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Analysis Of A Dual-Server Production Inventory System With Multiple Server Downtimes

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Abstract:
This article analyses an $M/M/2$ inventory system where production processes, inventory management, and server availability are integrated to optimise performance measures and the total expected costs. In this model, one server takes multiple vacations and the other server remains in the system even when the system is empty. The study compares how a heterogeneous and homogeneous system within a production inventory system affects efficiency and total system cost.

***Keywords**: Cost Analysis; Matrix-Analytic Method; Heterogeneous Servers; Homogeneous Server; Multiple Vacations.*

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I. Introduction

Server downtime may be due to server failure, lack of work or other tasks assigned to the servers. A diversified service system allows customers to receive different levels of service. If the servers are allowed to take vacation, this time can be used to perform some secondary tasks that can improve the profitability of an organization. The primary server provides a reliable backbone for the system, ensuring that critical processes are always running. The secondary server can be a backup during peak times, enhancing system reliability. This setup can be easily scaled by adding more vacationing servers, which can be activated or deactivated based on the production needs. This combination of servers allows for a balance between efficiency, and adaptability to changing workloads, making it a practical approach in many production inventory systems. Krishnamoorthy and Jose [5] discussed three production inventory systems with service time, loss and customer re-inquiry. Krishnamoorthy and Srinivasan [6] analysed an M/M/2 queuing system with heterogeneous servers including a vacationing server. Beena and Jose [2] discussed an M/M/2 production inventory system with multiple holidays. One server remains static even when the system is vacant, while the other considers multiple vacations. Junfang Yu and Yuanyuan Dong [1] discussed a numerical solution for a two-stage production and inventory system with random demand arrivals. For further discussion of server vacations, readers may refer to Doshi's survey paper [3]. Latouche and Ramaswami [7] derived a logarithmic reduction algorithm for quasi-birth-death processes. Sennott et al. [9] derived mean drifts and the non-ergodicity of Markov chains. Jose and Beena [4] studied a production inventory system with multiple server vacations and retrials of customers.

The rest of the paper is organized as follows. A description of the model is given in section 2. Mathematical modelling and analysis are given in section 3. In section 4, the stability condition and Steady-state probability vector are studied. Some performance measures are derived in section 5. Numerical experiments are given in section 6. The conclusion is given in section 7.

II. Description Of The Model

This article considers a production inventory system with two servers in which the secondary server takes multiple vacations and the primary server remains in the system even when the system is empty. Inter-arrival times of demand are according to the Poisson process with rate λ. This article performs a comparative analysis of system metrics and total cost estimation using heterogeneous and homogeneous servers. The secondary server goes on multiple vacations whenever no customer is waiting for service or no inventory is available. Two servers provide heterogeneous exponential service to customers with service rates μ_1 and μ_2 respectively. Then consider the case of homogeneous servers where the service rate is the average service rate of both servers. The period during which a secondary server is taken vacation is exponentially distributed with rate θ. Items are produced one unit at a time according to (s, S) policy. The system initiates production when the inventory level drops to s>0. Production stops when the inventory level is restored to S. The time required to produce a product in a manufacturing company is exponential with rate β.

III. Mathematical Modelling And Analysis

The following are the notations used in this model.
$N(t)$: Number of customers in the system at time t.
$I(t)$: Inventory level at time t.

$$C(t) = \begin{cases} 0 & if\ production\ is\ OFF \\ 1 & if\ production\ is\ ON \end{cases} ; \qquad J(t) = \begin{cases} 0 & if\ server\ 2\ is\ on\ vacation \\ 1 & if\ server2\ \ is\ busy \end{cases}$$

e: $(1,1,\dots\dots.1)'$, a column vector of 1's of appropriate dimension.

Then $\{N(t), C(t)\ , J(t), I(t)\}\ t \geq 0$, is a Level level-independent quasi-birth Death (LIQBD) process on the state space

$$\{(0,0,0,k), s+1 \leq k \leq S\} \cup \{(0,1,0,k), 0 \leq k \leq S-1\} \cup \begin{Bmatrix} (i,0,j,k), i \geq 1, j = 0,1 \\ s+1 \leq k \leq S \end{Bmatrix}$$
$$\cup \{(i,1,j,k),\ i \geq 1, j = 0,1, 0 \leq k \leq S-1\}$$

The rate matrix Q of this Markov chain is given by

$$Q = \begin{bmatrix} A_{00} & A_{01} & & & & \\ A_{10} & A_{11} & A_0 & & & \\ O & A_2 & A_1 & A_0 & & \\ & & A_2 & A_1 & A_0 & . \\ & & & & \cdot\cdot & \cdot\cdot \\ & & & & \cdot\cdot & \cdot\cdot \end{bmatrix}$$

A_{00}, A_{01}, A_{10}, A_{11} are block matrices of orders $(2S-s), (2S-s) \times (4S-2s), (4S-2s) \times (2S-s), (4S-2s)$ respectively and A_0, A_1, A_2 are square matrices of order $(4S-2s)$.

$$A_{01} = \begin{cases} \lambda & 1 \leq p \leq S-s, q = p \\ \lambda & S-s+1 \leq p \leq 2S-s, 2S-2s+1 \leq q \leq 3S-2s \\ 0 & otherwise \end{cases}$$

$$A_2 = \begin{cases} \mu_1 & 2 \leq p \leq S-s, q = p-1 \\ \mu_1 & p = 1, q = 2S-s+1 \\ \mu_1+\mu_2 & S-s+2 \leq p \leq 2S-2s, q = p-1 \\ \mu_1+\mu_2 & p = S-s+1, q = 3S-s+1 \\ \mu_1 & 2S-2s+2 \leq p \leq 3S-2s, q = p-1 \\ \mu_1+\mu_2 & 3S-2s+2 \leq p \leq 4S-2s, q = p-1 \\ 0 & otherwise \end{cases}$$

$$A_0 = (\lambda I)$$

$$A_{00} = \begin{cases} -\lambda & q = p, 1 \leq p \leq S-s \\ \beta & p = 2S-s, q = S-s \\ -(\lambda+\beta) & S-s+1 \leq p \leq 2S-s, q = p \\ \beta & S-s+1 \leq p \leq 2S-s-1, q = p+1 \\ 0 & otherwise \end{cases}$$

$$A_{11} = \begin{cases} -(\lambda+\mu_1) & 1 \leq p \leq S-s, q = p \\ -(\lambda+\mu_1+\mu_2) & S-s+1 \leq p \leq 2S-2s, q = p \\ -(\lambda+\beta) & p = 2S-2s+1, q = p \\ -(\lambda+\beta+\mu_1) & 2S-2s+2 \leq p \leq 3S-2s, q = p \\ \beta & 2S-2s+1 \leq p \leq 3S-2s-1, q = p+1 \\ -(\lambda+\beta) & p = 3S-2s+1, q = p \\ \beta & p = 3S-2s, q = S-s \\ -(\lambda+\beta+\mu_1+\mu_2) & 3S-2s+2 \leq p \leq 4S-2s, q = p \\ \beta & 3S-2s+1 \leq p \leq 4S-2s-1, q = p+1 \\ \beta & p = 4S-2s, q-2S-2s \\ 0 & otherwise \end{cases}$$

$$A_1 = \begin{cases} -(\lambda+\mu_1+\theta) & 1 \le p \le S-s, q = p \\ \theta & 1 \le p \le S-s, S-s+1 \le q \le 2S-2s \\ -(\lambda+\mu_1+\mu_2) & S-s+1 \le p \le 2S-2s, q = p \\ -(\lambda+\beta+\theta) & p = 2S-2s+1, q = p \\ \beta & 2S-2s+1 \le p \le 3S-2s-1, q = p+1 \\ \theta & 2S-2s+1 \le p \le 3S-2s, q = S+p \\ \beta & p = 3S-2s, q = S-s \\ \beta & p = 4S-2s, q = 2S-2s \\ -(\lambda+\beta+\theta+\mu_1) & 2S-2s+2 \le p \le 3S-2s, q = p \\ -(\lambda+\beta) & p = 3S-2s+1, q = p \\ -(\lambda+\beta+\mu_1+\mu_2) & 3S-2s+2 \le p \le 4S-2s, q = p \\ \beta & 3S-2s+1 \le p \le 4S-2s-1, q = p+1 \\ 0 & otherwise \end{cases}$$

$$A_{10} = \begin{cases} \mu_1 & 2 \le p \le S-s, q = p-1 \\ \mu_1 & p = 1, q = S+1 \\ \mu_1+\mu_2 & S-s+2 \le p \le 2S-2s, 1 \le q \le S-s \\ \mu_1+\mu_2 & p = S-s+1, q = S+1 \\ \mu_1 & 2S-2s+2 \le p \le 3S-2s, S-s+1 \le q \le 2S-s-1 \\ \mu_1+\mu_2 & 3S-2s+2 \le p \le 4S-2s, S-s+1 \le q \le 2S-s-1 \\ 0 & otherwise \end{cases}$$

IV. System Stability And Steady-State Probability Vector

For the Markov chain to be stable, it is necessary and sufficient that $\boldsymbol{\pi} A_0\, \boldsymbol{e} < \boldsymbol{\pi} A_2 \boldsymbol{e}$, where $\boldsymbol{\pi}$ is the unique stationary vector satisfying $\boldsymbol{\pi} A = 0$ *and* $\boldsymbol{\pi e} = 1$, where $A = A_0 + A_1 + A_2$.

Let $\boldsymbol{\pi} = (\pi_0, \pi_1, \dots\dots)$ be the steady-state probability vector of Q. The objective is to find out the stationary probability vector $\boldsymbol{\pi}$ from the system of equations $\boldsymbol{\pi}\, Q = 0$. Neuts [8] introduced a powerful technique, the Matrix Geometric Method for finding the steady-state distribution of Markov chains. The matrix, R satisfies

$\pi_i = \pi_{i-1} * R, for\ i = 2,3,4 \dots\dots$. R can be computed from $R^2A_2 + RA_1 + A_0 = 0$. The rate matrix R is given by

$R = -A_0(A_1)^{-1} - R^2A_2(A_1)^{-1}$. This leads to the successive substitution procedure proposed by Neuts [8] namely $R_0 = 0$, $R_{k+1} = -A_0(A_1)^{-1} - {R_k}^2 A_2\,(A_1)^{-1}, k = 0,1,2,3 \dots\dots$. The process is halted once successive differences are less than a specified tolerance criterion.

From $\boldsymbol{\pi} Q = 0$

$$\left.\begin{aligned} \pi_0 A_{00} + \pi_1 A_{10} &= 0 \\ \pi_0 A_{01} + \pi_1 A_{11} + \pi_2 A_2 &= 0 \end{aligned}\right\} \quad (1)$$

Replacing π_2 with $\pi_1 R$, we get the homogeneous equations

$$\left.\begin{aligned} \pi_0 A_{00} + \pi_1 A_{10} &= 0 \\ \pi_0 A_{01} + \pi_1 [A_{11} + RA_2] &= 0 \end{aligned}\right\} \quad (2)$$

$$\pi_i = \pi_{i-1} * R, for\ i = 2,3,4 \dots\dots\dots \quad (3)$$

The normalizing equation is

$$\pi_0 e + \{\pi_1[(I-R)^{-1}]\}\boldsymbol{e} = 1 \quad (4)$$

The boundary probabilities π_0, π_1, and the probabilities π_i, $for\ i \ge 2$ can be obtained using equations (2), (3) and (4).

V. System Performance Measures

Let $\boldsymbol{\pi}$ be the steady-state probability vector of Q. $\boldsymbol{\pi}$ can be partitioned as $\boldsymbol{\pi} = (\phi_0, \phi_1, \phi_2, \dots)$, where

$\phi_0 = \{(y_{0,0,0,s+1,\dots}\ y_{0,0,0,S}), (y_{0,1,0,0,\dots}\ y_{0,1,0,S-1})\}$

$\phi_i = \{(y_{i,0,0,s+1,\dots}\ y_{i,0,0,S}), (y_{i,0,1,s+1,\dots}\ y_{i,0,1,S}), (y_{i,1,0,0,} \dots\dots y_{i,1,0,S-1})$
$(y_{i,1,1,0} \dots\dots y_{i,1,1,S-1})\}, for\ i \ge 1$

Now,

1) Expected Inventory level, EI, in the system is

$$EI = \sum_{i=0}^{\infty} \sum_{k=s+1}^{S} \sum_{j=0}^{1} k\, y_{i,0,j,k} + \sum_{i=1}^{\infty} \sum_{k=1}^{S-1} \sum_{J=0}^{1} k\, y_{i,1,j,k}$$

2) Expected number of customers, EC, in the system is

$$EC = (\sum_{i=0}^{\infty} i\phi_i)e = \phi_1 e + \{\phi_1 R[(I-R)^{-1} + (I-R)^{-2}]\}e$$

3) Expected reorder rate, ERO, is

$$ERO = \mu_1 \sum_{i=1}^{\infty} y_{i,0,0,s+1} + (\mu_1 + \mu_2) \sum_{i=1}^{\infty} y_{i,0,1,s+1}$$

4) Fraction of time the production process is ON

$$EON = \sum_{i=0}^{\infty} \sum_{k=0}^{S-1} y_{i,0,1,k} + \sum_{i=2}^{\infty} \sum_{k=1}^{S-1} y_{i,1,1,k}$$

5) Expected number of departures after completing service, EDS, is

$$EDS = \mu_1 \sum_{i=1}^{\infty} \sum_{k=s+1}^{S} y_{i,0,0,k} + \mu_1 \sum_{i=1}^{\infty} \sum_{k=0}^{S-1} y_{i,1,0,k} + (\mu_1 + \mu_2) \sum_{i=1}^{\infty} \sum_{k=s+1}^{S} y_{i,0,1,k}$$

$$+(\mu_1 + \mu_2) \sum_{i=1}^{\infty} \sum_{k=0}^{S-1} y_{i,1,1,k}$$

VI. Numerical Experiments

To construct a cost function, define the following cost

$$C = Fixed\ cost$$
$$c_1 = Production\ cost\ per\ unit\ per\ unit\ time$$
$$c_2 = Holding\ cost\ of\ inventory\ per\ unit\ per\ unit\ time$$
$$c_3 = Holding\ cost\ of\ customers\ per\ unit\ per\ unit\ time$$
$$c_4 = Cost\ due\ to\ service\ per\ unit\ per\ unit\ time$$

The expected total cost (ETC) of the system per unit per unit time is defined as

$$\mathrm{EXTC} = (\mathrm{C} + (\mathrm{S} - \mathrm{s})\mathrm{c}_1)\mathrm{ERO} + \mathrm{c}_2\ \mathrm{EI} + \mathrm{c}_3\ \mathrm{EC} + \mathrm{c}_4\mathrm{EDS}$$

Some numerical examples are given to analyze different system metrics and expected total costs under homogeneous and heterogeneous systems using different parameters.

Table 1: Effect of λ on various performance measures and EXTC using heterogeneous servers ;
$C = 450,\ c_1 = 50, c_2 = 3.6, c_3 = 1.5,\ c_4 = 1, S = 20, s = 5, \mu_1 = 3.6, \mu_2 = 2.2, \beta = 2.1, \theta = 1.5.$

	Heterogeneous Servers					
λ	EI	EC	ERO	EDS	EP	EXTC
0.2	12.8947	0.0553	0.0122	0.1993	0.0952	61.3754
0.3	12.8333	0.0829	0.0175	0.2976	0.1429	67.6080
0.4	12.7647	0.1107	0.0222	0.3944	0.1905	73.1092
0.5	12.6874	0.1386	0.0262	0.4893	0.2381	77.8570
0.6	12.5999	0.1668	0.0297	0.5818	0.2857	81.8290
0.7	12.4999	0.1955	0.0325	0.6716	0.3333	85.0013
0.8	12.3845	0.2248	0.0347	0.7583	0.3810	87.3484
0.9	12.2502	0.2550	0.0363	0.8417	0.4286	88.8423

Table 1I; Effect of λ on various performance measures and EXTC using Homogeneous servers ;
$C = 450,\ c_1 = 50, c_2 = 3.6, c_3 = 1.5,\ c_4 = 1, S = 20, s = 5, \mu_1 = \mu_2 = 2.9, \beta = 2.1, \theta = 1.5.$

	Homogeneous Servers					
λ	EI	EC	ERO	EDS	EP	EXTC
0.2	12.8947	0.0683	0.0123	0.1992	0.0952	61.3754
0.3	12.8333	0.1022	0.0176	0.2973	0.1429	67.6080
0.4	12.7646	0.1361	0.0224	0.3937	0.1905	73.1092
0.5	12.6873	0.1702	0.0266	0.4880	0.2381	77.8570
0.6	12.5997	0.2046	0.0301	0.5797	0.2857	81.8290
0.7	12.4996	0.2395	0.0331	0.6686	0.3333	85.0013
0.8	12.3841	0.2750	0.0354	0.7543	0.3810	87.3484
0.9	12.2945	0.3116	0.0370	0.8366	0.4286	88.8423

Comparing the performance metrics and overall cost under homogeneous and heterogeneous systems, it can be observed that the EXTC are lower in heterogeneous servers for these parameter values.

VII. Conclusion

This paper discusses a production inventory system with two servers and multiple vacations. The advantage of using primary and secondary servers is the primary server maintains the essential operations. In contrast, the secondary server reduces unnecessary costs during periods of low activity. The model was analysed using the Matrix-Analytic Method. For an extension of the present work, one may consider a multi-server production inventory system with the arrival process as MAP (Markovian Arrival Process), service time as Phase-type distributed and production process as MPP (Markovian Production Process).