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A Model-Independent Upper Bound For The Price Of Swiss Re Mortality Bond 2003 Through Lagrangian Optimization Technique

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Abstract:
The use of Catastrophic Mortality (CATM) Bonds (CMBs) as an alternative risk transfer (ART) mechanism is well established in the insurance and reinsurance fields. However pricing of these bonds is a complex problem and no closed form solution can be found in the existing literature. In this paper, we introduce an interesting model-independent upper bound for the price of the Swiss Re Mortality Bond 2003 by employing Lagrangian Optimization Technique. Swiss Re mortality bond was indeed the first catastrophic mortality bond to be launched in the insurance market. The bond mechanism relies upon the behaviour of a well-defined mortality index to generate pay-off for bondholders. We employ the methodology of expressing the pay-off of such a bond in terms of the pay-off of an Asian put option in a manner similar to Bahl and Sabanis (2021) and present an efficient model-independent upper bound. We carry out Monte Carlo simulations to compute the bond price and illustrate the quality of the bound for a variety of models.
***Key Words**: Mortality Risk, Catastrophic Mortality Bonds, model-independent bound, Asian options.*

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I. Introduction

The modern day world boasts of a very amicable living that has indeed been achieved through massive technological development. However everything comes at a cost. Mankind has been brutal with the earth's ecosystem battering the nature's equilibrium, disturbing the food chain, causing global warming and altering the weather. All this has taken a toll on the natural defense system of the earth thereby inviting natural disasters. These disasters, pandemics, wars, terrorism attacks and industrial accidents bring along their share of adversities for the individuals who face them as well as for the insurance companies who end up burning a hole in their balance sheets paying the huge claims that arise due to these events. The most serious fall out of these incidents is the loss of human lives. This is termed as ``mortality risk", i.e., the risk of dying earlier than expected or having a shorter life time than anticipated of an individual or group of individuals.

More explicitly, life insurance companies provide protection to their policyholders in the form of a pay-out made in the event of a policyholder's death, in exchange for a premium. Extreme mortality events, such as a severe pandemic or a natural catastrophe or a large terrorist attack, could result in a life insurance company needing to make sudden pay-outs to many policyholders. This large pay-out would be exacerbated in that the investment portfolio would not yet have delivered sufficient returns so that the pay-outs to policyholders are made sooner than expected. Therefore it is crucial for life insurers, and life re-insurers, to manage their exposure to extreme mortality risks where insurance portfolio diversification by itself is insufficient.

The International Actuarial Association defines four components of longevity/mortality risk viz. *level, trend, volatility and catastrophe.* The four components are classified into two groups which are *systematic risk* and *specific risk* or idiosyncratic risk (c.f. Crawford et al. (2008)). Systematic risk is defined as the underestimation or overestimation of the base assumption of mortality rates, including the level component and the trend component. Specific risk is taken to be the volatility around the base assumption, including the volatility component and the catastrophe component. According to the famous law of large numbers in Statistics, specific risks can be minimized by diversifying with a large pool of lives; however it is not possible to reduce systematic risk by diversification.

As a result alternative methodologies have to be adopted to circumvent such risks. We look at the available methodologies to tackle mortality risk in the next section. Section 3 throws light on various CMB transactions. Section 4 unravels the design of the Swiss Re Bond. Section 5 presents the foundation of this paper i.e., the pay-off of the Swiss Re Mortality bond in terms of an Asian put option. Section 6 portrays the put-call parity for this bond. Section 7 presents the focus of this paper i.e. deriving an upper bond for the price of Swiss

Re Bond by using Lagrangian optimization. Section 8 showcases numerical results. Section 9 then concludes this article.

II. Taming The CAT

When a fire breaks out in a city, there needs to be a prompt fire-fighting response to contain the fire and prevent it from spreading. The outbreak of a major fire is the wrong time to hold discussions on the pay of fire-fighters, to raise money for improving the fire service or to consider fire insurance. It is too late in the day to do all that.

Just like fire, infectious diseases also spread at an exponential rate. On March 21, 2014, an outbreak of Ebola was confirmed in Guinea. In April, the World Health Organization (WHO) declared that it would cost a modest sum of $5 million to control the disease. In July this cost of control touched $100 million and by October it had ballooned to $1 billion. Ebola acted both as a serial killer and loan shark. If money in not made available readily enough to deal with the outbreak of an epidemic, its magnitude and intensity may go out of hand. However in general this scenario has been repeating itself with many pandemics, the latest being COVID-19 which has broken all records and the largest casualty is the insurance business.

Several possibilities exist relating to the risk reduction arising from catastrophes. A good reference in this direction is Huynh et al. (2013). We list below the possible remedies employed by insurers and re-insurers to safeguard themselves from the calamity of increased claims caused due to a catastrophe and then discuss a few of them in detail with the first one being the most important in the context of this article.

- Catastrophic or Extreme Mortality Bonds
- Risk Transfer Mechanisms such as reinsurance or retrocession[1]
- Self-insuring or retaining the risk through holding greater levels of capital[2]
- Natural Hedging by balancing mortality risk with longevity risk
- Diversification along other lines of businesses

While the first method is a recent innovation, the others are traditional methods of risk mitigation. We will discuss the first one in detail in the next section. In particular, reinsurance or retrocession has been the most popular method of offloading the risk. This consists in transferring the risk from a smaller and less diversified insurer to a larger re-insurer with a more diversified portfolio. However, the ceding party ultimately lands up with the same risk it seeks to transfer, via the credit risk of the counter party re-insurer. This is due to the inherent possibility of reinsurer and retrocessionaire defaulting when faced with widespread catastrophic losses such as in a pandemic. Reinsurance is essentially pure mortality risk business, and the usual advantage conferred by re-insurers' geographical diversification is significantly lost in the event of a pandemic (APRA(2007); Dreyer et al.(2007); Cummins and Trainar (2009)) since an influenza pandemic is likely to affect multiple geographical regions around the world as seen during COVID-19, compared, for example, to a single earthquake. In other words, while trying to eliminate mortality risk, credit risk comes into the picture. Thus the capacity of reinsurance is rather limited. An alternative to reinsurance are catastrophic mortality bonds which are "zero-beta assets", which essentially eradicate credit risk. These catastrophic-mortality securitization instruments offer several advantages and disadvantages compared to reinsurance and are described in the next section.

III. Catastrophic Mortality Bonds (CMBs)

As mentioned above catastrophic mortality bonds offer mortality securitization. Securitization consists in the isolation of a pool of assets or rights to a set of cash flows and the repackaging of the assets or cash flows into securities that are traded in the capital markets (c.f. Cowley and Cummins (2005)). Insurance-linked securities (ILS) are instruments designed to transfer insurance risk to the capital markets (c.f. Cummins and Trainar (2009)). Life securitizations have been predominantly used as a financing tool although some have facilitated risk management. On the other hand, non-life securitizations such as earthquake bonds and wind-storm bonds have typically been used to transfer extreme risk arising due to a catastrophic event into the capital markets for a number of years (c.f. Ernst and Young (2011)).

The market for ILS has grown significantly in recent years, expanding at 40-50% per year since 1997 (Hartwig et al. (2008)). Since extreme mortality can be modelled in the same way as other catastrophic risks (see Johnson (2013)), it has become another of many offerings in a menu of perils from which investors choose and mix. To the end of 2023, there have been sixteen public catastrophic mortality bonds transactions with a total bond issuance value of approximately U.S. $3.5 billion, with the last issue being that of VITA VI in July 2021 by Swiss Re. In fact Swiss Re was the pioneer to launch the first catastrophic mortality bond VITA I in

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[1] Reinsurance refers to the insurance purchased by an insurer from a re-insurer to transfer risk. Retrocession refers to the purchase of insurance by re-insurers from other reinsurance companies to transfer risk (Bellis et al. (2010)).
[2] c.f. Baumgart et al. (2007)

December 2003 which was extremely successful. The reinsurance giant has undoubtedly dominated the market in this sector and has experimented even with a 'Longevity Trend Bond' called 'Kortis' in 2010 and a 'Multiple Peril Bond' called 'Mythen Re' in 2012 which was a hybrid of a hurricane and a mortality bond. Table 1 summarizes all the catastrophic mortality bonds issued in the pre-COVID-19 era. Only two mortality bonds have been launched after the outbreak of COVID, viz. La Vie Re by Minnesota LIC in October 2020 and VITA VI by Swiss Re in July 2021.

In this table the fourth column represents the maturity of the various tranches, where tranches are parts of a security that can be broken apart and sold in pieces. Catastrophic mortality bonds have primarily appealed to huge, globally diversified insurers and re-insurers, and have predominantly been used in developed countries. Undoubtedly, these bonds enhance the capacity of the life insurance industry to write mortality risk business by transferring catastrophic losses from the insurance industry to the capital markets (Lin and Cox (2008); Bouriaux and MacMinn (2009)).

Table no 5: Summary of Catastrophic Mortality Bond Transactions

Year	Special Purpose Vehicle	Sponsor	Maturity	Principal Amount (Millions)	S & P Rating at Issuance	Initial Spread to 3-Month LIBOR/ EURIBOR (bps)	Attachment/ Exhaustion Point (%)	Covered Area
2003	Vita Capital I	Swiss Re	4	U.S. $400	A+	135	130/150	U.S. 70%, U.K. 15%, France 7.5%, Italy 5% and Switzerland 2.5%
2006	Vita Capital II	Swiss Re	5 5 5	U.S. $62 U.S. $200 U.S. $100	A- BBB+ BBB-	90 140 140	120/125 115/120 110/115	U.S. 62.5%, U.K. 17.5%, Germany 7.5%, Japan 7.5% and Canada 5%
2006	Tartan Capital	Swiss Re	3 3	U.S. $75[3]	AAA BBB+	19 300	115/120 110/115	U.S. 100%
2006	Osiris Capital	Swiss Re	4 4 4 4	EUR 100[3] EUR 50 U.S. $150 U.S. $100	AAA A- BBB BB+	20 120 285 500	114/119 114/119 110/114 106/110	France 60%, Japan 25%, and U.S. 15%
2006	Vita Capital III	Swiss Re	4 4 4 4 4 5 5 5 5	U.S. $100[3] U.S. $100[3] U.S. $90 EUR 30 EUR 55[3] U.S. $100[3,4] EUR 55 U.S. $50[3] U.S. $50	AAA AAA A A AAA AAA AA- AAA A	21 21 110 110 22 20 80 21 112	125/45 125/145 120/125 120/125 120/125 125/145 125/145 120/125 120/125	U.S. 62.5%, U.K. 17.5%, Germany 7.5%, Japan 7.5\% and Canada 5%
2008	Nathan	Munich Re	5	U.S. $100	A-	135	120/130	U.S. 45%, U.K. 25%, Canada 25% and Germany 5%
2009 to 2011	Vita Capital IV	Swiss Re	5 4 5 5 5 5	U.S. $75 U.S.$50 U.S. $100 U.S.$75 U.S. $100 U.S.$80	BB+ BB+ BB+ BB+ BBB- BB+	650 525 375 370 N/A N/A	U.K. 112.5/120 & U.S. 105/110 U.K. 112.5/120 & U.S. 105/110 Japan 107.5/115 & U.S. 105/110 Canada 111.5/120 & Germany 110/115 Canada 120/130 & Germany 125/135 Canada/ Germany 110/115, U.K. 115/120 and U.S. 105/110	U.K. and U.S. U.K. and U.S. Japan and U.S. Canada; Germany Canada; Germany Canada; Germany U.K. and U.S.

[3] These tranches have been credit enhanced by "monoline" insurers who guarantee the interest and principal payment.

[4] Property Claim Services

2012	Vita Capital V D-1 E-1	Swiss Re	5 5	U.S.$125 U.S.$150	BB- BB+	97 97	Australia 135 & Canada 120 Australia 120, Canada 110 & U.S. 105	Australia & Canada Australia, Canada & U.S.
2012	Mythen Re U.S. Hurricane U.K. Mortality	Swiss Re	4 5	U.S. $120 U.S. $80	B+ B-	8.50% 11.75%	U.S. PCS[4] 620.2/845.5 & U.K. 125/135 U.S.PCS 372.7/511.4	U.S. & U.K. U.S.
2013	Atlas IX	SCOR Re	5	U.S.$180	BB	3.25%	102/104	U.S.
2015	Benu Capital Excess Mortality	AXA	5 5	EUR 135 EUR 150	BB+ (sf) BB (sf)	2.55% 3.35%	France 116/152.5 Japan 116/140.8 & U.S. 108/120.4	Drop Down Level France 110%, Japan 110% & U.S. 106%
2015	Vita Capital VI	Swiss Re	5	U.S. $100	BB (sf)	2.90%	Australia 120, Canada 115 & U.K. 120	Australia, Canada & U.K.
2020	La Vie Re	Wills Re Securities	3	U.S. $100	NR	2.85%	110	U.S.
2021	Vita Capital VI (Series 2021-1)	Swiss Re	4	U.S. $120	NR	3%	106/116	Australia, Canada, U.K. & U.S.

IV. Design Of The Swiss Re Bond

As pointed out in the introduction, the financial capacity of the life insurance industry to pay catastrophic death losses from natural or man-made disasters is limited. To increase its capacity to pay catastrophic mortality losses, Swiss Re obtained about 400 million in coverage from institutional investors in lieu of its first pure mortality security. The reinsurance giant issued a three year bond in December 2003 with maturity on January 1, 2007. To carry out the transaction, Swiss Re set up a special purpose vehicle (SPV) called Vita Capital Ltd. This enabled the corresponding cash flows to be kept off Swiss Re's balance sheet. The principal is subject to mortality risk which is defined in terms of an index q_{t_i} in the year t_i. This mortality index was constructed as a weighted average of mortality rates (deaths per 100,000) over age, sex (male 65% and female 35%) and nationality (US 70%, UK 15%, France 7.5%, Italy 5% and Switzerland 2.5%) and is given below.

$$q_{t_i} = \sum_j C_j \sum_k A_k \left(G^m q^m_{k,j,t_i} + G^f q^f_{k,j,t_i}\right) \quad (4.1)$$

where q^m_{k,j,t_i} and q^f_{k,j,t_i} are the respective mortality rates (deaths per 100,000) for males and females in the age group k for country j, C_j is the weight attached to country j, A_k is the weight attributed to age group k (same for males and females) and G^m and G^f are the gender weights applied to males and females respectively.

The Swiss Re bond was a principal-at-risk bond. If the index q_{t_i} (t_i = 2004, 2005 or 2006 for i=1, 2, 3 respectively) exceeds K_1 of the actual 2002 level, q_0, then the investors will have a reduced principal payment. The following equation describes the principal loss percentage, in the year t_i:

$$L_i = \begin{cases} 0 & \text{if } q_{t_i} \le K_1 q_0 \\ \frac{(q_{t_i} - K_1 q_0)}{(K_2 - K_1) q_0} & \text{if } K_1 q_0 \le q_{t_i} \le K_2 q_0 \\ 1 & \text{if } q_{t_i} \ge K_2 q_0 \end{cases} \quad (4.2)$$

In particular, for the case of Swiss Re Bond, K_1=1.3 and K_2=1.5. In lieu of having their principal at risk, investors received quarterly coupons equal to the three-month U.S. LIBOR plus 135 basis points. There were 12 coupons in all with a coupon value of

$$CO_j = \begin{cases} \left(\frac{SP + LI_j}{4}\right).C & \text{if } j = \frac{1}{4}, \frac{2}{4}, \dots, \frac{11}{4} \\ \left(\frac{SP + LI_j}{4}\right).C + X_T & \text{if } j = 3 \end{cases} \quad (4.3)$$

where SP is the spread value which is 1.35%, LI_j are the LIBOR rates, C=\$400 million, $T = t_3$ and X_T is a random variable representing the proportion of the principal returned to the bondholders on the maturity date such that

$$X_T = C\left(1 - \sum_{i=1}^{3} L_i\right)^{+}, \tag{4.4}$$

where $\sum_{i=1}^{3} L_i$ is the aggregate loss ratio at t_3. However, there was no catastrophe during the term of the bond. The discounted cash flow (DC) of payments is given by

$$\mathrm{DC(r)} = \sum_{i=1}^{12} \frac{CO_{\frac{i}{4}}}{\left(1 + \frac{r}{4}\right)^{i}}, \tag{4.5}$$

where r is the nominal annual interest rate.

Further define

$$Y_T = -\int_0^T \rho(t)dt$$

where $\rho(t)$ is the US LIBOR at time t. As a result, the risk-neutral value at time 0 of the random principal returned at the termination of the bond is

$$P = \mathrm{E}_Q[e^{-Y_T} X_T]$$

where Q is the risk-neutral measure. However, under the assumption of independence of Y_T and X_T, this reduces to

$$P = \mathrm{E}_Q[e^{-Y_T}]\mathrm{E}_Q[X_T].$$

The conditions under which it is possible (or not) to transfer the independence assumption from the physical world measure $\mathbb{P}$ to Q have been discussed extensively in Dhaene et al. (2013). Henceforth, in this incomplete market, we choose to price under a risk neutral measure that preserves independence between market and mortality risks. In order to proceed, we represent $\mathrm{E}_Q[e^{-Y_T}]$ as e^{-rT}, which implies

$$P = e^{-rT}\mathrm{E}_Q[X_T] \tag{4.6}$$

where r is the risk-free rate of interest. In subsequent writing, we drop Q from the above expression.

V. The Principal Pay-Off Of Swiss Re Bond As That Of An Asian-Type Put Option

In the same spirit as Bahl and Sabanis (2021), we can write X_T given in equation (4.4) in a more compact form similar to the pay-off of the Asian put option as shown below:

$$X_T = D\left(q_0 - \sum_{i=1}^{3} 5\left(q_{t_i} - 1.3q_0\right)^{+}\right)^{+} \tag{5.1}$$

with

$$D = \frac{C}{q_0} \tag{5.2}$$

and the strike price equal to q_0. For the sake of simplicity, we use q_i in place of q_{t_i} and define

$$S_i = 5(q_i - 1.3q_0)^{+} \tag{5.3}$$

and

$$S = \sum_{i=1}^{3} S_i \tag{5.4}$$

Using equations (5.3)-(5.4) in equation (5.1) and plugging the result into equation (4.6), we have:

$$P = De^{-rT}E[(q_0 - S)^{+}] \tag{5.5}$$

It is naturally assumed that the inequalities $S \geq q_0$ almost surely (a.s.) and $S \leq q_0$ a.s. do not hold, otherwise the problem has a trivial solution. This means that $q_0 \in \left(F_S^{-1+}(0), F_S^{-1}(1)\right)$, where as in Dhaene et al. (2002), F_X^{-1} is the generalized inverse of the cumulative distribution function (c.d.f.), i.e.,

$$F_X^{-1} = \inf\{x \in \mathbb{R} | F_X(x) \geq p\}, \quad p \in [0,1] \tag{5.6}$$

and F_X^{-1+} is a more sophisticated inverse defined as

$$F_X^{-1+} = \sup\{x \in \mathbb{R} | F_X(x) \leq p\}, \quad p \in [0,1] \tag{5.7}$$

Our interest lies in the calculation of a reasonable upper bound for *P*. In order to obtain a lower bound for *P*, we consider the call counterpart of the pay-off of Swiss Re Bond rather than equation (5.5). We nomenclate this pay-off as P_1, i.e., we have

$$P_1 = De^{-rT}E[(S - q_0)^+] \tag{5.8}$$

We then exploit the put-call parity for Asian options to achieve the bounds for the pay-off in question.

VI. Put-Call Parity For The Swiss Re Bond

We now derive the put-call parity relationship for the Swiss Re Bond. For any real number a, we have:

$$(a)^+ - (-a)^+ = a \tag{6.1}$$

So we obtain

$$e^{-rT}\left(\sum_{i=1}^{3} S_i - q_0\right)^+ - e^{-rT}\left(q_0 - \sum_{i=1}^{3} S_i\right)^+ = e^{-rT}\left(\sum_{i=1}^{3} S_i - q_0\right) \tag{6.2}$$

On taking expectations on both sides, we obtain

$$e^{-rT}\mathrm{E}\left[\left(\sum_{i=1}^{3} S_i - q_0\right)^+\right] - e^{-rT}\mathrm{E}\left[\left(q_0 - \sum_{i=1}^{3} S_i\right)^+\right] = e^{-rT}\mathrm{E}\left[\sum_{i=1}^{3} S_i - q_0\right]$$

Finally, on multiplying by D and expanding the definition of S_i, we have

$$P_1 - P = De^{-rT}\mathrm{E}\left[\left(\sum_{i=1}^{3} 5(q_i - 1.3q_0)^+ - q_0\right)\right]$$

$$\Rightarrow P_1 - P = De^{-rT}\mathrm{E}\left[5\sum_{i=1}^{3} e^{rt_i}C\,(1.3q_0, t_i) - q_0\right], \tag{6.3}$$

where $C(K, t_i)$ depicts the price of a European call on the mortality index with strike *K*, maturity t_i and current mortality value q_0. As in Bahl and Sabanis (2021), we note that this option would be in-the-money if the mortality index is greater than $1.3q_0$ which is the trigger level of Swiss Re bond. Clearly, such instruments are not available for trading in the market at present. But a more comprehensive life market is developing and we feel such securities will soon be available (c.f. Bahl and Sabanis (2021), Blake et al. (2013) and Blake et al. (2008)). The pay-off structures, i.e. the design of the issued securities and the mortality contingent payments should be developed to appear attractive to investors and the re-insurer. Although, the Swiss Re bond was fully subscribed and press reports show that investors were quite satisfied with it (e.g. *Euroweek*, 19 December 2003), the market for mortality linked securities still needs innovations such as vanilla options on mortality index to provide flexible hedging solutions. Investors of the Swiss Re bond included a large number of pension funds as they could view this bond as a powerful hedging instrument. The underlying mortality risk associated with the bond is correlated with the mortality risk of the active members of a pension plan. If a catastrophe occurs, the reduction in the principal would be offset by reduction in pension liability of these pension funds. Moreover, the bond offers a considerably higher return than similarly rated floating rate securities (c.f. Blake et al. (2006)). In a manner similar to Bauer (2008), we feel the success of the life market hinges upon flexibility. As a result, such option-type structures enable re-insurer to keep most of the capital while at the same time being hedged against catastrophic mortality situation. Cox et al. (2006) present an interesting note on the trigger level of $1.3q_0$ in context of 2004 tsunami in Asia and Africa. A mortality option of the above type would become extremely useful in such a case. Tsai and Tzeng (2013) and Cheng et al. (2014) decompose the terminal pay-off of the Swiss Re bond into two call options.

Equation (6.3) gives the required put-call parity relation between the Swiss Re mortality bound and its call counterpart. Define

$$G = De^{-rT}\mathrm{E}\left[5\sum_{i=1}^{3} e^{rt_i} C\,(1.3q_0, t_i) - q_0\right] \tag{6.4}$$

Clearly, if we bound P_1 by bounds l_1 and u_1, then the corresponding bounds for the Swiss Re mortality bond are as follows

$$(l_1 - G)^+ \leq P \leq (u_1 - G)^+ \tag{6.5}$$

VII. An Upper Bound $\mathbf{SWUB_1}$ For The Price Of Swiss Re Bond

This section will focus on finding the upper bound SWUB_1 for the price of Swiss Re Bond 2003 by employing a Lagrange optimization technique. Before formally beginning the derivation of the upper bound, we throw light on an interesting proposition regarding the convexity of the value of a call option which would help us to arrive at the requisite upper bound.

Proposition 1. The pay-off of the call option is a convex function[5] of the strike price, i.e., $\mathrm{E}[(X-x)^+]$ is convex in *x*.

Proof.

$$\begin{aligned}\mathrm{E}\left[\left(X-(ax+(1-a)y)\right)^+\right] &= \mathrm{E}\left[\left((a+1-a)X-(ax+(1-a)y)\right)^+\right]\\ &= \mathrm{E}\left[\left(a(X-x)+(1-a)(X-y)\right)^+\right]\\ &\leq \mathrm{E}[a(X-x)^+ + (1-a)(X-y)^+]\\ &= a\mathrm{E}[(X-x)^+] + (1-a)\mathrm{E}[(X-y)^+]\end{aligned} \qquad \square$$

Now, to begin, consider a vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $\lambda_i \in \mathbb{R}$ and $\sum_{i=1}^{n} \lambda_i = 1$. Now, with the help of $\boldsymbol{\lambda}$ we can write the pay-off of the Asian-type call option as shown below.

$$P_1 = Ce^{-rT}\mathrm{E}\left[\left(\sum_{i=1}^{n}\left(5\left(\frac{q_i}{q_0} - 1.3\right)^+ - \lambda_i\right)\right)^+\right]. \tag{7.1}$$

Using the above proposition, equation (7.1) implies that the upper bound for the above Asian-type call can be expressed as follows:

$$P_1 \leq Ce^{-rT}\sum_{i=1}^{n}\mathrm{E}\left[\left(5\left(\frac{q_i}{q_0} - 1.3\right)^+ - \lambda_i\right)^+\right] \tag{7.2}$$

$$\begin{aligned}\Rightarrow P_1 &\leq 5De^{-rT}\sum_{i=1}^{n}\mathrm{E}\left[\left(q_i - q_0\left(1.3 + \frac{\lambda_i}{5}\right)\right)^+\right]\\ &= 5De^{-rT}\sum_{i=1}^{n}\mathrm{e}^{rt_i}\, C\left(q_0\left(1.3 + \frac{\lambda_i}{5}\right), t_i\right)\end{aligned} \tag{7.3}$$

As the λ_i are arbitrary, the goal is then to minimise this bound over all possible $\boldsymbol{\lambda}$ ensuring that $q_0\left(1.3 + \frac{\lambda_i}{5}\right) > 0$. This in turn is equivalent to minimising the sum $\sum_{i=1}^{n} \mathrm{e}^{rt_i}\, C\left(q_0\left(1.3 + \frac{\lambda_i}{5}\right), t_i\right)$. In order to achieve this, we assume that the European call option pay-off viz. $C(K,T) > 0$ for every positive *K*, T and that $C(K,T) \downarrow 0$ as $K \to \infty$. Then C is a convex, strictly decreasing function of K with a continuous, strictly increasing derivative $\frac{\partial C}{\partial K} < 0$. We define

$$d_i = q_0\left(1.3 + \frac{\lambda_i}{5}\right);\, i = 1,2,\ldots,n \tag{7.4}$$

Next, we define the Lagrangian as

$$L(\boldsymbol{\lambda}, \emptyset) = \frac{5}{q_0}\sum_{i=1}^{n}\mathrm{e}^{rt_i}\, C(\,d_i, t_i) + \emptyset\left(\sum_{i=1}^{n}\lambda_i - 1\right),$$

[5] A function $f: I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, is convex if and only if $f(ax + (1-a)y) \leq af(x) + (1-a)f(y) \;\; \forall a \in [0,1]$ and any pair of elements $x, y \in I$.

where $\emptyset$ is the Lagrange's multiplier. We wish to find λ_i for each i, that minimises L. Differentiating L w.r.t λ_i, we obtain

$$\frac{\partial L}{\partial \lambda_i} = -P[q_i \geq d_i] + \emptyset.$$

Thus, it is evident that the function L has a point of maxima or minima when λ_i solves the following equation for every i, i.e.,

$$\lambda_i = \frac{5}{q_0}\left(F_{q_i}^{-1}(1-\emptyset) - 1.3q_0\right) \tag{7.5}$$

where $F_{q_i}^{-1}$ is the inverse distribution function of the mortality index q_i. Further more, since $q_i > 0$, we have that the strike prices of the call viz. $d_i = q_0\left(1.3 + \frac{\lambda_i}{5}\right) > 0, \forall i$. The next aim is to check that the constraint $\sum_{i=1}^{n} \lambda_i = 1$ is satisfied. For this, we define H as

$$H(\emptyset) = \sum_{i=1}^{n} \lambda_i - 1 = \frac{5}{q_0}\left(F_{q_i}^{-1}(1-\emptyset) - 1.3q_0\right) - 1 \tag{7.6}$$

Under the aforesaid assumptions, H is a continuous function of $\emptyset$. Moreover, since by assumption F_{q_i} is injective for all $t_i, i = 1,2,\dots,n$, it follows that H is strictly decreasing in $\emptyset$. Hence, a solution to $H(\emptyset) = 0$ exists if $\inf H(\emptyset) < 0 < \sup H(\emptyset)$. For $\emptyset = 1$, $H(\emptyset) = -6.5n - 1$ and for the Swiss Re bond as $n = 3$, we have $H(1) = -20.5$. As far as searching for such a value of ϕ is concerned, for which $H(\emptyset) > 0$, we can immediately see that $F_{q_i}(K) = 1$, only when $K \to \infty$. Thus, $\lim_{\emptyset \downarrow 0} H(\emptyset) = \infty$ and so the application of intermediate value theorem ensures that we can find $\emptyset^*$ that satisfies $H(\emptyset^*) = 0$. Also $\emptyset^*$ is unique since H is strictly decreasing.

The final task is to check that the stationary point of L, which is obtained when $\boldsymbol{\lambda} = \lambda(\emptyset^*)$, is a point of minima. This is indeed straightforward because $\frac{\partial C}{\partial K}$ is strictly increasing. This implies that on using equation (7.5) in conjunction with equation (7.3), a minimal upper bound for the call counterpart of the Swiss Re bond is given by

$$P_1 \leq 5De^{-rT}\sum_{i=1}^{n} \mathrm{e}^{rt_i}\, C\left(F_{q_i}^{-1}(1-\emptyset^*), t_i\right). \tag{7.7}$$

Evidently, the argument of $F_{q_i}^{-1}$ in this result is identical to the one in equation (7.6) and this allows us to rewrite the upper bound as

$$P_1 \leq 5De^{-rT}\sum_{i=1}^{n} \mathrm{e}^{rt_i}\, C\left(F_{q_i}^{-1}(x), t_i\right) =: \mathrm{ub}_1. \tag{7.8}$$

where $x \in (0,1)$ is the solution of the equation

$$\sum_{i=1}^{n} F_{q_i}^{-1}(x) = \frac{q_0}{5}(1 + 6.5n) \tag{7.9}$$

which is a direct consequence of equation (7.6).

Now, invoking the put-call parity of section 6, we have for the Swiss Re bond

$$P \leq (ub_1 - G)^+ =: \mathrm{SWUB}_1 \tag{7.10}$$

where G is defined in equation (6.4). This provides an alternative methodology in comparison to comonotonicity approach employed in Bahl and Sabanis (2021) to obtain SWUB_1.

VIII. Performance Of SWUB_1

We present below in tables that follow the values of the upper bound vis-a-vis the well-known Monte Carlo (MC) Estimates for the price of the Swiss Re bond for a variety of models.

In tables 2 and 3, we assume that the mortality evolution process $\{q_t\}_{t\geq 0}$ obeys the Black-Scholes model, specified by the following stochastic differential equation (SDE)

$$dq_t = rq_t dt + \sigma q_t dW_t.$$

In order to simulate a path, we will consider the value of the mortality index in the three years that form the term of the bond, i.e., $n = 3$. In fact we consider the time points as $t_1 = 1, \dots, t_n = T = 3$. We invoke the following equation to generate the mortality evolution:

$$q_{t_j} = q_{t_{j-1}} \exp\left[\left(r - \frac{1}{2}\sigma^2\right)\delta t + \sigma\sqrt{\delta t} Z_j\right] \qquad Z_j \sim N(0,1),$$
$$j = 1,2,\dots,n \qquad (8.1)$$

We highlight below the parameter choices in accordance with Lin and Cox (2008). The value of the interest rate is varied in table 2 while table 3 experiments with the variation in the base value of the mortality index while assuming a zero interest rate.

Parameter choices for tables 1 and 2 with *t* specified in terms of years are:

$$q_0 = 0.008453, T = 3, t_0 = 0, \ n = 3, \ \sigma = 0.0388.$$

We further depict the results of tables 2 and 3 in figures 1-3. While figures 1 and 2 depict comparisons between the bounds, figure 3 portrays the price bounds for the Swiss Re bond generated by the Black-Scholes model. We will let MC denote the Monte Carlo estimate for the Swiss Re bond.

Table 2 reflects that the relative difference $\left(= \frac{|bound - MC|}{MC}\right)$ between the lower bound and the benchmark Monte Carlo estimate increases with an increase in the interest rate for a fixed value of the base mortality index q_0. This observation is echoed by figure 1. On the other hand, figure 2 depicts the difference between the Monte Carlo estimate of the Swiss Re bond and the derived bound. The absolute difference between the estimated price and the bounds increase as the value of the base mortality index is increased and then there is a switch and this gap begins to diminish. This observation is supported by the fact that an increase in the starting value of mortality increases the possibility of a catastrophe which leads to the washing out of the principal or in other words the option goes out of money.

Table no 2: SWUB_1 for the Swiss Re Mortality Bond under the Black-Scholes Model with $q_0 = 0.008453$ and $\sigma = 0.0388$ in accordance with Lin and Cox (2008).

r	SWUB_1	MC	S.E. of M.C.
0.035	0.899131637780	0.899131338643	0.000007814868
0.030	0.913324320930	0.913324365180	0.000005483857
0.025	0.927447619324	0.927447582074	0.000003766095
0.020	0.941626384749	0.941626356704	0.000002549695
0.015	0.955935736078	0.955935715489	0.000001673442
0.010	0.970419129772	0.970419112046	0.000001032941
0.005	0.985101141738	0.985101142704	0.000000646744
0.000	0.999995778584	0.999995770298	0.000000405336

Table no 3: SWUB_1 for the Swiss Re Mortality Bond under the Black-Scholes Model with $r = 0.0$ and $\sigma = 0.0388$ in accordance with Lin and Cox (2008).

q_0	SWUB_1	MC	S.E. of M.C.
0.008	0.999999915253	0.999999915033	0.000000052478
0.009	0.999822875816	0.999822630214	0.000003051524
0.010	0.986262918347	0.978782997810	0.000042738093
0.011	0.877336305502	0.652245039892	0.000090193709
0.012	0.395672911251	0.094677358603	0.000089559585
0.013	0.083466184427	0.001665407936	0.000011391823
0.014	0.008942985848	0.000002890238	0.000000379522

In our next example, we assume that the mortality rate 'q' obeys the four-parameter transformed Normal (S_u) Distribution (for details see Johnson(1949) and Johnson et al. (1994)) which is defined as follows

$$sinh^{-1}\left(\frac{q-\alpha}{\beta}\right)$$
$$= x \sim N(\mu, \sigma^2), \qquad (8.2)$$

where α, β, μ and σ are parameters ($\beta, \sigma > 0$) and $sinh^{-1}$ is the inverse hyperbolic sine function.

For table 4, we vary the interest rate as in table 2 and use the parameter set employed by Tsai and Tzeng (2013). The aforesaid authors use the mortality catastrophe model of Lin and Cox (2008) to generate the data and then utilize the quantile-based estimation of Slifker and Shapiro (1980) to estimate the parameters of the S_u-fit. The initial mortality rate and time points are same as for tables 2 and 3. The following arrays present the values of the parameters for the three years 2004, 2005 and 2006 that were covered by the Swiss Re bond.

α =[0.008399, 0.008169, 0.007905],
β =[0.000298, 0.000613, 0.000904],

μ =[0.70780, 0.58728, 0.58743]

and

σ =[0.67281, 0.50654, 0.42218].

Table no 4: SWUB_1 for the Swiss Re Mortality Bond under the S_u distribution with $q_0 = 0.008453$ and parameter choice in accordance with Tsai and Tzeng (2013).

r	SWUB_1	MC	S.E. of M.C.
0.035	0.88680657	0.88468962	0.00006349
0.030	0.90548179	0.90422765	0.00004987
0.025	0.92275950	0.92201394	0.00003804
0.020	0.93901043	0.93863396	0.00002794
0.015	0.95458265	0.95441569	0.00001956
0.010	0.96977488	0.96968765	0.00001352
0.005	0.98482046	0.98478917	0.00000859
0.000	0.99988427	0.99987622	0.00000513

Finally in tables 5 and 6, we experiment with log gamma distribution by varying the interest rate in table 4 and the base mortality rate in the latter. The parameters are chosen as in Cheng et al. (2014) who employ an approach similar to Tsai and Tzeng (2013) outlined above with q_0=.0088 but use maximum likelihood estimation to obtain the parameters of the fitted log gamma distribution. As before, the following arrays present the year wise parameters

p=[61.6326, 64.2902, 71.8574],
a=[0.0103, 0.0098, 0.0080],
μ=[-5.2452, -5.4600, -5.7238]

and

$\sigma = [7.4\text{x}10^{-5}, 9.5\text{x}10^{-5}, 9.4\text{x}10^{-5}]$.

Tables 5 and 6 clearly show that even for non-normal universe, the bound is extremely precise.

Table no 5: SWUB_1 for the Swiss Re Mortality Bond under the transformed Gamma distribution with $q_0 = 0.0088$ and parameter choice in accordance with Cheng et al. (2014).

r	SWUB_1	MC	S.E. of M.C.
0.035	0.86610436	0.85408651	0.00049859
0.030	0.88724013	0.87815608	0.00044050
0.025	0.90728309	0.90050920	0.00038741
0.020	0.92636640	0.92103020	0.00034012
0.015	0.94463331	0.94092949	0.00028650
0.010	0.96223065	0.95947457	0.00024259
0.005	0.97930297	0.97748291	0.00020357
0.000	0.99598733	0.99466024	0.00016677

Table no 6: SWUB_1 for the Swiss Re Mortality Bond under the Black-Scholes Model with $r = 0.0$ and parameter choice in accordance with Cheng et al. (2014).

q_0	SWUB_1	MC	S.E. of M.C.
0.008	0.99977956	0.99978465	0.00003227
0.009	0.99338335	0.99003596	0.00023335
0.010	0.95818959	0.89137680	0.00077924
0.011	0.83720797	0.56844674	0.00128761
0.012	0.61383872	0.20822580	0.00105003
0.013	0.38182244	0.04612178	0.00052388
0.014	0.21222938	0.00673234	0.00019165

IX. Conclusions

Mortality forecasts are extremely important in the management of life insurers and private pension plans. Securitization and construction of mortality bonds has become an important part of capital market solutions. In the era prior to the introduction of the Swiss Re bond in 2003; life insurance securitization was not designed to handle mortality risk.

This article proposes a model independent upper price bound for the Swiss Re mortality bond 2003. As stated in Deng et al. (2012), an incomplete mortality market that has no arbitrage opportunities guarantees the existence of at least one risk-neutral measure termed the equivalent martingale measure Q that can be used for calculating the fair prices of mortality securities. We rely on this fact and devise the upper bound for the mortality security in question without assuming any particular model. Model-specific bounds can then be achieved by plugging in the requisite models into the general bounds. The bound is extremely tight around the Monte Carlo values as can be compared from the respective tables for all three models.

The fact that a well-known technique of optimization in Mathematics viz. the Lagrangian technique has been utilized to obtain the bound with simplicity and ease, it makes the bound much more appealing for the practitioners to apply and the academicians to access.

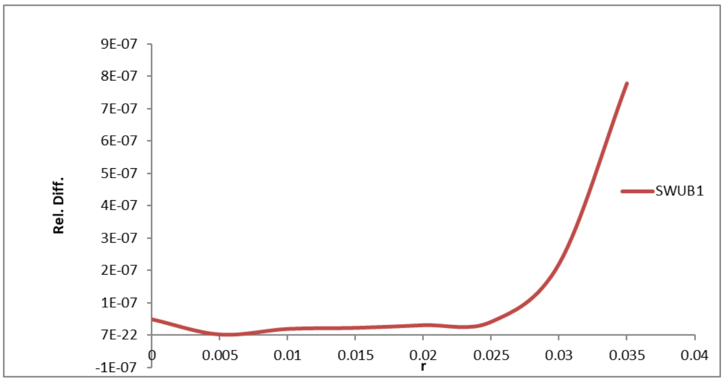

Figure 1: Relative Difference of $SWUB_1$ w.r.t. MC estimate under Black-Scholes Model

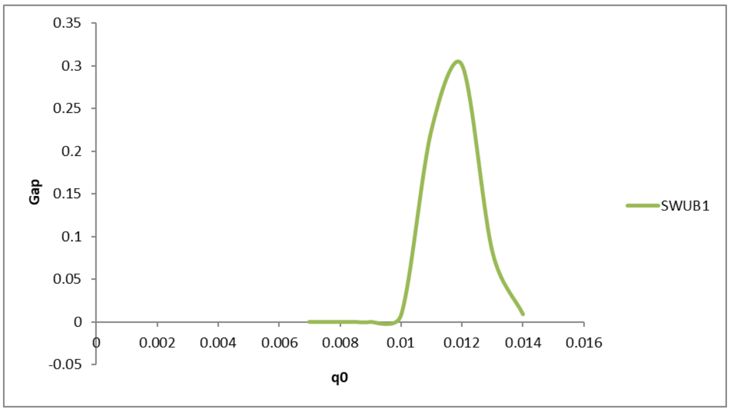

Figure 2: Performance of Upper bound under B-S model in terms of difference from MC estimate for r=0

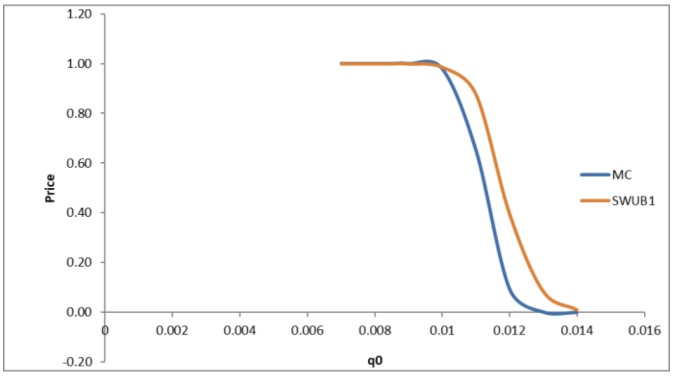

Figure 3: Upper Price Bound under Black-Scholes Model for the parameter choice of Lin and Cox(2008) Model