IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-0661, p-ISSN: 2278-8727, Volume 20, Issue 6, Ser. 2 (Nov. – Dec. 2024), PP 01-05
www.iosrjournals.org

A Review Of Domination Strategies In Fuzzy Graphs

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Abstract:
This article presents a review of domination strategies in fuzzy graphs, with a focus on recent developments in the field. It summarizes outcomes from both foundational and recent developments in the literature. Numerous advancements have emerged in recent years, expanding its theoretical and practical applications. This review consolidates the most relevant and recent findings, providing an overview of current strategies and methodologies in domination theory for fuzzy graphs. The article also elaborate on the concept of fz-domination in fuzzy graphs. The fz-domination framework considers both vertex and edge membership values, ensuring a more realistic and effective measure of domination in fuzzy environments.
Keywords*: Fuzzy graphs; Dominating sets; domination number.*

Date of Submission: 01-12-2024 Date of Acceptance: 10-12-2024

I. Introduction

The first definition of a fuzzy graph was given by Kaufmann [1] in 1973, based on Zadeh's fuzzy relations [2]. However, it was Azriel Rosenfeld [3] in 1975 who developed the theory of fuzzy graphs by considering fuzzy relations on fuzzy sets. Rosenfeld [3] introduced the concept of fuzzy graphs as a powerful framework to address uncertainties in real-life situations. Fuzzy graphs provide a generalized structure where edges and vertices are characterized by membership functions, enabling the representation of partial or uncertain relationships. Rosenfeld explored several fuzzy analogues of classical graph-theoretic concepts, including paths, cycles, trees, and connectedness, laying the foundation for extensive research in this field. Over the years, numerous studies have expanded upon these ideas, applying fuzzy graphs to various domains such as social networks, transportation systems, communication networks, and decision-making processes.

Domination plays a significant role in designing cost-effective network monitoring systems. A dominating set in a graph represents a subset of nodes such that every other node in the graph is either in the subset or adjacent to at least one node in the subset. In network monitoring, these dominating nodes can represent locations where monitoring devices, such as sensors or routers, are installed. By minimizing the size of the dominating set, it becomes possible to monitor the entire network efficiently while reducing the number of devices needed, thus lowering costs. This is particularly useful in large-scale networks such as telecommunications, power grids, or transportation systems, where deploying monitoring devices at every node is impractical.

When dealing with real-world networks, relationships between nodes often involve uncertainty or imprecision, such as fluctuating signal strengths or probabilistic connections. Fuzzy graphs extend the concept of domination to such scenarios by allowing edges and vertices to have membership values that represent the degree of connection or presence. In this context, domination in fuzzy graphs helps identify the most influential or critical nodes under uncertain conditions. For instance, in a fuzzy communication network, dominating nodes can ensure effective monitoring or control even when connections are unreliable. By incorporating fuzzy graph theory, domination becomes a more realistic and applicable tool for optimizing network monitoring in uncertain and dynamic environments.

Domination in fuzzy graphs was first introduced by A. Somasundaram and S. Somasundaram [6]. They defined domination in fuzzy graphs using the concept of effective edge in a fuzzy graph. A. Nagoorgani and V.T Chandrasekaran [8] defined domination in fuzzy graphs using strong edges. In [9], O.T Manjusha and M.S. Sunitha defined strong domination in fuzzy graphs using membership values of strong edges.

In 2020, Akul Rana [4] published the article 'A Survey on the Domination of Fuzzy Graphs', providing a comprehensive overview of the results and techniques related to domination in fuzzy graphs up to that time. While this survey served as a significant reference for researchers in the field, numerous advancements have been made since then, particularly in developing new methodologies, refining existing concepts, and applying domination in fuzzy graphs to emerging real-world problems. This article builds upon Rana's work by focusing specifically on the recent developments in domination theory for fuzzy graphs. It reviews the progress made since 2020, highlighting novel approaches, extended applications, and open problems that have emerged in this rapidly evolving area.

II. Preliminaries

In this section, we review some fundamental definitions and notations related to fuzzy graphs. For detailed terminology, we refer the reader to Mordeson and Nair [5]. Let S be a finite non-empty set. A fuzzy subset of is defined as a mapping $\mu : S \to [0,1]$. Given two fuzzy subsets μ and ν of S, the following hold:
(i) $\mu \subseteq \nu$ *if* $\mu(x) \leq \nu(x)$ *for all* $x \in S$.
(ii) $\mu \subset \nu$ *if* $\mu(x) \leq \nu(x)$ *for all* $x \in S$ and there exists at least one $x \in S$ such that $\mu(x) < \nu(x)$.

A fuzzy graph $G = (V, \mu, \sigma)$ is a non empty set V together with a pair of functions $\mu : V \to [0,1]$ and $\sigma : V \times V :\to [0,1]$ such that for all $u, v \in V$, $\sigma(u,v) \leq \mu(u) \wedge \mu(v)$. μ is called fuzzy vertex set of G and σ is called the fuzzy edge set of G respectively.

The order p and size q of a fuzzy graph $G = (V, \mu, \sigma)$ are defined to be $p = \sum_{x \in V} \mu(x)$ and $q = \sum_{x \in V \times V} \sigma(x, y)$.
Let $S \subseteq V$. Then the scalar cardinality of S is defined as $\sum_{x \in S} \mu(x)$ and is denoted by |S|.

A path P of length n is a sequence of distinct vertices $u_{0,}\ u_{1,}, u_{2,}, \ldots, u_n$ such that $\sigma(u_{i-1}, u_i) > 0$ for $i = 1, 2, \ldots, n$. Strength of a path is the degree of membership of the weakest edge in P. The strength of connectedness between two vertices u and v is defined as the maximum of the strengths of all paths between u and v and is denoted by $CONN_G(u,v)$. A fuzzy graph $G = (\mu, \sigma)$ is connected if for every $u, v \in V$, $CONN_G(u,v) > 0$.

An edge $e = (u,v)$ of a fuzzy graph is called an effective edge if $\sigma(u,v) = \mu(u) \wedge \mu(v)$. Then u and v are called effective neighbors. The set of all effective neighbors of u is called effective neighbourhood of u and is denoted by EN(u). Bhutani and Rosenfeld introduced the concept of strong edges of a fuzzy graph. An edge (u,v) is strong if $\sigma(u,v) = CONN(u,v)$. If edge (u,v) is strong, then the vertex v is a strong neighbor of vertex u.

III. Domination In Fuzzy Graphs

Domination in fuzzy graphs was first introduced by A. Somasundaram and S. Somasundaram [6]. They defined domination in fuzzy graphs using the concept of effective edge in a fuzzy graph. According to A. Somasundaram and S. Somasundaram [6], u dominates v in G if σ(u, v) = μ(u) ∧ μ(v). Then v dominates u also. A subset S of V is called a dominating set in G if for every $v \notin S$, there exists u ∈ S such that u dominates v. The minimum scalar cardinality of a dominating set in G is called the domination number of G and is denoted by γ(G) or γ. They verified that certain properties of domination in crisp graphs also hold for fuzzy graphs. In their paper, they also defined concepts such as independent domination, total domination, independent domination number, and total domination number for fuzzy graphs. The effect of vertex removal and some other graph operations on the domination number of fuzzy graphs are studied in [7].

Gani and Chandrasekaran [8] defined that a vertex u dominates v, if (u,v) is a strong edge. A vertex u dominates itself and its strong neighbors. A set S of vertices of $G = (V, \mu, \sigma)$ is a strong dominating set if every vertex of $V(G) - S$ is a strong neighbor of some vertex in S. The minimum scalar cardinality taken over all strong dominating sets of a fuzzy graph G is the strong domination number. They presented various properties of dominating sets and determined the domination number for certain fuzzy graphs. Additionally, they established some bounds for the domination number in fuzzy graphs.

O.T Manjusha and M. S Sunitha [9] defined different types of edges, neighborhoods and neighborhood degree of a vertex in a fuzzy graph. In [10] they defined strong domination in fuzzy graphs using membership values of strong edges. The weight of a strong dominating set D is defined as $W(D) = \sum_{u \in D} m(u,v)$, where $m(u,v)$ is the minimum of the membership values of the strong edges incident on u. The strong domination number of the fuzzy graph G is the minimum weight of the strong dominating sets in G.

None of the previously mentioned papers address the concept of a fuzzy subset of the vertex set for domination in fuzzy graphs. However, Arumugam et al. in [11] introduced the concept of (r, s)-fuzzy domination in fuzzy graphs as a fuzzy subset of the vertex set.

Definition 3.1 [11] *Let $G = (V, \mu, \sigma)$ be fuzzy graph. If $r, s \in [0,1]$ and if $r < s$, then a fuzzy subset μ_1 of μ is called an (r, s) −fuzzy dominating set of $G = (V, \mu, \sigma)$ if for all $v \in V$,*

$$\sum_{\sigma(u,v) \geq r} \mu_1(u) + \mu_1(v) \geq s.$$

The authors also introduce the (r,s)-fuzzy domination number as the minimum sum of membership values of vertices in an (r,s)-fuzzy dominating set. The condition for minimal (r,s)-fuzzy dominating set of a fuzzy graph is given in the following theorem.

Theorem 3.2 [11] *Let μ_1 be an (r,s)- fuzzy dominating set of a fuzzy graph $G = (V, \mu, \sigma)$. Then μ_1 is a minimal (r,s)-fuzzy dominating set if and only if for every $v \in V$ with $\mu_1(v) > 0$ there exists $w \in V$ such that $\sigma(v,w) \geq r$ and $\sum_{\sigma(u,w)\geq r} \mu_1(u) + \mu_1(w) = s$.*

The condition in the above theorem leads to the definition of an (r,s) irredundant set in a fuzzy graph.

Definition 3.3 [11] *A fuzzy subset μ_1 of μ is called an $(r,s) -$ irredundant set of G if, for every $v \in V$ with $\mu_1(v) > 0$ there exists $w \in V$ such that $\sigma(v,w) \geq r$ and $\sum_{\sigma(u,w)\geq r} \mu_1(u) + \mu_1(w) = s$.*

The authors also defined an (r,s)-fuzzy independent set in a fuzzy graph.

Definition 3.4 [11] *A fuzzy subset μ_1 of μ is called an $(r,s) -$ independent set in G if, for every $v \in V$ with $\mu_1(v) \geq r > 0$, the following holds:*

$$\sum_{\sigma(u,v)\geq r} \mu_1(u) + \mu_1(v) = s$$

Furthermore, the authors established the (r, s)-fuzzy domination chain, which is analogous to the domination chain in crisp graphs.

IV. The Fz-Domination Framework: Definition, Benefits And Properties

In [12], the authors introduce a new concept of domination in fuzzy graphs, emphasizing that the domination of a vertex is influenced by its own membership value and the minimum of the membership value of an adjacent vertex and the edges connecting them. Consider a fuzzy graph where a vertex v has a membership value of 0.8, neighboring vertex u has a membership value of 0.9 and the edge between u and v has a membership value of 0.6. The contribution of u to the domination of v is the minimum of u's membership value (0.9) and the edge's membership value (0.6). Thus, u contributes 0.6 to the domination of v, as the edge limits the maximum influence u can transmit. The total domination of v is then a combination of its self-domination and the contributions from all adjacent vertices like u.

This approach ensures that domination in fuzzy graphs is realistically modeled, taking into account not only the strength of vertices but also the capacity of edges connecting them. A vertex cannot exert more influence than what the edge allows, reflecting real-world scenarios where transmission or interaction is constrained by link quality. This makes the definition robust and practical for applications involving uncertain or limited interactions.

Definition 4.1 [12] *Let $G = (V, \mu, \sigma)$ be fuzzy graph. A fuzzy subset μ' of μ is called a fuzzy dominating set or fz- dominating set of G if for every $v \in V$,*

$$\mu'(v) + \sum_{x\in V} (\sigma(x,v) \wedge \mu'(x)) \geq \mu(v).$$

A fuzzy subset μ' is minimal fz- dominating set if $\mu'' \subset \mu'$ is not a fuzzy dominating set.

Definition 4.2 [12] *Fuzzy domination number or fz- domination number of a graph G is defined as $\gamma_{fz}(G) = min\{|\mu| : \mu' \text{ is a minimal } fz - \text{dominating set of } G\}$.*

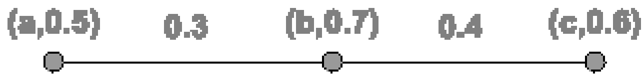

Fig. 1. Fuzzy graph, G.

For the fuzzy graph in Figure 1, $\mu_1 = \{(a, 0.2), (b, 0.3), (w, 0.3)\}$ is a minimal fz- dominating set of minimum scalar cardinality and $\gamma_{fz}(G) = 0.8$.

In the above definition, the domination contribution of an adjacent vertex is constrained by edge strength. A neighboring vertex can only contribute to the domination of another vertex up to the extent allowed by the strength of the edge connecting them. If the edge connecting two vertices has a low membership value, it restricts the amount of influence or "domination power" that can be transmitted, regardless of how strong the neighboring vertex itself is. Here the edge strength acts as a limiting factor. It acts like a pipeline through which domination is transmitted. If the "capacity" of this pipeline (the edge membership value) is low, even a very strong vertex cannot exert its full influence. This ensures that domination is not overestimated due to a high vertex membership value when the connecting edge is weak.

Ihe contribution from a neighboring vertex to the domination of a given vertex is calculated as the minimum of the edge's membership value and the neighboring vertex's membership value. This reflects the reality that influence is bounded by the weaker of the two components—the vertex's strength or the edge's capacity.

Definition 4.3 [12] For a fuzzy dominating set μ' of G, the boundary of μ' is defined as

$$B_{\mu'} = \left\{ v \in V : \mu'(v) + \sum_{x \in V} \big(\sigma(x, v) \wedge \mu'(x)\big) = \mu(v) \right\}$$

And the positive set of μ' is defined as

$$P_{\mu'} = \{v \in V : \mu'(v) > 0\}$$

Theorem 4.4 [12] *A fuzzy dominating set μ' of a fuzzy graph G is a minimal fuzzy dominating set if and only if each $v \in P_{\mu'}$ is either in $B_{\mu'}$ or there exists a vertex $u \in B_{\mu'}$ such that $\sigma(u, v) \geq \mu'(v)$.*

The above theorem can also be utilized to define fuzzy irredundant sets in fuzzy graphs. In [12], the authors further provide several bounds for the fz-domination number in terms of various graph parameters, including the order and size of the fuzzy graph, the maximum degree of vertices, the maximum membership values of vertices, and other related properties.

In [13], the authors explore various properties of the fz-domination number in fuzzy graphs as a result of different fuzzy graph operations. They establish bounds for the fz-domination number for certain graph products and investigate the conditions under which these bounds are attained with equality. Some important results in this article are listed below.

Theorem 4.5 [13] *For any two non- trivial fuzzy graphs G and H, the following results hold.*
1.$\gamma_{fz}(G \cup H) \leq \gamma_{fz}(G) + \gamma_{fz}(H)$
2.$\gamma_{fz}(G + H) \leq \max\,\{\gamma_{fz}(G), \gamma_{fz}(H)\}$

Conditions for equality and examples of graphs that attain these bounds are provided in the article. In general, the second result is true. However, examples of graphs for which $\gamma_{fz}(G + H) \leq \min\,\{\gamma_{fz}(G), \gamma_{fz}(H)\}$ are provided in the article. The article also provides several lower and upper bounds for $\gamma_{fz}(G + H)$ in terms of various graph parameters.

The authors establish the following results regarding the fz-domination number of the corona product:

1. The fz-domination number of the corona product satisfies $\gamma_{fz}(G \circ K_1) \geq \gamma_{fz}(G)$
2. $\gamma_{fz}(G \circ K_1) \leq \gamma_{fz}(G) + n\,\nu(u)$, where n is the number of vertices in G and $\nu(u)$ is the membership value of the vertex u in K_1.
3. If $\upsilon(u) \geq \mu(v)$ for all $v \epsilon V(G)$, then $\gamma_{fz}(G \circ K_1) = n\,\upsilon(u)$.

The authors establish the following results regarding the fz-domination number of the Cartesian product:

Theorem 4.6 [13] *For any two non-trivial fuzzy graphs G and H, with m and n representing the the number of vertices with non- zero membership values in G and H respectively, the following inequality holds:*

$$\gamma_{fz}(G \boxdot H) \leq \min\,\{n\,\gamma_{fz}(G), m\,\gamma_{fz}(H)\}.$$

V. Conclusion

This article presents a comprehensive survey of domination in fuzzy graphs, highlighting key definitions, results, and advancements in this domain. Special emphasis is given to the fz-domination framework, which addresses several real life situations by incorporating both vertex and edge membership values. Additionally, the properties of fz-domination under various fuzzy graph operations are discussed, along with bounds and conditions for sharpness. These contributions enhance the understanding of domination in fuzzy graphs and open opportunities for future research. Potential directions include studying fz-domination in dynamic fuzzy graphs, exploring its applications in network optimization, and extending it to other graph-theoretical concepts in fuzzy environments. This work is intended to serve as a valuable resource for researchers and inspire further exploration in this field.

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