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Reversible pebbling number of Kragujevac Trees

C. Muthulakshmi@Sasikala[1] and A. Arul Steffi[2]
[1] *Department of Mathematics,*
Sri Paramakalyani College, Alwarkurichi – 627 412 kalasasispkc@gmail.com
[2] *Department of Mathematics,*
St. Xavier's College (Autonomous),
Palayamkottai- 627 002

Abstract:
Starting with a pebble free graph, our aim is to pebble the target vertex t of any DAG G, and pebbles can be placed or removed from any vertex according to certain rules. In this paper we find the reversible pebbling number of Krajgujevac trees with a fixed number of branches.

Keywords: *Kragujevac tree, reversible pebbling number, root, branch.*

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I. Introduction.

Here the vertex set V is the union of source vertices S, target vertices T, and intermediate vertices I, that is $V = S \cup T \cup I$ and the set E is a set of ordered pairs of vertices (v_i, v_j) such that $i \neq j$ and $v_i, v_j \in V$. We say that a vertex v_i is a direct predecessor or in-neighbour of a vertex v_j, if there is a directed edge from v_i to v_j and the edge v_iv_j is called the incoming edge of v_j and outgoing edge of v_i. A vertex in a DAG with no incoming edges is called a source vertex and a vertex with no outgoing edges is called a target vertex.

A connected acyclic graph is called a tree. The number of vertices of a tree T is its order, denoted by $n(T)$. A rooted tree is a tree in which one particular vertex is distinguished this vertex is referred to as the root of the rooted tree. Bennett introduced the reversible pebble game.

Given any DAG G, with a source vertex s, the reversible pebble game starts with no pebbles on G and terminates with a pebble (only) on the target vertex r. Pebbles can be placed or removed from any vertex according to the following rules:

1. At any time source vertex can be pebbled. That is, a pebble can always be placed on an empty source vertex.
2. To pebble v, all in-neighbours of v must be pebbled.
3. To unpebble v, all in-neighours of v must be pebbled.
4. At any time, source vertex can be unpebbled, that is a pebble can always be removed from a source vertex.

The goal of the game is to pebble the targe vertex r (only) using the minimum number of pebbles (also using the minimum number of steps). Minimum number of pebbles needed to place a pebble on the targe vertex r is called the reversible pebbling number of a graph G and is denoted by $R(G)$.

Let P_3 be the 3-vertex path rooted at one of its terminal vertices. For $k = 2,3, \dots$ construct the rooted tree B_k by identifying the roots of k copies of P_3. The vertex obtained by identifying the roots of P_3 trees is the root of B_k.

A Kragujevac tree T is a tree possessing a vertex of degree $d \geq 2$, adjacent to the roots of $B_{p_1}, B_{p_2}, \dots, B_{p_d}$ where $p_1, p_2, \dots, p_d \geq 2$. This vertex is said to be the central vertex of T, whereas d is the degree of T. The subgraphs $B_{p_1}, B_{p_2}, \dots, B_{p_d}$ are the branches of T. We denote the Kragujevac tree of degree d with branches $B_{p_1}, B_{p_2}, \dots, B_{p_d}$ by $K_g(p_1, p_2, \dots, p_d)$.

II. Reversible Pebbling Number.

Definition 2.1. The reversible pebble game on G is the following one player game. At any time i of the game, we have a pebble configuration $p_i \subseteq V$. A pebble configuration p_{i-1} can be changed to p_i by applying exactly one of the following rules:

Pebble placement on v:

If all direct predecessors of an empty vertex v have pebbles on them, a pebble may be placed on v. In particular, a pebble can always be placed on an empty source vertex s, since predecessors of s in G is $\emptyset$.

Reversible pebble removal from v:

If all direct predecessors of a pebbled vertex v have pebbles on them, the pebble on v may be removed. In particular, a pebble can always be removed from a source vertex s.

A reversible pebbling of G is a sequence of pebble configurations $p = \{p_0, p_1, \ldots, p_t\}$ such that $p_0 = \emptyset$ and $p_t = \{r\}$ and for all $i = 1,2,3, \ldots, t-1$, it holds that p_i can be obtained from p_{i-1} by applying exactly one of the above stated rules.

Definition 2.2. The time of a reversible pebbling $p = \{p_0, p_1, \ldots, p_t\}$ is time (p)=t and the space of it is space $(p) = \max_{i\in\{1,2,3,\ldots,t-1\}} |p_i|$.

III. Reversible Pebbling number of Kragujevac tree. Theorem 3.1.

Reversible pebbling number of path on 3 vertices, $R(P_3) = 3$.

Three pebbles are needed to pebble the target vertex and unpebbled the other vertices of P_3.

Theorem 3.2.

Reversible pebbling number of the rooted tree B_2 obtained by identifying the roots of 2 copies of P_3 is $R(B_2) = 4$.

Proof. B_2 is the rooted tree obtained by identifying the roots of 2 copies of P_3.

$$V(B_2) = \{s_1, s_2, x_1, x_2, t\}$$

$$E(B_2) = \{s_1x_1, x_1t, s_2x_2, x_2t\}$$

s_1, s_2 are the source vertices, t is the target vertex, x_1 *and* x_2 are the intermediate vertices.

Consider the following pebble configurations p_i of 3 pebbles.

p_0: $\emptyset$

p_1: $\{s_1\}$ (s_1 is pebbled)

p_2: $\{s_1, x_1\}$ (since s_1 is pebbled, x_1 can also be pebbled since s_1 is the predecessor of x_1)

p_3: $\{s_2, x_1\}$ (s_1 is unpebbled and place that freed pebble in s_2)

p_4: $\{s_2, x_1, x_2\}$ (since predecessor of x_2 say s_2 is pebbled, a pebble can be placed at x_2)

p_5: $\{x_1, x_2, t\}$ (free the pebble from x_1 and place this pebble in t)

A pebble at x_1 cannot be freed since its predecessor s_1 has no pebble. Similarly a pebble at x_2 cannot be freed. Hence $R(B_2) \geq 4$.

Consider the following pebble configurations p_i of 4 pebbles.

p_0: $\emptyset$, p_1: $\{s_1\}$, p_2: $\{s_1, x_1\}$, p_3: $\{s_2, x_1\}$, p_4: $\{s_2, x_1, x_2\}$, p_5: $\{s_2, x_1, x_2, t\}$,

p_6: $\{s_2, x_1, s_1, t\}$, p_7: $\{s_2, s_1, t\}$, p_8: $\{s_1, t\}$, p_9: $\{t\}$.

So $R(B_2) \leq 4$. Hence $R(B_2) = 4$.

Theorem 3.3.

Reversible pebbling number of B_k, a rooted tree obtained by identifying the roots of $k$copies of P_3 is $R(B_k) = k + 2$.

Proof. Consider $V(B_k) = \{s_1, s_2, \ldots, s_k, x_1, x_2, \ldots, x_k, t\}$

$E(B_k) = \{s_1x_1, x_1t, s_2x_2, x_2t, \ldots, s_kx_k, x_kt\}$

Let the source vertices be $s_1, s_2, \ldots, s_k$ and intermediate vertices be $x_1, x_2, \ldots, x_k$ and target vertex be t.

Consider the following pebbling configurations p_i on $k + 2$ pebbles.

p_0: $\emptyset$, p_1: $\{s_1\}$, p_2: $\{s_1, x_1\}$, p_3: $\{s_2, x_1\}$, p_4: $\{s_2, x_2, x_1\}$, p_5: $\{s_3, x_2, x_1\}$,

p_6: $\{s_3, x_1, x_2, x_3\}$,…,p_{2k}: $\{s_k, x_1, x_2, \ldots, x_k\}$,$p_{2k+1}$: $\{s_k, x_1, x_2, \ldots, x_k, t\}$,

p_{2k+2}: $\{s_k, x_1, x_2, \ldots, x_{k-1}, s_{k-1}, t\}$,$p_{2k+3}$: $\{s_k, s_{k-1}, s_{k-2}, x_1, x_2, \ldots, x_{k-2}, t\}$,

p_{2k+4}: $\{s_k, s_{k-1}, s_{k-2}, s_{k-3}, x_1, x_2, \ldots, x_{k-2}, x_{k-3}, t\}$,…,

p_{3k-2}: $\{s_k, s_{k-1}, \ldots, s_3, x_1, x_2, x_3, t\}$,$p_{3k-1}$: $\{s_k, s_{k-1}, \ldots, s_3, s_2, x_1, x_2, t\}$,

p_{3k}: $\{s_k, s_{k-1}, \ldots, s_3, s_2, s_1, x_1, t\}$,$p_{3k+1}$: $\{s_k, s_{k-1}, \ldots, s_3, s_2, s_1, t\}$.
Retaining one pebble at the target vertex and then removing the pebble from the source vertices s_k, followed by s_{k-1} and so on until s_1. So $R(B_k) \leq k+2$.
Put the first pebble in the source vertex s_1. Since s_1 is pebbled we can place a second pebble in x_1.
Free the pebble in s_1 and place this freed pebble in s_2, since s_2 is pebbled, we can place a third pebble in x_2 and so on, continuing like this, after pebbling x_{k-1}, leave the pebble in s_{k-1} and place this pebble in s_k. Since s_k is pebbled, x_k is pebbled with $(k+1)^{th}$ pebble. If we free the pebble in s_k and place this freed pebble in t, we are unable to free the pebbles in $x_1, x_2, \ldots, x_k$ since its predecessors s_1, $s_2, \ldots, s_k$ are unpebbled.
Hence $R(B_k) \geq k+2$. $R(B_k) = k+2$.

Theorem 3.4.
The reversible pebbling number of $k_g(2,2,2,\ldots,2)$ is $R\,(k_g(2,2,2,\ldots,2)) = n+3$ where $n \geq 2$.
Proof. $s_1, s_2, \ldots, s_{2n}$ be the source vertices t be our target. Any $k_g(2,2,2,\ldots,2)$ consists of n $B's$ namely $(B_2)_1, (B_2)_2, \ldots, (B_2)_n$. Each $(B_2)_i$ has x_{i1} as the root vertex where $i = 1,2,\ldots,n$ and 2
x_{i2}, x_{i3} where $i = 1,2,\ldots,n$ are the intermediate vertices.
Consider $(B_2)_1$.
By placing one pebble on each of the source vertices s_1 *and* s_2, we can pebble intermediate vertices x_{12} *and* x_{13}. Since x_{12} *and* x_{13} are pebbled, place a pebble on the root vertex x_{11} of $(B_2)_1$. Retaining one pebble on x_{11}. Freeing the pebbles from $(B_2)_1$ and placing the pebbles in $(B_2)_2$.
Now freeing the pebble at x_{12} of $(B_2)_1$ and place it on s_3 of $(B_2)_2$ and freeing the pebble at x_{13} and place this freed pebble in s_4 of $(B_2)_2$ unpebble the vertex s_1 and pebble it on vertex x_{22} of $(B_2)_2$ and unpebble the vertex s_2 and pebble it on x_{23}. Now placing sixth pebble on root vertex x_{21} of $(B_2)_2$ and keep it as fixed.
Similarly, freeing the pebble at x_{22} and place it on s_5 and freeing the pebble at x_{23} and place it on s_6. Removing the pebble at s_3 and place it on x_{32} and removing the pebble at s_4 and place it on x_{33} and place seventh pebble at x_{31} and keep it as fixed. Continuing like this, from $(B_2)_{n-1}$ free the pebble at $x_{(n-1)2}$ and place it on s_{2n-1} and free the pebble at $x_{(n-1)3}$ and place it on s_{2n}.
Free the pebble at s_{2n-3} and place it on x_{n2} and free the pebble at s_{2n-2} and place it on x_{n3}.
Free the pebble from s_{2n-1} and place it on x_{n1}, and remove a pebble from s_{2n} and place it on the target vertex t. Since $x_{11}, x_{21}, \ldots, x_{n1}$ are pebbled, we are able to place a pebble on the target vertex.
Free the pebble at x_{n1} and place it on s_{2n-1}. Free the pebble at $x_{(n-1)1}$ and place it on s_{2n}. Now we are able to free the pebble at $x_{(n-1)2}$ and place it on $x_{(n-2)1}$. Free the pebble at s_{2n-1} and place it on s_{2n-2}. Free the pebble at s_{2n} and place it on $x_{(n-2)2}$.
Free the pebble at $x_{(n-2)1}$ and place it on s_{2n-3}. Free the pebble at $x_{(n-2)2}$ and place it on $x_{(n-3)2}$. And proceeding like this until pebbles are freed from intermediate vertices and source vertices, keeping the pebble at the target vertex fixed.

IV. Conclusion:

In this paper we find the reversible pebbling number of Kragujevac trees $k_g(2,2,2,\ldots,2)$. To find $R(k_g(2,3,\ldots,n)$ is an open problem.

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