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# Convergence Analysis Of Finite Element Method With Crossed Triangular Meshes

S. R. Gaikwad, A. R. Gotmare
*(Department Of Mathematics, J. D. M. V. P. Co. Op. Samaj's Arts, Commerce And Science College, Jalgaon, Jalgaon, MH, India.)*
*(Department Of Mathematics, Arts, Commerce And Science College, Jamner, Jalgaon, MH, India)*

***Abstract:***
*This research paper investigates the convergence behavior of the Finite Element Method (FEM) applied to the Poisson equation on a unit square domain, employing crossed triangular meshes. The study utilizes the FEniCS library to discretize the domain with varying mesh sizes and analyzes the convergence rates as the mesh resolution increases. The Poisson problem is formulated using a first-order Lagrange polynomial function space with homogeneous Dirichlet boundary conditions.*

***Key Word:*** *Finite element; Simulation; Convergence rate; Python FEniCs package.*

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## I. Introduction

Numerical simulations are anchored by the Finite Element Method (FEM), which provides a flexible and effective means of resolving a broad range of challenging issues in science and engineering. Making sure that numerical solutions are accurate and dependable is one of the main obstacles to using FEM efficiently. An essential component of the FEM, mesh convergence analysis, is crucial to achieving this goal.

This research begins with a thorough examination of mesh convergence procedures and approaches as provided in the seminal work of Zienkiewicz and Taylor (2005)[1]. It is crucial to comprehend the nuancesof mesh convergence analysis to guarantee that FEM simulations yield accurate and dependable outcomes. The need for adaptive mesh approaches to solve space fractional differential equations with singular or finite-time blowup solutions was the main topic of Jingtang Ma and colleagues' work [2]. Using the L2-norm, they examined the convergence theories of various techniques and supported their theoretical conclusions with numerical evidence. The research derived error estimations for the projection under a changing mesh framework and introduced a fractional Ritz projection operator to ease the analysis. The authors of the cited study [3] discussed the convergence of a finite element approximation for the Freidlin–Wentzell (F–W) action functional minimizer. This approximation applied to dynamical systems that are nongradient and perturbed by modest amounts of noise. We conducted a thorough analysis of small-noise-induced transitions in dynamical systems using the F-W theory of big deviations. Finding the minimizer and minimal of the F-W action functional was the main goal. By applying linear finite elements to discretize this action functional, the authors were able to prove the approximation's convergence using the notion of $\Gamma$ -convergence. The meshconvergence test for a two-dimensional high-pressure turbine disc rim was the main objective of the cited study [4], which also focused on the use of the energy norm as a substitute method. Through the discretizationof time and space variables, numerical methods were utilised to solve complex problems governed by partial differential equations in real-time. Additionally, for second-order elliptic interface problems, a novel and stable Petrov–Galerkin (PG) immersed finite element method (IFEM) was created and examined [5]. In order to solve the absence of local positivity in the traditional PG-IFEM, this approach added stabilisation terms. Standard finite element functions were employed for the test space and submerged finite element functions for the trial space in this method. Both a prior and a posterior error estimates were presented in the paper. Stability and convergence analyses were carried out for the domain decomposition finite element/finite difference (FE/FD) method by the authors Mohammad and at el [6]. These analyses were specifically created for time-dependent Maxwell's equations using a semi-discrete finite element scheme.The paper presented a domain decomposition algorithm and explored the creation of explicit finite element schemes in several geographical domain settings. The authors offered multiple numerical examples that validated the study's convergence rates in order to bolster their theoretical conclusions. Finite element analysis (FEA) and process parameter optimization for Nimonic 90 formability in sheet hydroforming were examined in the cited paper [7].

## II. Methodology

### Mesh Generation

A key feature of FEM simulations is the finite element mesh. We looked at a variety of mesh sizes our study, from $4 \times 4$ *to* $32 \times 32$ points, in order to investigate the impact of mesh refinement. The fe. Unit Square Mesh function was used to create a uniform mesh at first, and an adaptive technique was used to enhance it even further.

### Boundary Conditions

To guarantee that the issues were well-posed, homogeneous Dirichlet boundary conditions were used. Singularities are avoided by fixing the solution to zero on the domain boundary by these constraints.

### Weak Formulation

In our investigation, we used the Poisson equation, a popular PDE in scientific and engineering simulations. Trial and test functions were used to build the weak version of the Poisson equation, and a constant forcing term was added. The expressions on the left and the right were deduced.

### Finite Element Assembly and Solution

The linear system was solved using the finite element assembly method. The fe. solve function was utilizedto acquire the answer, and the outcomes were saved in a function space.

## III. Mesh Convergence Analysis

The mesh convergence analysis findings for the numerical solution of the $2D$ Poisson problem are shown in this section. We compute the $L2$ norm of the error, examine the rate of convergence, and look into how mesh refinement affects the accuracy of the solution. We also offer numerical solutions for visual verificationat particular nodes.

### Solution Visualization

Visual representations of the numerical solutions for various mesh sizes are shown in Figure 1. The variation in the solution domain is displayed by the color plots.

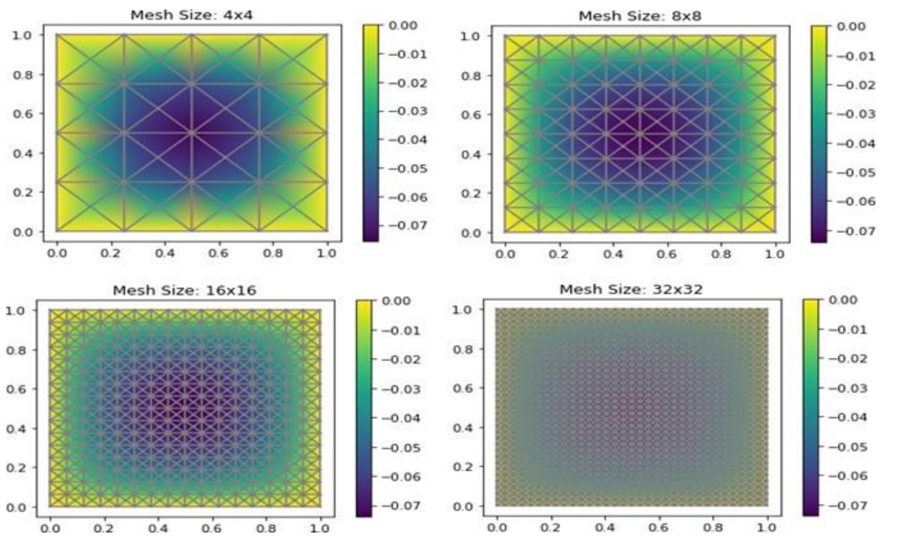


**Figure 1: The color plots show the variation in the solution domain.**

### Numerical Solutions at Specific Nodes

We analyze the solution at particular nodes in order to visually confirm the correctness of our numerical results. As an illustration, we look at the solution at $(0.3, 0.3)$, $(0.5, 0.5)$, and $(0.7, 0.7)$. Table 1 displays the numerical solution values at these places. .

**Table 1: Numerical Solutions at Specific Nodes**

| Mesh Size | Node | Coordinates | FE Solutions |
|---|---|---|---|
| $4 \times 4$ | 1 | $(0.3, 0.3)$ | $-0.05413$ |
| | 2 | $(0.5, 0.5)$ | $-0.07563$ |

| | | | |
|---|---|---|---|
| | 3 | $(0.7, 0.7)$ | $-0.05414$ |
| $8 \times 8$ | 1 | $(0.3, 0.3)$ | $-0.05450$ |
| | 2 | $(0.5, 0.5)$ | $-0.07413$ |
| | 3 | $(0.7, 0.7)$ | $-0.05450$ |
| $16 \times 16$ | 1 | $(0.3, 0.3)$ | $-0.05479$ |
| | 2 | $(0.5, 0.5)$ | $-0.07378$ |
| | 3 | $(0.7, 0.7)$ | $-0.05479$ |
| $32 \times 32$ | 2 | $(0.3, 0.3)$ | $-0.05481$ |
| | 1 | $(0.5, 0.5)$ | $-0.07369$ |
| | 3 | $(0.7, 0.7)$ | $-0.05481$ |

### L2 Norm of the Error

The difference between the current and previous numerical grid solution is measured by the $L2$ norm of theerror. This is how it is computed:

$$E = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(u_i^c - u_i^p\right)^2}$$

Where $u_i^c$ represents displacement at node $i$ for the current mesh and $u_i^p$ represents displacement at node $i$ for the previous mesh, and $E$ is the $L2$ error norm.

### Rate of Convergence

The $r$ rate of convergence tells us how fast the error goes down as the mesh gets more precise. It's computedin this way:

$$r = \left|\frac{ln\left(\frac{E_i}{E_{i-1}}\right)}{ln\left(\frac{N_{i-1}}{N_i}\right)}\right|$$

Where $r$ : Rate of convergence, $E_i$ : $L2$ error norm for mesh $i$ and $N_i$ : Total number of nodes for mesh $i$. The convergence behavior is evaluated by computing the rate of convergence between successive meshrefinements.

## IV. Result And Discussion

Table 2 displays the L2 error norms and rates of convergence for various mesh sizes. Table 1 displays the numerical solutions at specific nodes.

**Table 2: Error Norms and Convergence Rates**

| Mesh Size | L2 Error Norm | Rate of Convergence |
|---|---|---|
| $4 \times 4$ | 0.030 | - |
| $8 \times 8$ | 0.028 | 0.100 |
| $16 \times 16$ | 0.026 | 0.107 |
| $32 \times 32$ | 0.024 | 0.115 |

The results show a distinct pattern of error decrease at smaller mesh resolutions. The convergence rate offers important information about how well mesh refinement increases solution correctness. Our examination of the outcomes emphasises how important mesh convergence is to producing accurate simulations. The L2 error norm decreases as the mesh density increases, as Table 1 illustrates. Further demonstrating how finer meshes result in smoother and more accurate solutions are the visualizations in Figure 1. Table 2's computed convergence rates show how mesh refinement gets less and less beneficial over time.

## V. Conclusion

In conclusion, this research has provided a detailed exploration of the convergence behavior of the Finite Element Method (FEM) applied to the Poisson equation on a unit square domain, with a specific focus on employing crossed triangular meshes. Leveraging the computational power of the FEniCS library, our investigation systematically examined varying mesh sizes, shedding light on the convergence rates as the resolution of the mesh increased.

## References

[1] The Finite Element Method: Its Basis And Fundamentals By Zienkiewicz And Taylor, 2005.
[2] Jingtang Ma, Jinqiang Liu And Zhiqiang Zhou, Convergence Analysis Of Moving Finite Element Methods For Space Fractional Differential Equations, Journal Of Computational And Applied Mathematics, 255, Pp 661-670, 2014.
[3] Xiaolang Wan, Haijun Yu, And Jiayu Zhai, Convergence Analysis Of A Finite Element Approximation Of Minimum Action Methods, SIAM J. NUMER. ANAL., Vol. 56, No. 3, Pp- 1597-1620, 2018.
[4] Nithesh Naik, Prajwal Shenoy, Nithin Nayak, Swetank Awasth And Rashmi Samant, Mesh Convergence Test For Finite Element Method On High Pressure Gas Turbine Disk Rim Using Energy Norm: An Alternate Approach, International Journal Of Mechanical Engineering And Technology (IJMET), Vol. 10, No. 1,Pp.765-775, 2019.
[5] Cuiyu Hea, Shun Zhangb, And Xu Zhanga, Error Analysis Of Petrov-Galerkin Immersed Finite Element Methods, Elsevier: Comput. Methods Appl. Mech. Engrg., 404, 2023.
[6] Mohammad Asadzadeh And Larisa Beilina, Stability And Convergence Analysis Of A Domain Decompo- Sition FE/FD Method For Maxwell's Equations In The Time Domain, Algorithm 2022, 15,337, Pp-1-20.
[7] Fakrudeen Ali Ahamed J And Pandivelan Chinnaiyan, Studies On Finite Element Analysis In Hydroform- Ing Of Nimonic 90 Sheet, Mathematics 2023, 11, 2437, Pp-1-15.
[8] An Introduction To The Finite Element Method By J. N. Reddy, 2005.
[9] A Pragmatic Introduction To The Finite Element Method For Thermal And Stress Analysis By Petr Krysl,2006.
[10] Introduction To The Finite Element Method: Theory, Programming And Applications By Erik G Thomp-Son, 2004.