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Some Results On Artinian Banach-Jordan Systems

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Abstract
The main objective of this research article consists in giving a characteriza− tion of a non−degenerate Artinian complex Banach−Jordan pair. Goncretely me shom that under some related conditions, the pair in question is finite dimen− sional . Similar results for Banach−Jordan algebras hold to be true.
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1. Introduction

An associative or alternative algebra A is Artinian (Noetherian) if it satisfies the descending (ascending) chain condition on left ideals. Sinclair and Tullo [15] showed that a complex Noetherian Banach associative algebra is finite dimensional. The same result was obtained for complex Banach alternative algebras [2] . In 2020, the authors of research article [1] has shown that a quotient algebra $\frac{A}{Rad(A)}$ is finite-dimensional provided that the Banach algebra A satisfies the descending chain condition on closed left ideals. Here $Rad(A)$ denotes the Jacobson radical of A.

In virtue of what it is recalled, the question which naturally springs to mind is: under the descending chain condition on inner ideals, does the main result in [15] still holds in the more general framework of Banach-Jordan structures, specially in the case of ternary rings? our task consists in giving an affirmative answer to this question.

For a Jordan algebra J or a Jordan pair $P = (P^+, P^-)$, the suitable Artinian (Noetherian) concept is the descending (ascending) chain condition on inner ideals $I = (I^+, I^-)$ of $P = (P^+, P^-)$, in the the sense that $Q_{I^\sigma} P^{-\sigma} \subset I^\sigma$ ($\sigma = \pm$). Of particular relevance to this work, the result asserting that a complex Noetherian Banach-Jordan algebra is finite dimensional [3]. The same result was extended in [4] to complex Noetherian Banach-Jordan pairs. The present paper is devoted to the study of the inverse situation. That is, we are dealing with complex Artinian Banach-Jordan systems. Concretely, we prove that every non-degenerate Artinian complex Banach-Jordan pair is finite dimensional. The result with respect to Artinian associative or alternative complex Banach-algebras is actually a direct consequence of our main theorem. Our approach consists in using Loos's fundamental structure theorem for central Jordan pairs of finite capacity [8] but as presented by McCrimmon in [14] together with techniques related to local algebras [5].

After a preliminary section whose main interest is to fix notations and recall some basic results on Jordan algebras and pairs, the second section is devoted to the main results.

2. Preliminaries.

In this paper we shall deal with Jordan pairs and Jordan algebras over a commutative ring of scalars Υ. Nevertheless, we shall be mainly interested in the *linear* case ($\frac{1}{2} \in \Upsilon$), and very specially in the case that Υ is the real or complex field. The reader is referred to [8] for further details. However, we shall record in this section some notations and results.

Given a Jordan pair $P = (P^+, P^-)$ over a commutative ring R of characteristic not two of two R-modules P^σ, $\sigma = \pm$ equipped with a couple (Q^+, Q^-) of quadratic operators $Q^\sigma : P^\sigma \longrightarrow Hom_\Upsilon(P^{-\sigma}, P^\sigma)$ such that the following identities hold for all $(x, y) \in P^\sigma \times P^{-\sigma}$

$$\begin{aligned} V^\sigma_{(x,y)} Q^\sigma_x &= Q^\sigma_x V^{-\sigma}_{(y,x)}, \\ V^\sigma_{(Q_x y, y)} &= V^\sigma_{(x, Q^{-\sigma}_y x)}, \end{aligned}$$

where

$$V^\sigma_{(x,y)} z = Q^\sigma_{(x,z)} y = \{x, y, z\}_\sigma \ (\sigma = \pm)$$

are the triple products of $P = (P^+, P^-)$ and

$$Q^\sigma_{(x,z)} = Q^\sigma_{x+z} - Q^\sigma_x - Q^\sigma_z,$$

and $\{x, y, x\}_\sigma = 2Q^\sigma_x y$ [8].

An example of Jordan pairs over a field K of characteristic not two is given by taking $P = (P^+, P^-)$ such that

$$P^+ = M_{pq}(R), \ P^- = M_{pq}(R), \ (p, q \in \mathbb{N}^*)$$

the linear vector spaces of rectangular matrices with entries in an associative algebra R over the field K. The multiplication in P is defined by:

$$Q_u v = uvu, \; 6(u, v) \in P^{\sigma} \times P^{-\sigma},$$

the usual matrix product.

Any associative, alternative or Jordan algebra A gives rise to a Jordan pair (A, A) with a quadratic multiplication xyx or $U_x y$, with U denoting the usual U-operator of a Jordan algebra defined by

$$U_x y = 2x(xy) - x^2 y.$$

In the opposite direction, given a Jordan pair $P = (P^+, P^-)$ and an element $u \in P^{-\sigma}$, we can define a Jordan algebra on the vector space P^{σ} by

$$U_a = U_a^{(u)} = Q_a Q_u,$$

and the square

$$a^{(2,u)} = Q_a u.$$

This Jordan algebra, denoted by $P^{\sigma(u)}$, is called the u-*homotope* of P at u. We shall write $\{.,.,.\}^{(u)}$to denote the triple product in $P^{\sigma(u)}$. If P is a linear Jordan pair, we just need to define the linear product in $P^{\sigma(u)}$ by $a._u b = \frac{1}{2}\{a, u, b\}$.

Gocal algebra of a Jordan pair. Let $P = (P^+, P^-)$ be a Jordan pair and $y \in P^{-\sigma}$ by [8] the set

$$Ker(y) = \{x \in P^{\sigma} : Q_y x = Q_y Q_x y = 0\}$$

turns out to be an ideal of $P^{\sigma(y)}$ and the quotient $P^{\sigma(y)}/Ker(y)$ becomes a Jordan algebra called the *local algebra* of P at y, which we denote by P_y. As pointed out in [5, 1.2.4(ii)] the condition $Q_y Q_x y = 0$ is superfluous if P is linear or *non−degenerate* in the sense that P has no absolute zero divisor, that is: $Q_x = 0$ implies $x = 0$.

Ideal and inner ideal of a Jordan pair. Let $P = (P^+, P^-)$ be a Jordan pair. The couple $I = (I^+, I^-)$ of submodules of P is said to be an *ideal* of $P = (P^+, P^-)$ if

$$Q_{I^{\sigma}} P^{-\sigma} + Q_{P^{\sigma}} I^{-\sigma} + \{I^{\sigma}, P^{-\sigma}, P^{\sigma}\} \subseteq I^{\sigma}.$$

If $\frac{1}{2} \in R$, then every ideal of $P = (P^+, P^-)$ has an easy condition:

$$Q_{P^{\sigma}} I^{-\sigma} + \{I^{\sigma}, P^{-\sigma}, P^{\sigma}\} \subseteq I^{\sigma}.$$

I^{σ} is said to be an *inner ideal* of $P = (P^+, P^-)$ if

$$Q_{I^{\sigma}} P^{-\sigma} \subseteq I^{\sigma}.$$

If $I^{-\sigma} \subseteq P^{-\sigma}$ is an inner ideal of P then for every $x \in P^{\sigma}$, $y \in P^{-\sigma}$ $Q_x I^{-\sigma}$ and $B_{(y,x)} I^{-\sigma}$ are inner ideals of $P = (P^{+}, P^{-})$, where

$$B_{(x,y)} = Id_{V^{\sigma}} - V_{(x,y)}^{\sigma} + Q_x^{\sigma} Q_y^{-\sigma},$$

for each $(x, y) \in P^{\sigma} \times P^{-\sigma}$, denotes the so called *Bergman operator*. Such operator is of great interest in Jordan theory. In particular, $Q_x P^{-\sigma}$ is an inner ideal of $P = (P^{+}, P^{-})$, it is called a *principal inner ideal*, for more details see [8].

Gomer radical. The *lomer radical* of P, denoted by

$$rad(P) = \left(rad(P^{+}), rad(P^{-}) \right),$$

is defined as the smallest ideal of P among its ideals I such that P/I is non-degenerate. We also have that P is non-degenerate if and only if $rad(P) = 0$.

Jacobson radical. A couple (x, y) is *quasi−invertible* in a Jordan pair $P = (P^{+}, P^{-})$ if x is quasi-invertible in the homotope $P^{-(y)}$. This is equivalent that the Bergman operator $B_{(x,y)}$ is invertible in the operator algebra $L(P^{+})$. Also, x is said to be *properly quasi−invertible* if it is quasi-invertible in all homotopes of P.

Following [8], the *Jacobson radical* of a Jordan pair $P = (P^{+}, P^{-})$ is defined as the ideal $Rad(P) = (Rad(P^{+}), Rad(P^{-}))$, where $Rad(P^{\sigma})$ is the set of *properly quasi−invertible* elements of P^{σ}, that is, those elements which are quasi-invertible in every homotope $P^{\sigma(u)}$. A Jordan pair is said to be *semiprimitive* if $Rad(P) = 0$.

Idempotent. Recall that an element $x \in P^{\sigma}$ is called *von Neumann regular* if there exists $y \in P^{-\sigma}$ such that $x = Q_x y$. A Jordan pair $P = (P^{+}, P^{-})$ is said to be *von Neumann regular* if and only if so are all its elements.

A pair of elements $e = (e_{+}, e_{-}) \in P^{+} \times P^{-}$ is called an *idempotent* if $Q_{e_+} e_{-} = e_{+}$ and $Q_{e_-} e_{+} = e_{-}$. It is an elementary but important fact that every von Neumann regular element gives rise to an idempotent.

A non-degenerate Jordan pair has *finite capacity* if it contains a *strong frame*, that is an *orthogonal system* $\{e_1, e_2, \ldots, e_n\}$ of *division idempotents* ($P_2(e_i)$ is a division Jordan pair) such that $\cap_{i=1}^{n} P_0(e_i) = 0$, equivalently the lengths of its chains of principal inner ideals are bounded [10]. In this case $e_1 + e_2 + \ldots + e_n = e$ is a *maximal idempotent of* P in the sense that the Peirce 0-space $P_0(e) = 0$.

For more details on idempotents and Peirce decomposition see [8].

By a *normed* Jordan pair we shall mean a Jordan pair $P = (P^{+}, P^{-})$ over 9 = (R or s) where the vector spaces P^{+} and P^{-} are respectively endowed with norms $\|.\|_{+}$ and $\|.\|_{-}$ satisfying

$$\|\{x, y, z\}_{\sigma}\|_{\sigma} \leq \|x\|_{\sigma} \|y\|_{-\sigma} \|x\|_{\sigma}$$

for every $(x, z) \in P^{\sigma}$ and $y \in P^{-\sigma}$ $\sigma =$.

If the norms $\|.\|_+$ and $\|.\|_-$ are complete, then P is said to be a *Banach–Jordan pair*.

If there is no confusion, the triple products $\{x, y, z\}_\sigma$ and the norms $\|.\|_\sigma$ of P are simply denoted by $\{x, y, z\}$ and $\|.\|$.

Socle. An inner ideal M of a Jordan pair $P = (P^+, P^-)$ is called *simple* if for every inner ideal $M' \subset M$ either $M' = 0$ or $M' = M$. For a non-degenerate Jordan pair P, $Soc(P) = (Soc(P^+), Soc(P^-))$, where $Soc(P^\sigma)$ denotes the sum of all *simple inner ideals* of P^σ, is an ideal called the *socle* of P. By [9, Theorem 2], for a non-degenerate Jordan pair

(i) *a simple element generates a simple ideal of* P.

(ii) $Soc(P)$ *is the direct sum of simple ideals of* P.

The following conditions on an inner ideal M are equivalent.

(i) M *is simple,*
(ii) $M \neq 0$, *and* $M = [x]$ *for all* $0 \neq x \in M$.

where $[x] = Q_x P^{-\sigma}$ denotes the principal inner ideal generated by x.
Note that, for $x \in P^+$, $(x) = 9x + Q_x P^-$ is the *inner ideal generated by* x. In general $x \notin [x]$, in fact, this is the case if and only if x is von Neumann regular, that is, $(x) = [x] = Q_x P^-$.

Annihilators. Following [8, pp. 104], the annihilators of a subset X of P^σ is the inner ideal $ann(X) \subset P^{-\sigma}$ of all $a \in P^{-\sigma}$ satisfying:

$$Q_a X = 0,\ Q_X a = 0,\ Q_a Q_X = V(a, X) = 0,\ and\ Q_X Q_a = V(X, a) = 0.$$

By [13], the annihilator of an ideal I of a non-degenerate Jordan algebra J has an easy expression:

$$ann_J(I) = \{a \in J : U_a I = 0\} \text{ and}$$

$$I \cap ann_J(I) = 0.$$

It is not difficult to check (see [12]) that the correspondence $X \longrightarrow ann(X)$ is a Galois one:

$$X \subset ann(ann(X)), \text{ and}$$

$$ann(ann(ann(X))) = ann(X).$$

By [7, Corollary 8], every non-degenerate Jordan algebra satisfying the descending chain condition dcc on principal inner ideals, equivalently, coinciding with its socle, also satisfies the ascending chain condition acc on annihilators of its principal inner ideals. This result is a consequence of the double annihilator property

$$ann(ann([x])) = [x]$$

of the elements of the socle of a Jordan algebra [7, Theorem 7], which also holds for elements of the socle of a non-degenerate Jordan pair containing an invertible element [7, Corollary 8]. However, in general socle elements of a non-degenerate Jordan pair do not have this property [7, Remark 10 (2)].

3. Main results

In this section we will give a characterization of non-degenerate Artinian Banach-Jordan pairs. First of all, we will specially be interested in a class of Jordan pairs, namely the pairs $A(M, R, ')$ and (J, J) for a Jordan algebra J.

Let begin by the pair $A(M, R, ')$. Consider a unital associative algebra R over a field 9, and $M = (M^+, M^-)$ a couple of 9-vector spaces such that M^+ is a left R-module and M^- is a right R-module, and $' : M^+ \times M^- \longrightarrow R$ is an R-bilinear form in the sense that

$$'(ax, yb) = a\,'(x, y)b,$$

for every $a, b \in R$, and $(x, y) \in M^+ \times M^-$.

The mapping $'$ is said to be non-degenerate if it satisfies the conditions:

$$'(M^+, y) = 0 \text{ implies } y = 0, \text{ and } '(x, M^-) = 0 \text{ implies } x = 0,$$

for every $x \in M^+$ and $y \in M^-$. The pair of modules (M^+, M^-) becomes an associative pair over 9, denoted by $A = A(M, R, ')$ with the triple products:

$$\begin{array}{llll} <.>_+: & M^+ \times M^- \times M^+ & \longrightarrow & M^+ \\ & (x, y, z) & \longrightarrow & <xyz> = '(x, y)z \end{array}$$

and

$$\begin{array}{llll} <.>_-: & M^- \times M^+ \times M^- & \longrightarrow & M^- \\ & (y, x, t) & \longrightarrow & <yxt> = t\,'(x, y) \end{array}$$

satisfying the identities

$$<uv<xyz>> = <u<yxv>z> = <<uvx>yz>,$$

for every $x, u, z \in M^+$ and $y, v \in M^-$.

$A(M, R, ')$ can be seen as a Jordan pair, denoted by: $P = A(M, R, ')^J$, with quadratic maps $Q^\sigma : P^\sigma \longrightarrow Hom_9(P^{-\sigma}, P^\sigma)$ defined by

$$Q^+_x y = <xyx> = '(x, y)x \text{ and } Q^-_y x = <yxy> = y\,'(x, y),$$

for every $(x, y) \in M^+ \times M^-$.

The Jordan pair $P = A(M, R, ')^J$ is non-degenerate if and only if so is $'$.

Lemma 3.1. *Get $P = A(M, R, ')^J$ be the complex Banach−Jordan pair, with R simple Artinian algebra and $'$ non−degenerate. If P satisfies the dcc on inner ideals then it is finite−dimensional.*

Proof. Since every inner ideal of the associative pair $A(M, R, ^*)$ is an inner ideal of the Jordan pair $P = A(M, R, ^*)^J$ then $A(M, R, ^*)$ has dcc on inner ideals. By means of Loos's result in [8, pp. 128] , we get:

$$P \cong (M_{pq}(D), M_{qp}(D^{op})),$$

where D is a complex division associative algebra.
Moreover, by D'Amour and MCcrimon [6, 4.1] ,

$$D \cong (P^{\sigma})_b.$$

where $(P^{\sigma})_b$ is the local algebra for certain rank one element b of $P^{-\sigma}$. Therefore, Mazur-Gelfand Theorem for Jordan algebras applies to deduce that D is isomorphic to the complex field, say

$$D \cong s.$$

This proves that P is finite-dimensional. ■

Let us consider the second class of Jordan pairs (J, J) where $J = 9 \; X$ also denoted by $J(X, f)$ is a Jordan algebra defined by a 9—vector space X and a symmetric bilinear form f. The binary product of $J(X, f)$ is defined as follows:

$$(\; + x)(\beta + y) = \; \beta + f(x, y) + \beta x + \; y,$$

for all $\; , \beta \in 9$ and $x, y \in X$.

Lemma 3.2. *Get $J = J(X, f)$ be the Banach−Jordan algebra determined by a continuous non−degenerate symmetric bilinear form f defined on a complex Banach vector space X. Suppose that J is Artinian then it is finite dimensional.*

Proof. Suppose that X is infinite dimensional, and let $\{e_n\} \subset X$ be a countable orthogonal system with respect to f . Let us take the sequence $(u_n)_n$ with

$$u_n = e_{2n} + e_{2n+1}.$$

It's easily seen that

$$f(u_n, u_m) = 0 \text{ for every } n, m \in \mathbb{N}^*.$$

Consider the principal inner ideals I_n ($n = 1, 2 ...$) defined as follows

$$I_n = Q_{u_1+\ldots+u_n}(J).$$

By the formula

$$\{x, \; + y, z\} = \; f(x, z) + f(x, y)z + f(y, z)x - f(z, x)y,$$

satisfied for every $x, y, z \in X$ and $\; \in s$, $(I_n)_n$ forms an increasing sequence of inner ideals of J. But $(ann(I_n))_n$ consists of a decreasing sequence of inner

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ideals and hence it is stationary since J is Artinian. That is, there exists a natural number n_0 such that

$$ann(I_{n_0}) = ann(I_n),$$

for every $n \geq n_0$. This clearly yields

$$ann(ann(I_{n_0})) = ann(ann(I_n)).$$

This double property of annihilator and the fact that the elements u_n lie in $soc(J)$ since J is Artinian, enable to have

$$I_{n_0} = I_n \text{ for every } n \geq n_0,$$

which is a contradiction. ■

The following Theorem [8] will be needed in our main result.

Theorem 3.3. [8, 10.14] . *Get P be a semi−simple Jordan pair mith dcc on principal inner ideals and containing a maximal idempotent e ($V_0(e) = 0$). fhen P is a finite direct sum of simple ideals mith the same properties.*

Now all ingredients are gathered to establish the main result in this paper.

Theorem 3.4. *Get $P = (P^+, P^-)$ be a non−degenerate Artinian complex Banach−Jordan pair. fhen P is finite dimensional.*

Proof. Suppose that $P = (P^+, P^-)$ be a non-degenerate Banach-Jordan pair satisfying dcc on inner ideals. Then, by [10], $P = Soc(P)$. That is, the Banach-Jordan pair P is von Neumann regular and hence it has finite capacity [11, Theorem 11]. Therefore, the results stated in the preliminary section enable us to conclude that there exists a strong frame in P. That is, P contains an orthogonal system $\{e_1, e_2, \dots, e_n\}$ of division idempotents ($V_2(e_i)$ is a division Jordan pair) such that

$$\cap_{i=1}^{n} V_0(e_i) = 0.$$

The element

$$e = e_1 + \dots + e_n$$

is a maximal idempotent since

$$V_0(e) = \cap_{i=1}^{n} V_0(e_i) = 0.$$

The pair P is semi-simple since it is a non-degenerate Artinian Jordan pair [8, 10.8] .

Now all conditions of Theorem 3.3 are satisfied, that is, P is a semi-simple Jordan pair verifying dcc on principal inner ideals and containing a maximal idempotent. Then P is

$$P = \quad _{1\leq i\leq n} P_i$$

a finite direct sum of simple ideals P_i satisfying the same properties of P, in particular $Soc(P_i) = P_i$. We claim that P_i is central. Indeed, for every $b \in P_i^-$, the operator Q_b is continuous, then $Ker(b) = KerQ_b$ is a closed ideal of the homothope $P_i^{+(b)}$ endowed with the norm

$$|x| = \|x\|_* \|b\|\_$$

and then the local algebra $(P_i^+)_b = P_i^{+(b)}/KerQ_b$ is a Banach-Jordan algebra with respect to the canonical quotient norm. Now $(P^+)_b$ is also a division algebra by Mazur-Gelfand Theorem for Jordan algebras. Therefore

$$(P_i^+)_b \text{ ˆ } s.$$

On the other hand,

$$(P_i^+)_b \text{ ˆ } Q_b P_i^+,$$

as vector spaces. Therefore b is a reduced element of P_i in the sense that

$$Q_b P_i^+ = s.b,$$

and in virtue of [14, 2.10], P_i is central.

By Loos's classical list of central Jordan pairs of finite capacity [8, 12.12], but as presented in [14, 5.9], in order to identify the centroids, P_i is a central simple non-degenerate Banach-Jordan pair with finite capacity since $P_i = Soc(P_i)$, as just pointed out. Consequently, the pair P_i is one of the following cases:

1) $A(M, R, ')^J$ for a central simple Artinian associative algebra R w $M_n(\mathrm{A})$ with A a central division associative s-algebra and ' non-degenerate.

2) $(A_n(s), A_n(s))$ the Jordan pair of complex alternating matrices $n \times n$ with $n \geq 5$.

3) $(H_n(D, ^*), H_n(D, ^*))$ the Jordan pair associated with the Jordan algebra $H_n(D, ^*)$ of hermitian matrices of order n defined on a complex associative division *-algebra D ($n \geq 5$).

4) (J, J) the Jordan pair associated with the Jordan algebra $J = J(X, f)$ determined by a non-degenerate symmetric bilinear form f defined on a complex Banach vector space X.

5) $(M_{1,2}(\ ), M_{1,2}(\ ))$ where is the Cayley-Dickson algebra defined on the complex field s.

6) (A, A) the Jordan pair associated with the Albert algebra A of 27-dimension over s.

It's clear that P_i has finite dimension provided that P_i is one of the cases 2), 5) or 6).

If P_i is in the case 1) then by Lemma 3.1, $A(M, R, f)$ has finite dimension.

If P_i is in the case 4) then $P_i = (J, J)$ with $J = J(X, f)$ is the Banach-Jordan algebra associated with a symmetric bilinear form on the Banach space X and hence, by Lemma 3.2, P_i is finite dimensional.

Now if P_i is in the case 3) then the special Jordan algebra $H_n(D, ^*)$ is isomorphic to the Jordan algebra A^+ where A is an associative Banach algebra. As P_i

is equal to its socle, therefore ($A^+ = soc(A^+) = (socA)^+$). To conclude, it suffices to recall Tullo's result [16] , asserting that every Banach algebra coinciding with its socle is finite dimensional. This completes the proof. ■

Remark 3.5. We can apply a similar proof proposed in case 3) to the Jordan pair associated with the Jordan algebra $J = J(X, f)$, since the algebra in question is special.

Corollary 3.6. *Get J be a non−degenerate Artinian complex Banach−Jordan algebra. fhen J has finite dimension.*

Proof. It is a direct consequence of Theorem 3.4, since every Artinian Banach-Jordan algebra J gives rise to the Artinian Banach-Jordan pair $P = (J, J)$. ■

Corollary 3.7. *Every non−degenerate Artinian complex associative or al− ternative Banach algebra A has finite dimension.*

Proof. It suffices to consider the Banach-Jordan pair $P = (A, A)^J$ where A is a non-degenerate Artinian Banach associative or alternative algebra. Every inner ideal of the associative or alternative pair (A, A) is an inner ideal of the Jordan pair $P = (A, A)^J$, and then the Banach-Jordan pair P is non-degenerate and Artinian if and only if so is A. Hence Theorem 3.4 applies to the Jordan pair $P = (A, A)^J$ to see that the Banach algebra A is finite dimensional.

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