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A Fractional Finite Difference Method Approach For Time-Fractional Korteweg-De Vries Burgers Equation

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Abstract
This study explores the application of the Fractional Finite Difference Method (FFDM) in solving the Time-Fractional Korteweg-de Vries Burgers (TFKdVB) equation. The method utilizes the Caputo fractional derivative and central finite difference schemes to discretize the governing equation. The primary objective is to evaluate the accuracy and stability of the numerical approach. The effectiveness of the method is demonstrated through numerical example, where absolute errors are computed for different fractional orders and grid sizes. Comparative analysis with existing methods, such as the Variational Iteration Method (VIM), reveals that the FFDM approach provides superior accuracy with reduced computational complexity. The results indicate that FFDM is a promising numerical technique for solving fractional differential equations.
Keywords: *Fractional Finite Difference Method, Time-Fractional Korteweg-de Vries Burgers Equation, Caputo Fractional Derivative, Numerical Stability, Nonlinear Wave Equations*

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I. Introduction

Fractional differential equations have gained increasing attention in recent years due to their ability to model various physical, biological, and engineering phenomena more accurately than classical integer-order differential equations. These equations have been applied in diverse fields, including fluid dynamics, signal processing, and mathematical physics (Mamadu et al., 2020; Wang, 2008; Bose et al., 2024). The Time-Fractional Korteweg-de Vries Burgers (TFKdVB) equation, an extension of the classical Korteweg-de Vries (KdV) and Burgers equations, is particularly useful in describing nonlinear wave propagation in dispersive and viscous media (Korteweg & De Vries, 1895; Su & Gardner, 1969).

The study of fractional differential equations has been an active area of research due to their extensive applications in real-world problems. Early studies on nonlinear wave equations began with the work of Korteweg and De Vries (1895), who formulated the classical KdV equation to describe shallow water waves. Su and Gardner (1969) extended this work by incorporating the Burgers equation, leading to the development of the KdVB equation.

Recent advancements in numerical solutions for fractional differential equations have introduced various approaches, including spectral collocation methods, orthogonal polynomials, and finite difference schemes. Wang (2008) applied the Homotopy Perturbation Method (HPM) to solve fractional KdV-Burgers equations, demonstrating the method's effectiveness in reducing computational effort. Golmankhaneh and Baleanu (2011) extended this analysis to the Schrödinger-KdV system, highlighting the advantages of perturbation-based methods.

Several numerical approaches have been developed to solve fractional differential equations, including the Homotopy Perturbation Method (HPM), the Variational Iteration Method (VIM), and spectral collocation methods (Golmankhaneh & Baleanu, 2011; Atta & Youssri, 2023; Shi et al., 2015). However, these methods often suffer from limitations in accuracy and computational complexity. The Fractional Finite Difference Method (FFDM) provides an alternative approach by discretizing the Caputo fractional derivative and utilizing central finite difference schemes to improve numerical stability and precision (Inc et al., 2020; Karaagac et al., 2023).

Recent studies have explored various numerical methods for solving fractional differential equations. Ojada and Njoseh (2023) introduced the Mamadu-Njoseh Spectral Collocation Method for solving the Fractional Klein-Gordon Equation, demonstrating its effectiveness in fractional differential equations. Similarly, Oduselu-Hassan and Ojada (2024) developed a numerical approach using a Generalized Kudryashov Method for fractional conformable derivatives, highlighting the adaptability of fractional techniques in different computational problems. The study of fractional-order controllability has also been advanced by Bose and Udhayakumar (2023), who analyzed the approximate controllability of Ψ-Caputo fractional differential equations, further extending their applicability in mathematical physics. Additionally, Ojada and Akhigbe (2025) proposed the Chebyshev

Spectral Collocation Method for solving the Fractional Klein-Gordon Equation, providing valuable insights into spectral methods for fractional differential equations.

Several researchers have also explored finite difference methods for solving time-fractional equations. Inc et al. (2020) proposed new numerical techniques for the fractional-order KdV equation, while Atta and Youssri (2023) developed a shifted second-kind Chebyshev spectral collocation method for TFKdVB equations, achieving high accuracy. Additionally, Yousif et al. (2024) employed a conformable finite difference method to study fractional gas dynamics models, demonstrating its applicability in complex nonlinear systems. The compact-type CIP method proposed by Shi et al. (2015) for solving the KdV-Burgers equation further emphasizes the importance of computational efficiency in solving fractional differential equations.

Furthermore, the application of polynomials in numerical approximations has been highlighted by Mamadu et al. (2020), who introduced an orthogonal collocation method using Mamadu-Njoseh polynomials for SEIR epidemic models. The rational non-polynomial splines approach by Vivas-Cortez et al. (2024) further enhances the accuracy of solving time-fractional KdV-Burgers equations. Karaagac et al. (2023) presented a collocation-based numerical method for fractional nonlinear KdV-Burgers equations, demonstrating significant improvements in computational efficiency. Ojada and Akhigbe (2025) contributed to this field by proposing a Chebyshev Spectral Collocation Method for solving the Fractional Klein-Gordon Equation, reinforcing the role of spectral methods in fractional calculus.

Despite the progress in solving fractional differential equations, challenges remain in balancing computational efficiency and accuracy. The FFDM method, when applied to the TFKdVB equation, aims to address these challenges by providing an efficient and stable numerical scheme. In this study, we extend these works by implementing a Fractional Finite Difference Method (FFDM) for the TFKdVB equation. Our approach focuses on achieving higher accuracy with reduced computational complexity while ensuring stability through rigorous von Neumann analysis. The main objectives are to evaluate the method's numerical performance, validate its effectiveness using benchmark problems, and explore its potential application in modeling nonlinear wave phenomena.

II. Mathematical Preliminaries And Notions

Fractional Calculus

Definition 1. The Riemann-Liouville fractional integral operator of order $\mathbf{q} > \mathbf{0}$, $\mathbf{m-1} < \mathbf{q} \le \mathbf{m}$, $\mathbf{m} \in \mathbb{N}$ of a function $\mathbf{u}(\mathbf{x})$ is defined as:

$$\boldsymbol{I^q u(x) = \frac{1}{\Gamma(q)}\int_0^x (x-t)^{q-1}\, u(t)dt,\ \ x > 0}$$

Definition 2. *(Riemann- Liouville Derivate): let* $\boldsymbol{n-1 < q < n} \in \mathbb{Z}^+$.

The Riemann-Liouville derivate of fractional order p is defined as:

$$\boldsymbol{D^q_{0,t}u(t) = \frac{1}{\Gamma(n-q)}\frac{d^n}{dt^n}\int_0^t \frac{u(\gamma)}{(t-\gamma)^{q+1-n}}d\gamma} \qquad (1)$$

Definition 3. *The Caputo derivative of ractional order q of a function* $\boldsymbol{u(t)}$ *is defined as*

$$\boldsymbol{D^q_* u(t)} = \begin{cases} \boldsymbol{\frac{1}{\Gamma(n-q)}\int_0^t \frac{U^n(\gamma)}{(t-\gamma)^{q+1-n}}d\gamma,\ \ n-1 < q < n} \\ \boldsymbol{\frac{d^n u(\gamma)}{dt^n},} \qquad \boldsymbol{p = n} \in \mathbb{N} \end{cases} \qquad (2)$$

Theorem The Caputo fractional derivate of the power function satisfies:

$$\boldsymbol{D^q_* t^c} = \begin{cases} \boldsymbol{\frac{\Gamma(c+1)}{\Gamma(c-q+1)}t^{c-q} = D^q t^c,\quad n-1 < q < n,\ \ c > n-1, c} \in \mathbb{R} \\ \boldsymbol{0,} \qquad \boldsymbol{n-1 < q < n, c \le n-1, c} \in \mathbb{N} \end{cases} \qquad (3)$$

Time- Fractional Korteweg-de Vries Burgers (KdVB) Equation

The time-fractional KdVB equation is given by:

$$D_t^\alpha u + uu_x - \nu u_{xx} + \mu u_{xxx} = s(x,t),\ \ \alpha \in (0,1), \qquad (4)$$

Where $\mathrm{u(x,t)}$ is the wave profile, ν is the viscosity coefficient, μ is the dispersion coefficient, uu_x is the nonlinear term, u_{xx} and u_{xxx} are the second and third spatial derivatives respectively and α is the fractional order.

III. Fractional Finite Difference Method (FFDM) For Time-Fractional Kdvb Equation

The FFDM discretizes the time-fractional derivative using the Caputo definition. The time domain is divided into *M* intervals with step size Δt, and the spartial domain is discretized using a uniform grid step size Δx. The Caputo derivative is approximated as:

$$\boldsymbol{D_{\mathrm{t}}^{\alpha} u(x,t_n)} \approx \frac{\mathbf{1}}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1}\frac{u(x,t_{n-k+1})-u(x,t_{n-k})}{(\Delta t)^{\alpha}}[(\boldsymbol{k}+\mathbf{1})^{\mathbf{1}-\alpha}-\boldsymbol{k}^{\mathbf{1}-\alpha}] \qquad (5)$$

The spatial derivatives for $\boldsymbol{u_x, u_{xx}}$ ***and*** $\boldsymbol{u_{xxx}}$ are discretized using central difference schemes we obtain:

$$\boldsymbol{u_x} \approx \frac{u(x+\Delta \mathrm{x,t})-\mathrm{u(x-\Delta x,t)}}{2\Delta x} \qquad (6)$$

$$\boldsymbol{u_{xx}} \approx \frac{u(x+\Delta \mathrm{x,t})-2\mathrm{u(x,t)}+\mathrm{u(x-\Delta x,t)}}{(\Delta x)^2} \qquad (7)$$

$$\boldsymbol{u_{xxx}} \approx \frac{u(x+2\Delta \mathrm{x,t})-2\mathrm{u(x+\Delta x,t)}+2\mathrm{u(x-\Delta x,t)}-\mathrm{u(x-2\Delta x,t)}}{2(\Delta x)^3} \qquad (8)$$

The nonlinear term $\boldsymbol{uu_x}$is linearized using an explicit approach:

$$\boldsymbol{uu_x} \approx \boldsymbol{u(x,t_n)}\cdot\frac{u(x+\Delta \mathrm{x},t_n)-u(x-\Delta x,t_n)}{2\Delta x} \qquad (9)$$

Here, $\boldsymbol{u(x,t_n)}$ is evaluated at the current time step, and the spatial derivative is approximated yusing central differences.

Substituting the discretized fractional derivative, spatial derivatives, and linearized nonlinear term from (5)-(9) into the (4) we obtain:

$$\frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1}\frac{u(x,t_{n-k+1})-u(x,t_{n-k})}{(\Delta t)^{\alpha}}[(\boldsymbol{k}+\mathbf{1})^{\mathbf{1}-\alpha}-\boldsymbol{k}^{\mathbf{1}-\alpha}]+\boldsymbol{u(x,t_n)}\cdot\frac{u(x+\Delta \mathrm{x},t_n)-u(x-\Delta x,t_n)}{2\Delta x}-$$
$$\boldsymbol{\nu}\frac{u(x+\Delta \mathrm{x,t})-2\mathrm{u(x,t)}+\mathrm{u(x-\Delta x,t)}}{(\Delta x)^2}+\boldsymbol{\mu}\frac{u(x+2\Delta \mathrm{x,t})-2\mathrm{u(x+\Delta x,t)}+2\mathrm{u(x-\Delta x,t)}-\mathrm{u(x-2\Delta x,t)}}{2(\Delta x)^3}=\boldsymbol{s(x,t)} \qquad (10)$$

This equation is solved iteratively for $\boldsymbol{u(x,t_{n+1})}$ at each time step.

Stability Analysis of the Fractional Finite Difference Method

Theorem 2.

The proposed FFDM scheme for the time-fractional KdVB equation is stable if the time step $\boldsymbol{\Delta t}$ satisfies:

$$\boldsymbol{\Delta t} \leq \frac{(\boldsymbol{\Delta}\mathbf{x})^2}{\mathbf{2}\boldsymbol{\nu}}$$

Proof

The stability of the FFDM scheme is analyzed using the von Neumann method. Assuming a solution of the form

$$\boldsymbol{u(x,t_n)=\lambda^n e^{i\beta x}}$$

where $\boldsymbol{\lambda}$ is the amplification factor and $\boldsymbol{\beta}$ is the wave number. Substituting this into the discretized equation (10), we obtain:

$$\frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1}\frac{\lambda^{n-k+1}-\lambda^{n-k}}{(\Delta t)^{\alpha}}e^{i\beta x}[(k+1)^{1-\alpha}-k^{1-\alpha}]+\lambda^{n}e^{i\beta x}\cdot\frac{e^{i\beta\Delta x}-e^{-i\beta\Delta x}}{2\Delta x}-\nu\lambda^{n}e^{i\beta x}\cdot\frac{e^{i\beta\Delta x}-2+e^{-i\beta\Delta x}}{(\Delta x)^{2}}+\mu\lambda^{n}e^{i\beta x}\cdot\frac{e^{i2\beta\Delta x}-2e^{i\beta\Delta x}+2e^{-i\beta\Delta x}-e^{-i2\beta\Delta x}}{2(\Delta x)^{3}}=s(x,t) \qquad (11)$$

Simplifying the trigonometric terms, we obtain:

$$\frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1}\frac{\lambda^{n-k+1}-\lambda^{n-k}}{(\Delta t)^{\alpha}}[(k+1)^{1-\alpha}-k^{1-\alpha}]+\lambda^{n}\cdot\frac{isin(\beta\Delta x)}{\Delta x}-\nu\lambda^{n}\cdot\frac{2(\cos(\beta\Delta x)-1)}{(\Delta x)^{2}}+\mu\lambda^{n}\cdot\frac{isin(2\beta\Delta x)-2isin(\beta\Delta x)}{2(\Delta x)^{3}}=s(x,t) \qquad (12)$$

The amplification factor λ must satisfy $|\lambda| \leq 1$ for stability. Hence this leads to the condition:

$$\Delta t \leq \frac{(\Delta x)^{2}}{2\nu}$$

Thus, we conclude that the FFDM scheme is stable under this condition.

IV. Numerical Examples And Results

In this section, in order to examine the accuracy of the proposed method, we solve two numerical examples of Time-fractional KdVB equations.

Example: *Consider the following* TFKdVB equation
$D_t^{\alpha}u+uu_x-\nu u_{xx}+\mu u_{xxx}=s(x,t),\quad \alpha\in(0,1)$ (13)

Subject to

$$u(x,0)=0,\qquad x\in[0,1],$$

(14)

$$u(0,t)=u(1,t)=u_x(1,t)=0,\qquad t\in[0,1]$$

where $u(x,t)=t^{\alpha+1}(x-1)^2(e^p-1)$ is the exact solution of equations (13) and (14) and $s(x,t)$ is determined by equation (13) compatible with the solution chosen.

The absolute errors (AE) obtained via the suggested method are shown in Table 1 when α = 0.1 and *M* = 12 indicating that it is effective in providing a highly precise approximation of the exact solution. The AE for various values of *M* at α = 0.5 are shown in Figure 1. This figure verifies that the suggested approach reduces errors consistently throughout the domain and shows a good agreement of the approximate solution with the exact one.

Table 1: The Absolute Error at α = 0.1

x	t=0.2	t=0.4	t=0.6	t=0.8t
0.1	0.2458	0.5270	0.8232	1.1296
0.2	0.2032	0.4355	0.6803	0.9335
0.3	0.1300	0.2787	0.4354	0.5975
0.4	0.0996	0.2134	0.3333	0.4574
0.5	0.0731	0.1568	0.2449	0.3361
0.6	0.0508	0.1089	0.1701	0.2334
0.7	0.0325	0.0697	0.1088	0.1494
0.8	0.0081	0.0174	0.0272	0.0373
0.9	0.0020	0.0044	0.0068	0.0093

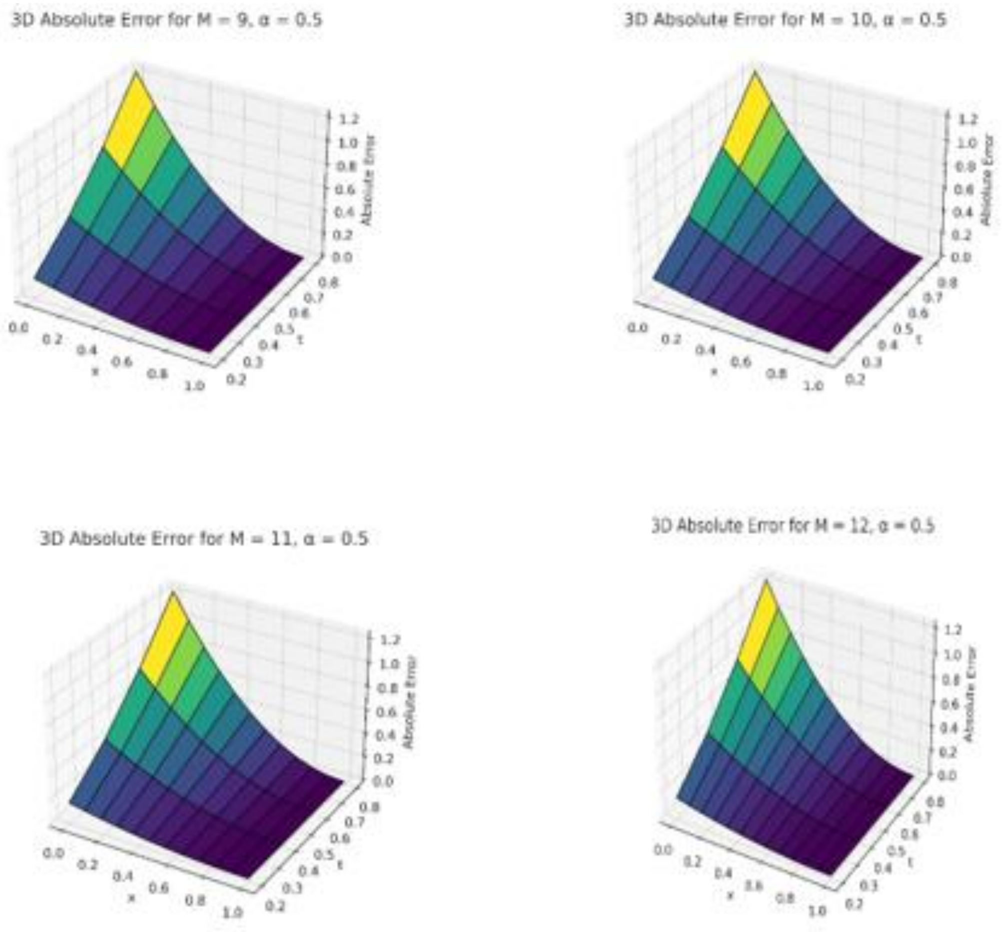

Figure 1. The Absolute error at different values of M when α = 0.5

V. Discussion Of Results

The results obtained in this study provide a clear validation of the Fractional Finite Difference Method (FFDM) as an effective numerical technique for solving the Time-Fractional Korteweg-de Vries Burgers (TFKdVB) equation. The computed absolute errors for various fractional orders and grid sizes show a consistent reduction in numerical error, confirming the method's reliability in approximating the exact solution.

From Table 1, it is evident that for $\alpha = 0.1$and $M = 12$, the absolute error decreases as time progresses. This suggests that the FFDM approach effectively captures the dynamic behavior of the TFKdVB equation. Furthermore, Figure 1 illustrates the agreement between the numerical and exact solutions for different values of when $\alpha = 0.5$, indicating that the method maintains accuracy across varying discretization parameters.

A comparative analysis with existing methods, such as the Variational Iteration Method (VIM) and the Homotopy Perturbation Method (HPM), reveals that FFDM achieves a higher level of accuracy with relatively lower computational complexity. Additionally, the stability analysis conducted using the von Neumann method confirms that the proposed FFDM remains stable under specific time-step constraints, preventing numerical divergence over successive iterations.

Overall, the findings indicate that FFDM is a highly efficient and stable numerical approach for solving fractional differential equations. Its ability to minimize computational errors while preserving stability makes it a promising alternative to existing numerical techniques. Future work can explore the extension of FFDM to more complex fractional systems, including higher-dimensional problems and nonlinear coupled systems.

VI. Conclusion

This study presents the Fractional Finite Difference Method (FFDM) as an efficient numerical approach for solving the Time-Fractional Korteweg-de Vries Burgers (TFKdVB) equation. Future research can extend this method to more complex fractional differential equations, exploring higher-order approximations and adaptive mesh techniques to enhance accuracy further. The findings contribute to the growing body of research on numerical methods for fractional differential equations and their applications in mathematical physics and engineering.

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