{k,t} \leq \frac{1}{\min_{1\leq t\leq M-1} S_t} = \frac{1}{B_{t_0}} \leq \frac{1}{|B_{t_0}|} \tag{5.9}$$

for some t_0 between 1 and $M - 1$.

We define $\|A^{-1}\| = \max_{1\leq t\leq M-1} \sum_{t=1}^{M-1} |\bar{m}_{kt}|$ and $\|T(h)\| = \max_{1\leq t\leq M-1} \sum_{t=1}^{M-1} |T_t(h)|$.

From (5.2),(5.6) and (5.8), we get

$$e_j = \sum_{t=1}^{M-1} \bar{m}_{kt} T_t(h), j = 1(1)M-1$$

And therefore,

$$e_j \leq \frac{kh^2}{|B_{t_0}|}, j = 1(1)M-1 \tag{5.10}$$

Where $k = \frac{f_t r_t}{12\mu}$.

Therefore, using equation (5.10), we have

$$\|E\| = O(h^2)$$

The above results can be summarized as follows:

Theorem 1 *Let $u(w)$ be the exact solution of singularly perturbed second-order delay differential equation (2.1) with the interval and boundary conditions (2.2) and let u_t be the numerical solution obtained by the scheme (3.21). Then, for sufficiently small h, (3.21) gives a second order convergent solution on uniform mesh.*

6 Numerical Illustrations :

The proposed technique is implemented on model problems of the type by equations (2.1) -(2.2), for different values of delay parameter $\delta$and perturbation parameter ε. Maximum absolute errors are computed in Tables: 1 to 6, tabulated and compared with the results of [38, 12, 39] for the left-end boundary layer problems and the right-end boundary layer problems. As the exact solution are not known, for different value of δ of $o(\varepsilon)$ hence the double mesh principle is used to calculate the MAEs by the following double mesh principle [23]:

$\sum_{\varepsilon,\delta}^{M} = \max_{1\le j\le M-1} \left|u_j^M - u_j^{2M}\right|,$

where, let u_j^M denote the computed solution of the problem on M number of mesh points and let u_j^{2M} denote the computed solution on a double number of mesh points $2M$ by including the mid-points $w_{j+1/2} = \frac{w_{j+1}+w_j}{2}$ into the mesh points.

The rate of convergence (ROC) is calculated using this formula as

$$Rate_{\varepsilon,\delta}^{M} = \log_2 \left|\frac{Error_{\varepsilon,\delta}^{M}}{Error_{\varepsilon,\delta}^{2M}}\right|,$$

It is evident from the computational results presented in tables 1, 3, 4, and 5 that the scheme is capable of providing highly accurate second-order convergent solutions when $h = \frac{1}{M}$ approaches zero for any value of ε ranging between 2^{-1} and 2^{-7}.

Example 1: Firstly, let us assume the following problem

$$\varepsilon u''(w) + u'(w-\delta) - u(w) = 0$$

$$\text{with } u(0) = 1 \text{ and } u(1) = 1$$

Example 2: Let us assume the following problem with variable coefficients

$$\varepsilon u''(w) + e^{(-0.25w)} u'(w-\delta) - u(w) = 0$$

$$\text{with } u(0) = 1 \text{ and } u(1) = 1$$

Example 3: Let us assume the following problem

$$\varepsilon u''(w) - u'(w-\delta) + u(w) = 0$$

$$\text{with } u(0) = 1 \text{ and } u(1) = 1$$

Example 4: Finally, let us assume the following problem with variable coefficients

$$\varepsilon u''(w) - e^{w} u'(w-\delta) - u(w) = 0$$

$$\text{with } u(0) = 1 \text{ and } u(1) = 1$$

Four model problems with their solution exhibiting a boundary layer are considered. Examples 1 and 2 have the boundary layer on the left side of the domain and Examples 3 and 4 have it on the right side of the domain. In Table 1 and 5, the MAE and rate of convergence of the scheme of model problems 1 and 4 are given for different values of ε, M and $\delta = 0.5 * \varepsilon$ ranging from $2^{-1} \to 2^{-7}$ and similarly Table 3 and 4, the MAE and rate of convergence of the scheme of model problems 2 and 3 are given for different values of ε and M ranging from $2^{-1} \to 2^{-7}$ with fixed $\delta = 0.03$. It is observed that the scheme gives a quadratic order of convergence. In Tables 2, 3, 4 and 6, the comparison of the proposed scheme with the result in [38, 12, 39] is given for model problems 1, 2, 3 and 4 respectively. As we observe, the MAE of the proposed scheme is much better than that of in [38, 12, 39] for model problems 1, 2, 3 and 4 respectively.

Figures 1-2 and 3-4 demonstrate the impact of δ on the behavior of Example 1 and 2 solutions, respectively. In the left boundary layer problem, it is observed that as the δ increases, the thickness of the boundary layer decreases. As the value of εdecreases the solution graphs exhibit a pronounced/strong boundary layer. Similarly Figures 5-6 and 7-8 demonstrate the impact of δ on the behavior of Example 3 and 4 solutions, respectively. In the right boundary layer problem, it is observed that as the δ increases, the thickness of the boundary layer increases. As the value of ε decreases the solution graphs exhibit a pronounced/strong boundary layer.

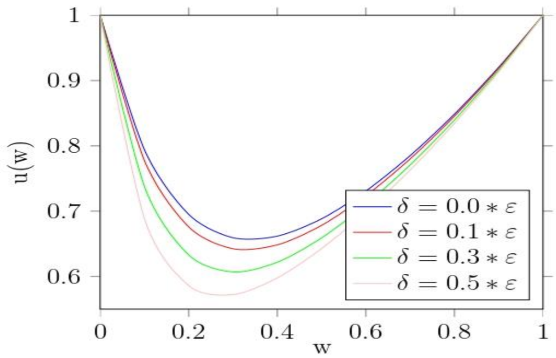

Fig. 1 Numerical solution of problem 1 for various δ values using M=10 and ε =0.1

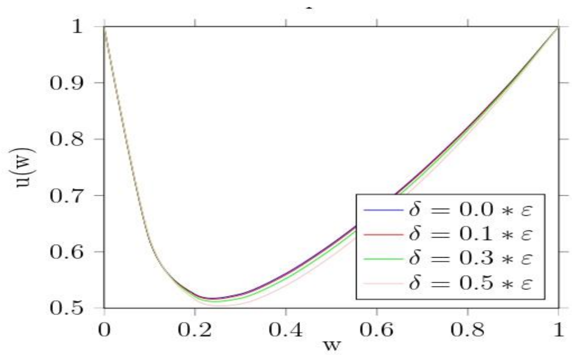

Fig. 2 Numerical solution of problem 1 for various δ values using M=10 and ε =0.01

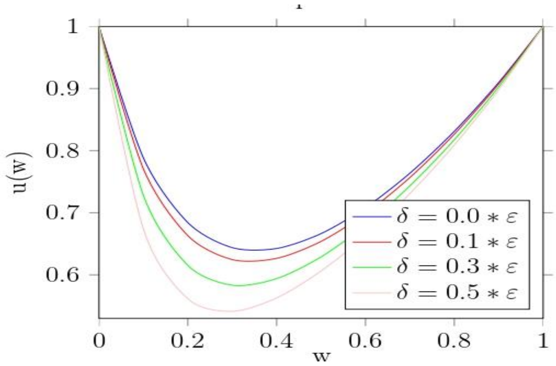

Fig. 3 Numerical solution of problem 2 for various δ values using M=10 and ε =0.1

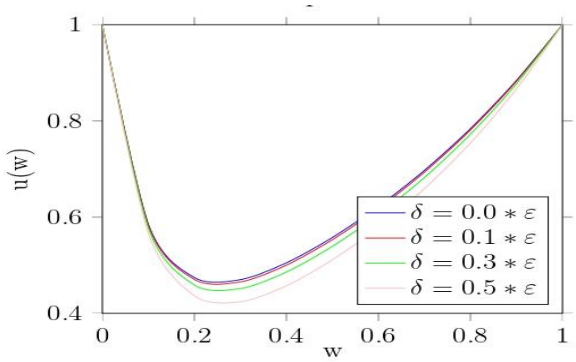

Fig. 4 Numerical solution of problem 2 for various δ values using M=10 and ε =0.01

Table 1 The $Rate^M_{\varepsilon,\delta}$ (ROC) and the $E^M_{\varepsilon,\delta}$ (MAE) of example-1 for varying $\varepsilon, \delta = 0.5 * \varepsilon$ and M.

$\varepsilon \downarrow$	$M = 64$	$M = 128$	$M = 256$	$M = 512$	$M = 1024$	$M = 2048$
2^{-1}	2.8529E-05	7.1345E-06	1.7837E-06	4.4592E-07	1.1153E-07	2.7686E-08
$Rate^M_{\varepsilon,}$	1.9995	1.9999	2.0000	1.9994	2.0102	
2^{-2}	1.4689E-04	3.6761E-05	9.1913E-06	2.2979E-06	5.7451E-07	1.4350E-07
$Rate^M_{\varepsilon,}$	1.9985	1.9998	2.0000	1.9999	2.0013	
2^{-3}	6.3730E-04	1.5991E-04	3.9995E-05	9.9997E-06	2.5001E-06	6.2497E-07
$Rate^M_{\varepsilon,}$	1.9947	1.9994	1.9999	1.9999	2.0001	
2^{-4}	2.6108E-03	6.5728E-04	1.6465E-04	4.1196E-05	1.0300E-05	2.5751E-06
$Rate^M_{\varepsilon,}$	1.9899	1.9971	1.9988	1.9999	1.9999	
2^{-5}	1.0301E-02	2.6527E-03	6.6796E-04	1.6729E-04	4.1844E-05	1.0463E-05
$Rate^M_{\varepsilon,}$	1.9573	1.9896	1.9974	1.9993	1.9997	
2^{-6}	3.6591E-02	1.0384E-02	2.6749E-03	6.7364E-04	1.6871E-04	4.2198E-05
$Rate^M_{\varepsilon,}$	1.8171	1.9568	1.9894	1.9974	1.9993	

2^{-7} $Rate_{\varepsilon,}^{M}$	9.3276E-02 1.3451	3.6717E-02 1.8161	1.0427E-02 1.9565	2.6865E-03 1.9894	6.7658E-04 1.9974	1.6945E-04

Table 2 Comparison of $E_{\varepsilon,\delta}^{M}$ (MAE) of problem-1 for $\varepsilon = 0.1$and $\varepsilon = 0.01$different values of δ and M of proposed scheme and results in [38, 12]

$\delta \downarrow$	$M = 10^2$	$M = 10^3$	$M = 10^4$	$M = 10^5$
$\varepsilon = 10^{-1} \downarrow$				
My present method results				
$\delta = 0.1 * \varepsilon$	1.2251E-04	1.2257E-06	9.0688E-09	4.3209E-10
$\delta = 0.3 * \varepsilon$	2.0724E-04	2.0740E-06	2.0738E-08	3.8262E-10
$\delta = 0.6 * \varepsilon$	6.5158E-04	6.5375E-06	6.5378E-08	7.4076E-10
$\delta = 0.8 * \varepsilon$	2.6406E-03	2.6663E-05	2.6669E-07	3.0912E-08
Results by the method in Challa and Reddy [38]				
$\delta = 0.1 * \varepsilon$	0.011721	0.001225	0.000123	1.230e-005
$\delta = 0.3 * \varepsilon$	0.015056	0.001589	0.000159	1.599e-005
$\delta = 0.6 * \varepsilon$	0.025750	0.002812	0.000283	2.840e-005
$\delta = 0.8 * \varepsilon$	0.047807	0.005629	0.000573	5.747e-005
Results by the method in Kadalbajoo and Sharma [12]				
$\delta = 0.1 * \varepsilon$	0.011824	0.001229	0.000123	1.236e005
$\delta = 0.3 * \varepsilon$	0.015155	0.001593	0.000160	1.603e005
$\delta = 0.6 * \varepsilon$	0.025847	0.002816	0.000284	2.845e005
$\delta = 0.8 * \varepsilon$	0.083131	0.011103	0.001151	5.748e005
$\varepsilon = 10^{-2} \downarrow$				
My present method results				
$\delta = 0.1 * \varepsilon$	1.2658E-02	1.3315E-04	1.3322E-06	2.5941E-08
$\delta = 0.3 * \varepsilon$	1.9435E-02	2.2052E-04	2.2071E-06	3.4301E-08
$\delta = 0.6 * \varepsilon$	5.3006E-02	6.7651E-04	6.7815E-06	7.4653E-08
$\delta = 0.8 * \varepsilon$	1.1077E-01	2.6923E-03	2.7186E-05	2.7902E-07
Results by the method in Challa and Reddy [38]				
$\delta = 0.1 * \varepsilon$	0.090733	0.012286	0.001279	1.284e-004
$\delta = 0.3 * \varepsilon$	0.108033	0.015622	0.001644	1.653e-004
$\delta = 0.6 * \varepsilon$	0.127778	0.026309	0.002870	2.897e-004
$\delta = 0.8 * \varepsilon$	0.100404	0.048338	0.005688	5.794e-004
Results by the method in Kadalbajoo and Sharma [12]				
$\delta = 0.1 * \varepsilon$	0.090928	0.012290	0.001279	1.284e004
$\delta = 0.3 * \varepsilon$	0.108362	0.015626	0.001644	1.653e004
$\delta = 0.6 * \varepsilon$	0.128454	0.026314	0.002870	2.897e004
$\delta = 0.8 * \varepsilon$	0.101499	0.048347	0.005688	5.794e004

Table 3 $Rate_{\varepsilon,\delta}^{M}$ (ROC)and Comparison of$E_{\varepsilon,\delta}^{M}$ (MAE) of problem-2 for $\delta = 0.03, M$and varies εdifferent values of δ and M of proposed scheme and results in [39]

$\varepsilon \downarrow$	$M = 100$	$M = 200$	$M = 400$	$M = 800$	$M = 1600$
2^{-1} $Rate_{\varepsilon,\delta}^{M}$	1.9462E-06 2.0004	4.8643E-07 2.0000	1.2161E-07 2.0027	3.0346E-08 2.0016	7.5783E-09

2^{-2} $Rate^{M}_{\varepsilon,\delta}$	1.2495E-05 1.9998	3.1242E-06 1.9999	7.8112E-07 2.0005	1.9521E-07 1.9994	4.8823E-08
2^{-3} $Rate^{M}_{\varepsilon,\delta}$	6.6614E-05 1.9995	1.6659E-05 1.9997	4.1657E-06 1.9996	1.0417E-06	2.6005E-07
2^{-4} $Rate^{M}_{\varepsilon,\delta}$	2.9695E-04 1.9986	7.4311E-05 1.9994	1.8585E-05 1.9995	4.6477E-06 2.0010	1.1611E-06
2^{-5} $Rate^{M}_{\varepsilon,\delta}$	1.2289E-03 1.9953	3.0822E-04 1.9988	7.7119E-05 1.9995	1.9286E-05 2.0002	4.8207E-06
2^{-6} $Rate^{M}_{\varepsilon,\delta}$	4.9330E-03 1.9810	1.2496E-03 1.9953	3.1342E-04 1.9988	7.8420E-05 1.9998	1.9608E-05
2^{-7} $Rate^{M}_{\varepsilon,\delta}$	1.8419E-02 1.8893	4.9719E-03 1.9811	1.2594E-03 1.9953	3.1588E-04 1.9988	7.9033E-05
Results in Omkar and Phaneendra [39]					
2^{-1}	2.2573e-04	7.3086e-05	2.6515e-05	1.0740e-05	4.7393e-06
2^{-2}	4.2612e-04	1.3699e-04	4.9174e-05	1.9718e-05	8.6367e-06
2^{-3}	7.7550e-04	2.4419e-04	8.5262e-05	3.3306e-05	1.4307e-05
2^{-4}	1.4151e-03	4.1821e-04	1.3513e-04	4.8916e-05	1.9780e-05
2^{-5}	2.9428e-03	7.6106e-04	2.0667e-04	6.0787e-05	1.9899e-05
2^{-6}	7.6116e-03	1.8207e-03	3.9430e-04	7.5748e-05	1.5215e-05
2^{-7}	1.7691e-02	4.7307e-03	1.0985e-03	1.1542e-04	2.3304e-05

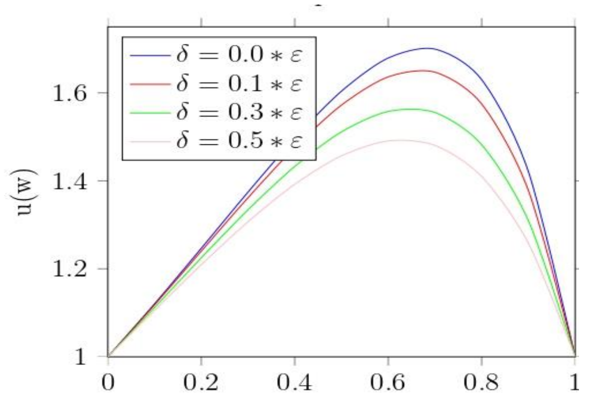

Fig. 5 Numerical solution of problem 3 for various δ values using M=10 and ε =0.1

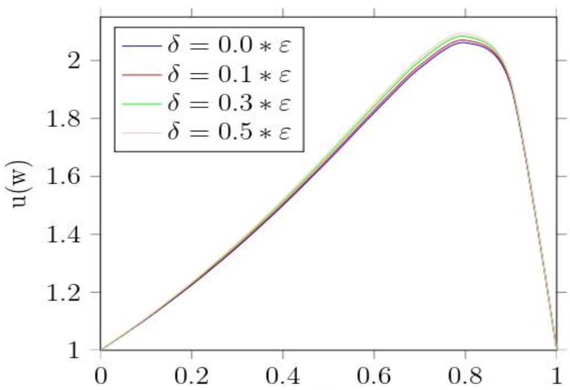

Fig. 6 Numerical solution of problem 3 for various δ values using M=10 and ε =0.01

Table 4 $Rate^{M}_{\varepsilon,\delta}$ (ROC)and Comparison of$E^{M}_{\varepsilon,\delta}$ (MAE) of problem-3 for fixed $\delta = 0.03, M$and varies ε of proposed scheme and results in [39]

$\varepsilon\downarrow$	$M = 100$	$M = 200$	$M = 400$	$M = 800$	$M = 1600$
My present method results					
2^{-1} $Rate^{M}_{\varepsilon,\delta}$	2.3459E-06 1.9999	5.8651E-07 2.0000	1.4663E-07 1.9999	3.6659E-08 1.9999	9.1654E-09
2^{-2} $Rate^{M}_{\varepsilon,\delta}$	1.7578E-05 1.9998	4.3951E-06 2.0000	1.0988E-06 1.9999	2.7471E-07 2.0000	6.8678E-08 1.9998
2^{-3} $Rate^{M}_{\varepsilon,\delta}$	1.0612E-04 1.9996	2.6537E-05 1.9999	6.6347E-06 2.0000	1.6587E-06 2.0000	4.1468E-07
2^{-4} $Rate^{M}_{\varepsilon,\delta}$	3.9483E-04 1.9992	9.8765E-05 1.9992	2.4695E-05 2.0000	6.1739E-06 2.0000	1.5435E-06

2^{-5} $Rate_{\varepsilon,\delta}^{M}$	8.9738E-04 1.9989	2.2451E-04 1.9998	5.6137E-05 1.9999	1.4035E-05 2.0000	3.5088E-06
2^{-6} $Rate_{\varepsilon,\delta}^{M}$	1.5448E-03 1.9967	3.8709E-04 1.9996	9.6801E-05 1.9998	2.4203E-05 1.9999	6.0510E-06
2^{-7} $Rate_{\varepsilon,\delta}^{M}$	2.1985E-03 1.9970	5.5077E-04 1.9988	1.3781E-04 1.9998	3.4457E-05 1.9999	8.6146E-06
Results in Omkar and Phaneendra [39]					
2^{-1}	1.6785e-04	4.1860e-05	1.0451e-05	2.6111e-06	6.5256e-07
2^{-2}	3.4720e-04	8.6528e-05	2.1595e-05	5.3939e-06	1.3479e-06
2^{-3}	6.8995e-04	1.7230e-04	4.3045e-05	1.0757e-05	2.6886e-06
2^{-4}	1.3788e-03	3.4472e-04	8.6138e-05	2.1527e-05	5.3806e-06
2^{-5}	3.5095e-03	8.8050e-04	2.2015e-04	5.5021e-05	1.3752e-05
2^{-6}	1.3569e-02	3.4562e-03	8.6809e-04	2.1726e-04	5.4328e-05
2^{-7}	3.3612e-02	9.6117e-03	2.4894e-03	6.2795e-04	1.5734e-04

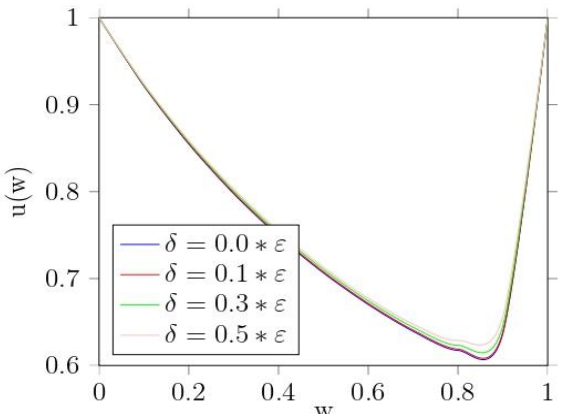

Fig. 7 Numerical solution of problem 4 for various δ values using M=10 and ε=0.1

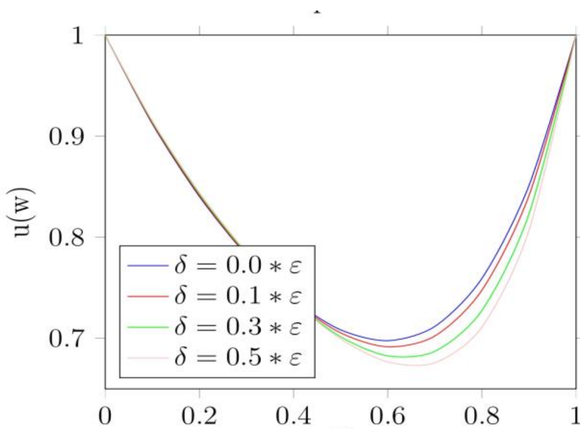

Fig. 8 Numerical solution of problem 4 for various δ values using M=10 and ε=0.01

Table 5 The$Rate_{\varepsilon,\delta}^{M}$ (ROC)and the$E_{\varepsilon,\delta}^{M}$ (MAE) of model problem-4 for verifying $\varepsilon, \delta = 0.5 * \varepsilon$, and M

$\varepsilon \downarrow$	$M = 64$	$M = 128$	$M = 256$	$M = 512$	$M = 1024$	$M = 2048$
2^{-1} $Rate_{\varepsilon,\delta}^{M}$	1.7121E-05 2.0000	4.2803E-06 2.0009	1.0694E-06 1.9977	2.6777E-07 1.9956	6.7145E-08 2.0166	1.6594E-08
2^{-2} $Rate_{\varepsilon,\delta}^{M}$	7.3102E-05 2.0015	1.8257E-05 2.0011	4.5609E-06 1.9984	1.1415E-06 1.9995	2.8547E-07 2.0153	7.0615E-08
2^{-3} $Rate_{\varepsilon,\delta}^{M}$	2.5496E-04 2.0071	6.3429E-05 2.0022	1.5833E-05 1.9993	3.9603E-06 1.9999	9.9013E-07 2.0057	2.4655E-07
2^{-4} $Rate_{\varepsilon,\delta}^{M}$	9.3068E-04 2.0504	2.2468E-04 2.0131	5.5662E-05 2.0019	1.3897E-05 2.0007	3.4725E-06 2.0010	8.6754E-07
2^{-5} $Rate_{\varepsilon,\delta}^{M}$	3.9744E-03 2.2185	8.5395E-04 2.0591	2.0492E-04 2.0057	5.1028E-05 2.0040	1.2722E-05 2.0009	3.1785E-06
2^{-6} $Rate_{\varepsilon,\delta}^{M}$	1.6449E-02 2.1083	3.8149E-03 2.2380	8.0871E-04 2.0071	1.9433E-04 2.0058	4.8386E-05 2.0040	1.2063E-05
2^{-7} $Rate_{\varepsilon,\delta}^{M}$	5.4558E-02 1.7650	1.6052E-02 2.1068	3.7267E-03 2.2491	7.8393E-04 2.0506	1.8923E-04 2.0113	4.6937E-05

Table 6 Comparison of $E_{\varepsilon,\delta}^{M}$ (MAE) of problem-4 for $\varepsilon = 0.1$ and varies different values of δ and M of proposed scheme and results in [38,12]

$\delta \downarrow$	$M = 10^2$	$M = 10^3$	$M = 10^4$
My present method results			
$\delta = 0.1 * \varepsilon$	2.7379E-04	2.7170E-06	2.7741E-08
$\delta = 0.3 * \varepsilon$	2.0145E-04	2.0036E-06	2.0532E-08
$\delta = 0.6 * \varepsilon$	1.3822E-04	1.3778E-06	1.4593E-08
$\delta = 0.8 * \varepsilon$	1.1181E-04	1.1177E-06	1.1668E-08
Results by the method in Challa and Reddy [38]			
$\delta = 0.1 * \varepsilon$	7.7065e-003	8.5743e-004	8-6724e-005
$\delta = 0.3 * \varepsilon$	5.5572e-003	6.0006e-004	6.0487e-005
$\delta = 0.6 * \varepsilon$	3.8911e-003	4.1085e-004	4.1314e-005
$\delta = 0.8 * \varepsilon$	3.2241e-003	3.3750e-004	3.3908e-005
Results by the method in Kadalbajoo and Sharma [12]			
$\delta = 0.1 * \varepsilon$	1.3426e002	1.5027e003	1.5211e004
$\delta = 0.3 * \varepsilon$	9.6462e003	1.0470e003	1.0560e004
$\delta = 0.6 * \varepsilon$	6.7256e003	7.1364e004	7.1798e005
$\delta = 0.8 * \varepsilon$	5.5701e003	5.8554e004	5.8854e005

7 Conclusion :

This paper envisages the use of cubic exponential differential-difference method to find the numerical solution of SPDEs involves a BL on the one end (left or right) of the suggested domain. First, using Taylor series, the given SPDE is approximated by an asymptotically equivalent singularly perturbation BVPs with a uniform type of mesh. A numerical technique in corporating a fitting parameter is devised to minimize error and regulate the layer structure in the solution. We examined the convergence of the proposed technique. Four model problems that possess BLs are being examined to confirm the theoretical conclusion. The tables and figures demonstrate that the scheme achieves convergence with a rate of convergence of second when h is less than ε. The results achieved in this work are more precise than the approaches described in the literature [38, 12] and [39]. Moreover, the system is straightforward in terms of its underlying concepts, highly effective, user-friendly, and readily adaptable for computerized implementation.

Declaration

Data Availability

No data was used for the research described in the article.

Ethics approval and consent to participate

Not applicable

Human and animal ethics

Not applicable

Conflict of interest

The author declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper

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