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Hilbert Graceful Labeling on Complete Bipartite Graph with Pendant Edges – Paper I

Dr. J. Suresh Kumar
Department of Mathematics,
St. Thomas College of Arts and Science, Koyambedu, Chennai, Tamil Nadu, India.
jskumar.robo@gmail.com

Abstract:
Let G be a simple, finite, connected, undirected, non-trivial graph with p vertices and q edges. $V(G)$ be the vertex set and $E(G)$ be the edge set of G. The n^{th} Hilbert number is denoted by H_n and is defined by $H_n = 4(n-1)+1$ where $n \geq 1$. A Hilbert graceful labeling is an injective function $\mathcal{H}$ from the vertex set $V(G)$ to a set of Hilbert number $\{x : x = 4(i-1)+1, 1 \leq i \leq 2q\}$ which induces a bijective function $\mathcal{H}^*$ from the set $E(G)$ to the set of number $\{1, 2, 3, 4, \ldots, q\}$, where for each edge $uv \in E(G)$ with $u, v \in V(G)$ applies $\mathcal{H}^*(uv) = \frac{1}{4}|\mathcal{H}(u) - \mathcal{H}(v)|$. A graph with Hilbert graceful labeling is called a Hilbert graceful graph. This research aims to construct some new graphs and prove that graphs are Hilbert graceful.
Keywords: *Hilbert numbers, Graceful labeling, Hilbert graceful labeling, Hilbert graceful graph.*

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I. Introduction

A graph G consists of a finite set of vertices $V(G)$ and a set of edges c consisting of distinct, unordered pairs of vertices 1]. The graph discussed in this paper is a simple, undirected, and finite graph. $|V(G)|$ represents the number of vertices on graph G, and the number of edges on graph G is represented by $|V(G)|$. Graph labeling has been studied since the 60s. Graph labeling is a branch of graph theory that continues to develop. Labeling on a graph is the assignment of an integer value to the elements of the graph, usually a positive integer. Alex Rosa first discovered graceful labeling in 1967 2]. Since this discovery, many researchers have been interested in looking for graceful labeling constructs and their variations. Several graphs with graceful labeling include tree graphs with vertices less or equal to 35, circle graph C_n for $n = 0(mod\ 4)$ or $n = 3(mod\ 4)$, and wheel graph W_n. Another class of graphs known to have graceful labeling can be seen in the survey conducted by Gallian 3]. The following shows some relevant research: graceful labeling of paths 4], graceful labeling of pendant graphs 5, 6, 7], vertex graceful labeling of caterpillar graphs 8], graceful labeling on torch graph 9], counting graceful labelings of trees 10], and other results on super graceful labeling of graphs 11]. Motivated by the above articles, the new type of graceful labeling called Hilbert graceful labeling is introduced in this paper and Hilbert graceful labeling of some complete bipartite graph is studied.

II. Definition

Defintion 2.1: A graph obtained by attaching l pendant edges to the vertex u_1 of the complete bipartite graph $K_{m,n}$ with the vertex set $\{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\}$ is denoted by $K_{m,n} \odot u_1(l)$.

Defintion 2.2: A graph obtained by attaching l_1 pendant edges to the vertex u_1 and attaching l_2 pendant edges to the vertex u_m of the complete bipartite graph $K_{m,n}$ with the vertex set $\{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\}$ is denoted by $K_{m,n} \odot u_{1,m}(l_1, l_2)$.

Defintion 2.3: A graph obtained by attaching l_1 , l_2 and l_3 pendant edges to the vertex u_1 , u_m and v_1 of the complete bipartite graph $K_{m,n}$ with the vertex set $\{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\}$ is denoted by $K_{m,n} \odot u_{1,m}v_1(l_1, l_2, l_3)$.

Defintion 2.4: A graph obtained by attaching l_1 , l_2 , l_3 and l_4 pendant edges to the vertex u_1 , u_m , v_1 and v_n of the complete bipartite graph $K_{m,n}$ with the vertex set $\{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\}$ is denoted by $K_{m,n} \odot u_{1,m}v_{1,n}(l_1, l_2, l_3, l_4)$.

III. Main Result

Theorem 3.1: The graph $K_{m,n} \odot u_1(l)$ admits Hilbert graceful labeling.
Proof: Let G be a $K_{m,n} \odot u_1(l)$ graph.
$V(G) = \{u_i : 1 \le i \le m\} \cup \{v_j : 1 \le j \le n\} \cup \{x_k : 1 \le k \le l\}$ and
$E(G) = \{u_i\, v_j : 1 \le i \le m;\ 1 \le j \le n\} \cup \{u_1\, x_k : 1 \le k \le l\}$.

$$|V(G)| = m + n + l$$
$$|E(G)| = m\,n + l$$

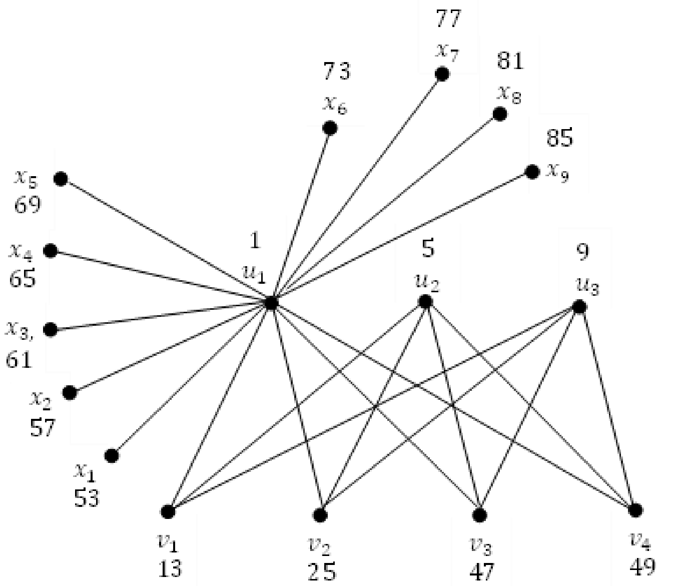
We define a function $\mathcal{H} : V(G) \to \{x : x = 4(i-1) + 1, 1 \le i \le 2q\}$
The vertex labeling is as follows:
$\mathcal{H}(u_i) = 4[i-1] + 1 \qquad 1 \le i \le m$
$\mathcal{H}(v_j) = 4[m\,j] + 1 \qquad 1 \le j \le n$
$\mathcal{H}(x_k) = 4[m\,n + k] + 1 \qquad 1 \le k \le l$
By above labeling pattern, we observed that function
$\mathcal{H} : V(G) \to \{x : x = 4(i-1) + 1, 1 \le i \le 2q\}$ is $1-1$.
From the induced function $\mathcal{H}^* : E(G) \to \{1, 2, 3, 4, \dots, q\}$, we get the edge labels as follows.

Table 1: Edge labels of the graph $k_{m,n} \odot u_1(l)$

$\mathcal{H}^*$	Edge Labels	Value of i, j and k
$\lvert\mathcal{H}(u_i) - \mathcal{H}(v_j)\rvert$	$[m\,j - i + 1]$	$1 \le j \le n, 1 \le i \le m$
$\lvert\mathcal{H}(u_1) - \mathcal{H}(x_k)\rvert$	$[m\,n + k]$	$1 \le k \le l$

From the above table 1, we observe that $\mathcal{H}^* : E(G) \to \{1, 2, 3, 4, \dots, q\}$ defined by $\mathcal{H}^*(uv) = \frac{1}{4}|\mathcal{H}(u) - \mathcal{H}(v)|$ is a bijective. Hence, $\mathcal{H}$ is Hilbert graceful labeling and the graph $K_{m,n} \odot u_1(l)$ is Hilbert graceful graph.

Example 3.1: Hilbert graceful labeling of the graph $K_{3,4} \odot u_1(9)$



Theorem 3.2: The graph $K_{m,n} \odot u_{1,m}(l_1, l_2)$ admits Hilbert graceful labeling.
Proof: Let G be a $K_{m,n} \odot u_{1,m}(l_1, l_2)$ graph.
$V(G) = \{u_i : 1 \le i \le m\} \cup \{v_j : 1 \le j \le n\} \cup \{x_k : 1 \le k \le l_1\} \cup \{y_k : 1 \le k \le l_2\}$ and
$E(G) = \{u_i\, v_j : 1 \le i \le m;\ 1 \le j \le n\} \cup \{u_1\, x_k : 1 \le k \le l_1\} \cup \{u_m\, y_k : 1 \le k \le l_2\}$.
$|V(G)| = m + n + l_1 + l_2$

$|E(G)| = m\,n + l_1 + l_2$
We define a function $\mathcal{H} : V(G) \rightarrow \{x : x = 4(i-1)+1, 1 \le i \le 2q\}$
The vertex labeling is as follows:
$\mathcal{H}(u_i) = 4[i-1]+1 \qquad 1 \le i \le m$
$\mathcal{H}(v_j) = 4[m\,j]+1 \qquad 1 \le j \le n$
$\mathcal{H}(x_k) = 4[m\,n+k]+1 \qquad 1 \le k \le l_1$
$\mathcal{H}(y_k) = 4[m\,n+l_1+m-1+k]+1 \qquad 1 \le k \le l_2$

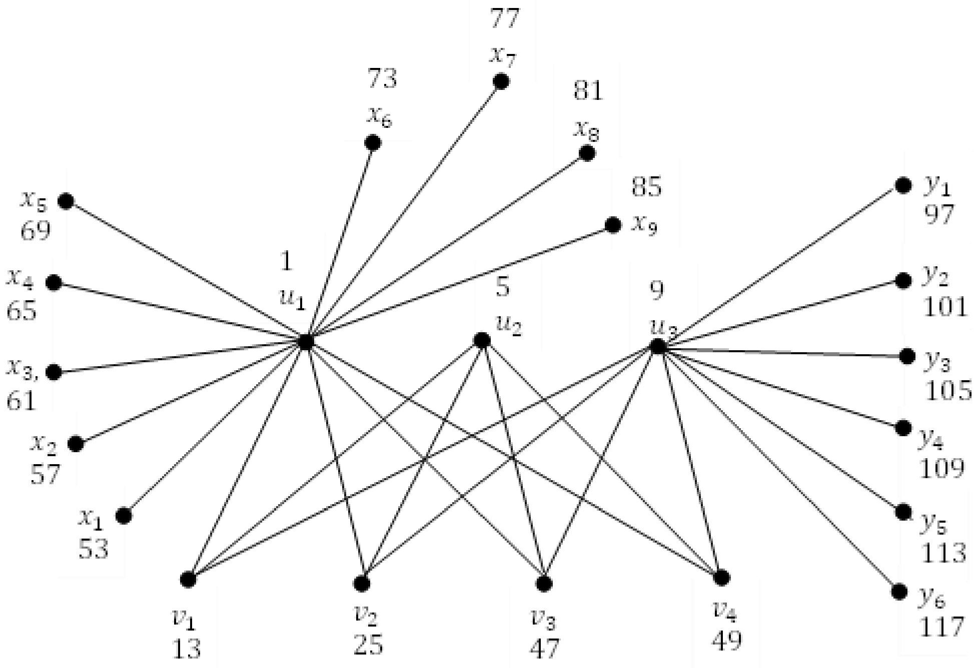
By above labeling pattern, we observed that function
$\mathcal{H} : V(G) \rightarrow \{x : x = 4(i-1)+1, 1 \le i \le 2q\}$ is $1-1$.
From the induced function $\mathcal{H}^* : E(G) \rightarrow \{1,2,3,4,\ldots,q\}$, we get the edge labels as follows.

Table 2: Edge labels of the graph $K_{m,n} \odot u_{1,m}(l_1, l_2)$

$\mathcal{H}^*$	Edge Labels	Value of i, j and k
$\lvert\mathcal{H}(u_i) - \mathcal{H}(v_j)\rvert$	$[m\,j - i + 1]$	$1 \le j \le n, 1 \le i \le m$
$\lvert\mathcal{H}(u_1) - \mathcal{H}(x_k)\rvert$	$[m\,n + k]$	$1 \le k \le l_1$
$\lvert\mathcal{H}(u_m) - \mathcal{H}(y_k)\rvert$	$[m\,n + l_1 + k]$	$1 \le k \le l_2$

From the above table 2, we observe that $\mathcal{H}^* : E(G) \rightarrow \{1,2,3,4,\ldots,q\}$ defined by $\mathcal{H}^*(uv) = \frac{1}{4}|\mathcal{H}(u) - \mathcal{H}(v)|$ is a bijective. Hence, $\mathcal{H}$ is Hilbert graceful labeling and the graph $K_{m,n} \odot u_{1,m}(l_1, l_2)$ is Hilbert graceful graph.

Example 3.2: Hilbert graceful labeling of the graph $K_{3,4} \odot u_{1,3}(9,5)$



Theorem 3.3: The graph $K_{m,n} \odot u_{1,m} v_1(l_1, l_2, l_3)$ admits Hilbert graceful labeling.
Proof: Let G be a $K_{m,n} \odot u_{1,m} v_1(l_1, l_2, l_3)$ graph.
$V(G) = \{u_i : 1 \le i \le m\} \cup \{v_j : 1 \le j \le n\} \cup \{x_k : 1 \le k \le l_1\} \cup \{y_k : 1 \le k \le l_2\}$
$\cup \{z_k : 1 \le k \le l_3\}$ and
$E(G) = \{u_i\, v_j : 1 \le i \le m;\ 1 \le j \le n\} \cup \{u_1\, x_k : 1 \le k \le l_1\} \cup \{u_m\, y_k : 1 \le k \le l_2\}$
$\cup \{v_1\, y_k : 1 \le k \le l_3\}$..
$|V(G)| = m + n + l_1 + l_2 + l_3$ and $|E(G)| = m\,n + l_1 + l_2 + l_3$
We define a function $\mathcal{H} : V(G) \rightarrow \{x : x = 4(i-1)+1, 1 \le i \le 2q\}$

The vertex labeling is as follows:

$\mathcal{H}(u_i) = 4[i-1]+1 \qquad 1 \le i \le m$

$\mathcal{H}(v_j) = 4[m\,j]+1 \qquad 1 \le j \le n$

$\mathcal{H}(x_k) = 4[m\,n+k]+1 \qquad 1 \le k \le l_1$

$\mathcal{H}(y_k) = 4[m\,n+l_1+m-1+k]+1 \qquad 1 \le k \le l_2$

$\mathcal{H}(z_k) = 4[m\,n+l_1+l_2+m+k]+1 \qquad 1 \le k \le l_3$

By above labeling pattern, we observed that function

$\mathcal{H} : V(G) \rightarrow \{x : x = 4(i-1)+1, 1 \le i \le 2q\}$ is $1-1$.

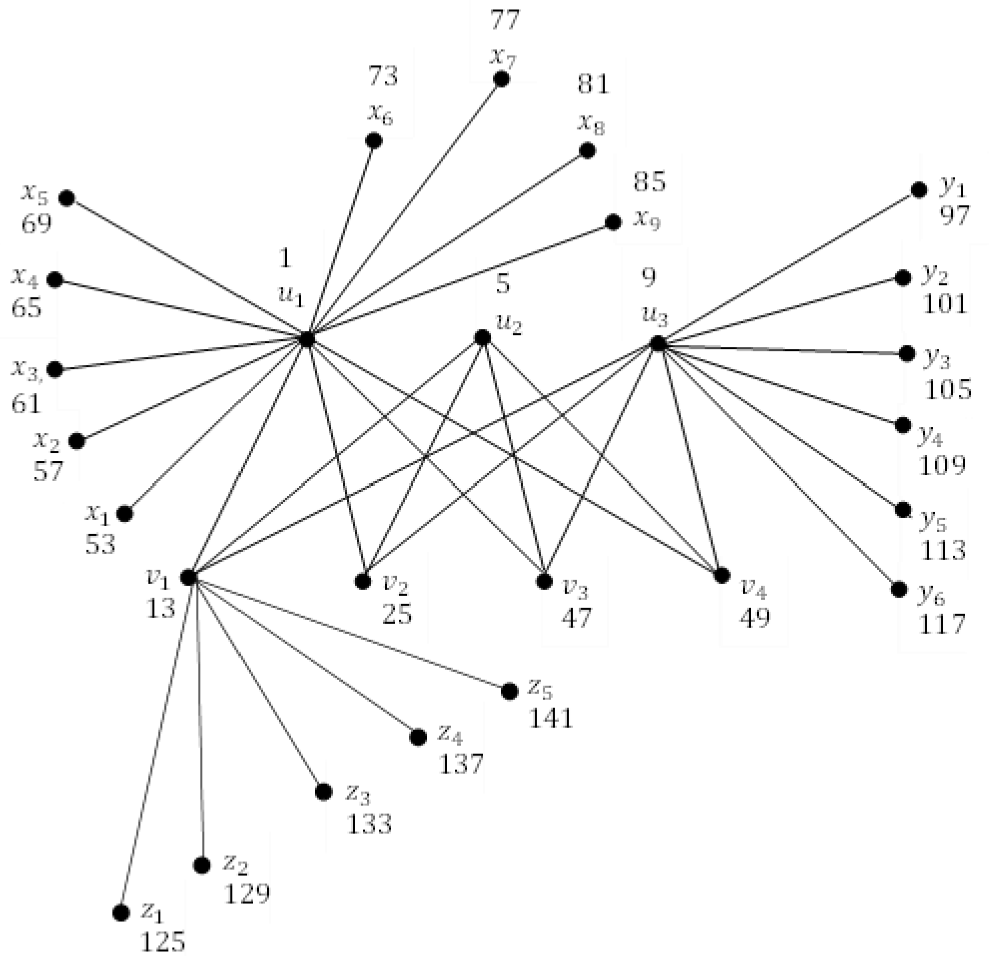
From the induced function $\mathcal{H}^* : E(G) \rightarrow \{1, 2, 3, 4, \dots, q\}$, we get the edge labels as follows.

Table 3: Edge labels of the graph $K_{m,n} \odot u_{1,m}v_1(l_1, l_2, l_3)$.

H^*	Edge Labels	Value of i, j and k
$\lvert\mathcal{H}(u_i) - \mathcal{H}(v_j)\rvert$	$[mj-i+1]$	$1 \le j \le n, 1 \le i \le m$
$\lvert\mathcal{H}(u_1) - \mathcal{H}(x_k)\rvert$	$[m\,n+k]$	$1 \le k \le l_1$
$\lvert\mathcal{H}(u_m) - \mathcal{H}(y_k)\rvert$	$[m\,n+l_1+k]$	$1 \le k \le l_2$
$\lvert H(v_1) - H(z_k)\rvert$	$[m\,n+l_1+l_2+k]$	$1 \le k \le l_3$

From the above table 3, we observe that $\mathcal{H}^* : E(G) \rightarrow \{1, 2, 3, 4, \dots, q\}$ defined by $\mathcal{H}^*(uv) = \frac{1}{4}\lvert\mathcal{H}(u) - \mathcal{H}(v)\rvert$ is a bijective. Hence, $\mathcal{H}$ is Hilbert graceful labeling and the graph $K_{m,n} \odot u_{1,m}v_1(l_1, l_2, l_3)$ is Hilbert graceful graph.

Example 3.3: Hilbert graceful labeling of the graph $K_{3,4} \odot u_{1,3}v_1(9, 6, 5)$.



Theorem 3.4: The graph $K_{m,n} \odot u_{1,m}v_{1,n}(l_1, l_2, l_3, l_4)$ admits Hilbert graceful labeling.

Proof: Let G be a $K_{m,n} \odot u_{1,m} v_{1,n}(l_1, l_2, l_3, l_4)$ graph.

$V(G) = \{u_i : 1 \le i \le m\} \cup \{v_j : 1 \le j \le n\} \cup \{x_k : 1 \le k \le l_1\} \cup \{y_k : 1 \le k \le l_2\}$
$\cup \{z_k : 1 \le k \le l_3\} \cup \{w_k : 1 \le k \le l_4\}$ and

$E(G) = \{u_i\, v_j : 1 \le i \le m;\ 1 \le j \le n\} \cup \{u_1\, x_k : 1 \le k \le l_1\} \cup \{u_m\, y_k : 1 \le k \le l_2\}$
$\cup \{v_1\, y_k : 1 \le k \le l_3\} \cup \{v_n\, w_k : 1 \le k \le l_4\}$.

$|V(G)| = m + n + l_1 + l_2 + l_3$ and $|E(G)| = m\,n + l_1 + l_2 + l_3$

We define a function $\mathcal{H} : V(G) \to \{x : x = 4(i-1) + 1, 1 \le i \le 2q\}$

The vertex labeling is as follows:

$\mathcal{H}(u_i) = 4[i-1] + 1$ $\quad 1 \le i \le m$

$\mathcal{H}(v_j) = 4[m\,j] + 1$ $\quad 1 \le j \le n$

$\mathcal{H}(x_k) = 4[m\,n + k] + 1$ $\quad 1 \le k \le l_1$

$\mathcal{H}(y_k) = 4[m\,n + l_1 + m - 1 + k] + 1$ $\quad 1 \le k \le l_2$

$\mathcal{H}(z_k) = 4[m\,n + l_1 + l_2 + m + k] + 1$ $\quad 1 \le k \le l_3$

$\mathcal{H}(w_k) = 4[2m\,n + l_1 + l_2 + l_3 + k] + 1$ $\quad 1 \le k \le l_4$

By above labeling pattern, we observed that function
$\mathcal{H} : V(G) \to \{x : x = 4(i-1) + 1, 1 \le i \le 2q\}$ is $1-1$.

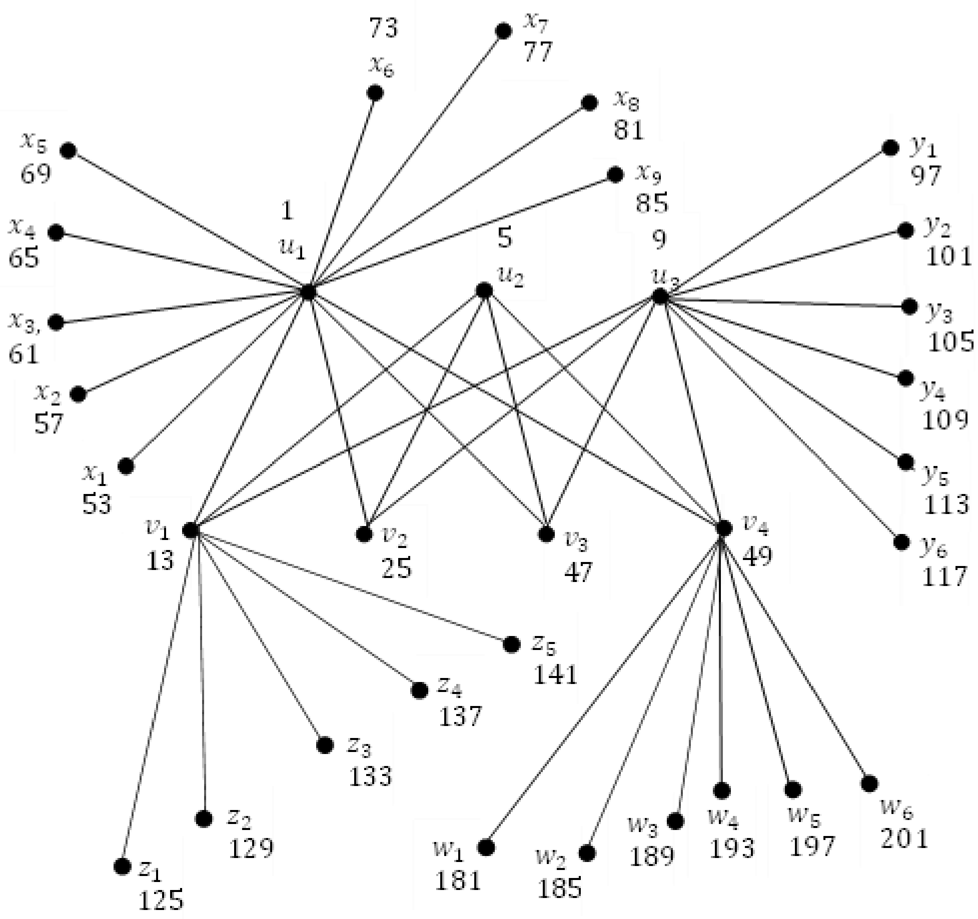
From the induced function $\mathcal{H}^* : E(G) \to \{1, 2, 3, 4, \dots, q\}$, we get the edge labels as follows.

Table 4: Edge labels of the graph $K_{m,n} \odot u_{1,m} v_{1,n}(l_1, l_2, l_3, l_4)$.

$\mathcal{H}^*$	Edge Labels	Value of i, j and k
$\lvert\mathcal{H}(u_i) - \mathcal{H}(v_j)\rvert$	$[m\,j - i + 1]$	$1 \le j \le n, 1 \le i \le m$
$\lvert\mathcal{H}(u_1) - \mathcal{H}(x_k)\rvert$	$[m\,n + k]$	$1 \le k \le l_1$
$\lvert\mathcal{H}(u_m) - \mathcal{H}(y_k)\rvert$	$[m\,n + l_1 + k]$	$1 \le k \le l_2$
$\lvert\mathcal{H}(v_1) - \mathcal{H}(z_k)\rvert$	$[m\,n + l_1 + l_2 + k]$	$1 \le k \le l_3$
$\lvert\mathcal{H}(v_n) - \mathcal{H}(w_k)\rvert$	$[m\,n + l_1 + l_2 + l_3 + k]$	$1 \le k \le l_4$

From the above table 4, we observe that $\mathcal{H}^* : E(G) \to \{1, 2, 3, 4, \dots, q\}$ defined by $\mathcal{H}^*(uv) = \frac{1}{4}|\mathcal{H}(u) - \mathcal{H}(v)|$ is a bijective. Hence, $\mathcal{H}$ is Hilbert graceful labeling and the graph $K_{m,n} \odot u_{1,m} v_{1,n}(l_1, l_2, l_3, l_4)$ is Hilbert graceful graph.

Example 3.4: Hilbert graceful labeling of the graph $K_{3,4} \odot u_{1,3} v_{1,4}(9, 6, 5, 6)$.



IV. Conclusion

In this paper, we proved some complete bipartite graph with pendant edges are Hilbert graceful graph. Labeling pattern is demonstrated by means of illustrations, which provide better understanding of derived results. Analysing Hilbert graceful on other families of graph are our future work.

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