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Line Graph Of A Colored Graph

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Abstract
In this article we introduce new class of graphs called line graph of colored graphs and is denoted by $L(Gc)$, involves two important graph parameters viz., $E(G)$ −edge set and vertex coloring $\chi(G)$ of graph. The line graph $L(Gc)$ of a colored graph Gc is the simple graph whose vertices are the edges of Gc with two vertices in $L(Gc)$ are adjacent, whenever the corresponding edges in Gc share at least one common colored vertex in Gc. We discuss some fundamental properties and characterizations of $L(Gc)$.
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1.0 Introduction

Graph theory is young and rapidly maturing subject. It is always changing considerably at every quarter of the century. We have found that there is a clear need for a text only to the well established results but too many of the newer developments as well. Basic concepts of graph theory are extraordinary simple and can be used to express problems from many different subjects.

A graph G is a pair of sets (V, E), where V is finite non-empty set of elements called vertices, and E is a set of unordered pairs of distinct vertices called edges. The sets V and E are vertex set and edge set of G and are often denoted by $V(G)$ and $E(G)$, respectively.

The concept of line graph of a given graph is so natural that it has been independently discovered by many researchers in [2,3,4,5,6,7].

Definition 1. (Line Graph/Edge Graph): Let G be a simple graph with $V(G) = \{v_1, v_2, \ldots v_n\}$ be the vertex set and $E(G) = \{e_1, e_2, \ldots e_m\}$ be edge set of G. The line graph $L(G)$ of graph G is a simple graph with the vertices $e_1, e_2, \ldots e_m$ in which e_i and e_j are adjacent whenever, the edges e_i and e_j share a common vertex in G. The $L(G)$ is called the line graph or edge graph of graph G.

Various properties and characterization of line graphs are developed in the literature, reader can refer [8-11].

The notion of graph coloring originates from coloring of countries of a map where each face is virtually colored. This was extended to coloring the faces of a graph embedded in a plane. By planar duality it become coloring the vertices of G and in this form it generalize to any graphs. The concept of graph coloring was introduced in 1852 with a four color problem and it becomes a famous problem in graph theory. Later graph coloring has been studied as an algorithm problem since 1970.

A proper coloring of G is an assignment of colors to each vertex so that adjacent vertices receive different colors, If the vertices have been colored with k colors then the graph is said to be k coloring. The chromatic number of G is the minimum number of colors required to color vertices of the graph, such that no two adjacent vertices have the same color. It is denoted by $\chi(G)$.

Note that, an efficient (polynomial time) algorithm is not yet found for proper coloring of arbitrary graphs.

We consider finite, undirected colored graphs throughout the paper. We have large enough literature on two graph parameters viz., edge set and vertex coloring $\chi(G)$, of graph. For more details on these two parameters see [11,12, 13].

Note 1: The graphs obtained by proper coloring is called colored graphs (denoted by G_c) and arbitrarily coloring is not unique.

Definition 2. (Hamiltonian Graph): A Hamiltonian cycle of a graph $G(V, E)$ is cycle of length $n = |V|$, i.e. the cycle goes through all vertices once. A graph is called Hamiltonian if it consists a Hamiltonian cycle.

This article is organized as follows: Section 1.0 deals with brief introduction of line graph and coloring of graph. Section 1.1 deals with definition of line graph of a colored graph. i.e., $L(G_c)$ and some fundamental properties of $L(G_c)$.

1.1 Main Results

In this section we define the concept called line graph of a colored graph and discuss some basic properties.

The line graph of a colored graph G_c is denoted by $L(G_c)$ and is the simple graph whose vertices are the edges of G_c with two colored vertices in $L(G_c)$ are adjacent whenever the corresponding edges in G_c share at least one common colored vertex.

Note that, in general the optimal coloring with $\chi(G)$ colors is not unique, so for different optimal coloring of graph, we get different line graphs of G_c. Trivial fact is, for any colored graph G_c with $\chi(G) \leq 3$, then $L(G_c)$ is a complete graph of order m, where m is the number of edges in G_c. It suffices that $L(G_c)$ will construct a huge class of isomorphic graphs from old non-isomorphic colored graphs, exclusively for G_c with $\chi(G) \leq 3$.

The concept of color line graph gives some quite surprising results and hopefully, that can be applied to some specific real world problems, like, analysis of storage problems, design theory etc., and also construction of new graphs from old graph, finding isomorphism between newly constructed graphs is of good academic interest, so we have define $L(G_c)$ and discuss some basic properties of $L(G_c)$. The subgraphs and forbidden subgraphs of $L(G_c)$, isomorphism between newly constructed $L(G_c)$ graphs are discussed in our forth coming two articles.

The article is designed as follows.

Definition 3. (Line Graph of a Colored Graph): Let G_c be a simple colored graph with $\chi(G)$ colors and $V(G_c) = \{v_1, v_2, \ldots v_n\}$ be the vertex set, $E(G_c) = \{e_1, e_2, \ldots e_m\}$ be edge set of G_c. The line graph $L(G_c)$ of a colored graph G_c is a simple graph with the vertices $e_1, e_2, \ldots e_m$ in which e_i and e_j are adjacent whenever, the edges e_i and e_j share at least one common colored vertex in G_c. The $L(G_c)$ is called the line graph or edge graph of colored graph G_c.

In other words, the line graph $L(G_c)$ of a colored graph G_c is the simple graph whose colored vertices are the edges of G_c with two colored vertices in $L(G_c)$ are adjacent, whenever the corresponding edges in G_c share at least one common colored vertex.

Note that $L(G_c)$ is always simple and isolated vertices of G_c do not have any bearing on $L(G_c)$, so we assume that G_c has no isolated vertices.

Properties of Line Graph of a Colored Graph.

In this section we discuss basic properties of line graph of colored graphs. Throughout the discussion for any edges e_i, $e_j \in (G_c)$ such that $e_i \cap e_j = \emptyset$ means no edges share at least one common colored vertex in G_c and $e_i \cap e_j \neq \emptyset$ share at least one common colored vertex in G_c.

Observations

1. Line graph is spanning graph of line graph of a colored graph G_c. i.e., $L(G) \subseteq L(G_c)$.
2. For any proper colored graph $G_c$with $\chi(G) = k$, for some $k \in Z^+$, then $L(G_c)$ is connected graph.
3. The maximum edges share a common colored vertex in G_c, givestise to a clique (complete subgraph) of line graph of $L(G_c)$.
4. If G_c be a complete colored graph of order n with $\chi(G) = n$ colors, then $L(G_c)$ has clique of order $n - 1$.

Proof of observation 4. Let $G_c = K_n$ with $\chi(G) = n$, and $d(v_i) = n - 1$, for all $(i = 1,2, \ldots, n)$, implies, $e_i \cap e_j \neq \emptyset$ except disjoint edges of $E(K_n)$. Hence by definition of $L(G_c)$ the corresponding vertices e_i and e_j are adjacent in $L(G_c)$ except edges e_i, e_j such that $e_i \cap e_j = \emptyset$. Therefore $L(G_c)$ has clique of order $n - 1$.

The following Theorem 1 is trivial.

Theorem 1. $L(G_c) \cong K_m$ if and only if with $\chi(G) \leq 3$.

Proof. Suppose $G_c(n, m)$ be a colored graph with $\chi(G) \leq 3$, implies $e_i \cap e_j \neq \emptyset$,for all e_i, $e_j \in E(G_c)$. Thus by definition of $L(G_c)$ the corresponding every pair of vertices e_i, e_j are adjacent in $L(G_c)$ for all e_i, $e_j \in V(L(G_c))$ and hence $L(G_c) \cong K_m$.

Conversely, Let $L(G_c) \cong K_m$, it follows that every pair of vertices in $L(G_c)$ are joined by an edge, implies $e_i \cap e_j \neq \emptyset$,for all $e_i, e_j \in E(G_c)$, whence $(G) \leq 3$.

On the contrary, suppose $\chi(G) \geq 4$, then there exist at least one pair of disjoint edge $e_i, e_j \in G_c$, such that $e_i \cap e_j = \emptyset$, implies the corresponding edges e_i and e_j are not adjacent in $L(G_c)$. This contradicts to the fact that $L(G_c) \cong K_m$. Therefore the colored graph G_c must be $\chi(G) \leq 3$.

Corollary 1. If $L(G_c) \ncong K_m$ if, and only if $\chi(G) \geq 4$.

By Theorem 1, it is evident that, if G_c is colored graph with $\chi(G) \leq 3$, then $L(G_c)$ is Hamiltonian.

Theorem 2. If $L(G_c)$ is line graph of colored graph G_c, then, the number of edges sharing a common colored vertices in G_c is $m(L(G_c)) = \frac{1}{2}[\sum_i d_i^2] - (m - q)$ and degree of e in $L(G_c)$ is $d_{G_c}(u) + d_{G_c}(v) - 2 + q$, where q is the number of disjoint edges in G_c those share at least one common colored vertex in G_c.

Proof. If $e = uv$ is an edge of simple colored graph G_c joining two colored vertices u and v, then degree of e in $L(G_c)$ is same as the number of edges sharing at least one common colored vertex in G_c. Therefore this number is precisely, $d_{G_c}(u) + d_{G_c}(v) - 2 + 2q$, where q is the number of disjoint edges of G_c those share at least one common colored vertex in G_c.

Hence $$d_{L(G_c)}(e) = d_{G_c}(u) + d_{G_c}(v) - 2 + 2q.$$

Now here two cases arises, case (i) $q = 0$ and (ii) $q > 0$.

case (i). If $q = 0$, then the line graph of a colored graph G_c coincides with the ordinary line graph of underlying graph G_c. Therefore $d_{L(G_c)}(e) = d_{G_c}(u) + d_{G_c}(v) - 2$ and

$$\sum_{e \in V(L(G_c))} d_{L(G_c)}(e) = \sum_{uv \in G} (d_{G_c}(u) + d_{G_c}(v) - 2)$$

Implies $$= \sum_i d_i^2 - 2(m),$$

where $d_1, d_2, \ldots, d_n$ is the degree sequence of the G_c and $m = m(G_c)$ is the number of edges in G_c. Therefore by Euler's theorem (i.e., sum of the degree of the vertices of a graph is equal to twice the number of edges), it follows that

$$m(L(G_c)) = \frac{1}{2}[\sum_i d_i^2] - m.$$

We have the following are some special class of colored graphs, whose $q = 0$.

(i). K_n, $K_{1,n}$ $K_{m,n}$ −complete bipartite colored graphs and $K_{n_1,n_2,\ldots,n_m}$ − complete multipartite colored graphs.

If $q \neq 0$, The similar argument as discussed above follows with addition of q.

$$d_{L(G_c)}(e) = (d_{G_c}(u) + d_{G_c}(v) - 2) + 2q \quad (1)$$

Now have again two cases,

Case a). Equation (1) holds if for any two disjoint edges $e_i, e_i \in E(G_c)$ such that $e_i \cap e_i \neq \emptyset$, then $q \neq 0$. Otherwise, equation (1) reduces to

$$d_{L(G_c)}(e) = (d_{G_c}(u) + d_{G_c}(v) - 2) \quad (2)$$

Case b). Let any two edges $e_i, e_i \in G_c$ do not share at least one common colored vertex in G_c. i.e., $e_i \cap e_i \neq \emptyset$, otherwise $q' > 0$.

$$\sum_{e \in V(L(G_c))} d_{L(G_c)}(e) = \sum_{uv \in G} (d_{G_c}(u) + d_{G_c}(v) - 2) + 2q$$

$$= [\sum_{uv \in G_c}(d_{G_c}(u)^2)] - 2m(G_c) + 2q \quad (3)$$

(Since uv belongs to star at u and v, and with addition of q), it implies

$$\sum_{e \in V(L(G_c))} d_{L(G_c)}(e) = [\sum_i d_i^2] - 2m + 2q$$

Implies $$\sum_{e \in V(L(G_c))} d_{L(G_c)}(e) = \sum_i d_i^2 - 2(m - q)$$

Again by Euler's Theorem and with addition of q in G_c, it follows that

$$m(L(G_c)) = \frac{1}{2}[\sum_i d_i^2] - (m - q). \quad (4)$$

This completes the proof of the theorem.

The following Theorem 3, gives us the bound for the number of vertices and edges in $L(G_c)$ in first Zagreb index $M_1(G)$ and $EM_1(G)$.

Theorem 3. For any colored graph $G(n, m)$, with $\chi(G) = k$, for all $k \geq 1, k \in Z^+$. Then

i. $V(L(G)) = m$

ii. $E(L(G_c)) \leq \frac{1}{2}M_1(G) - EM_1(G) + m(m-1)$ (5)

Where $M_1(G) = \sum_i d_i^2$ and $EM_1(G) = M_1(L(G))$

Proof: By the definition of $L(G_c)$, (i) follows.

To prove (ii). Let $G(n, m)$ be a graph with $\chi(G) = k,\ k \geq 1$, by observation1, $L(G)$ is subgraph of $L(G_c)$. Hence ,

$$L(G) \subseteq L(G_c). \text{ i.e., } \frac{1}{2}M_1(G) - m \leq E(L(G_c)). \quad (6)$$

Since every edge is non incident with exactly $m - 1 - (d_{G_c}(u) + d_{G_c}(v) - 2)$ edges. Therefore by (6), we have

$$E(L(G_c)) \leq \frac{1}{2}M_1(G_c) - m + \sum_{i=1}^{m}[m - 1 - d_{G_c}(u) + d_{G_c}(v) - 2]$$

$$= \frac{1}{2}M_1(G_c) - m + m(m-1) - \sum_{i=1}^{m} d_{G_c}(u) + d_{G_c}(v) + m$$

$$= \frac{1}{2}M_1(G) + m(m-1) - \sum_i d_i^2$$

$$= \frac{1}{2}M_1(G) - M_1(L(G)) + m(m-1)$$

$$\leq \frac{1}{2}M_1(G) - EM_1(G) + m(m-1).$$

This completes the proof of (ii).

Further if $\chi(G) \leq 3$, then by Theorem 1, equality holds in (i).

Corollary 1. Let $G(n, m)$ graph with $\chi(G) = k$ and clique no ξ, then

$$E(L(G_c)) \leq \frac{1}{2}M_1(G) - EM_1(G) + m(m-1) - \xi \quad (7)$$

Proof. Proof follows from observation 3 and (ii) of Theorem 3, above.

Conclusion: We assume and have strong belief that $L(G_c)$ will be more helpful in the analysis of storage problems, design theory etc., and, in future we work on its applications.

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