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Modified Jacobi Methods For Solving Linear Systems With M-Matrices

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Abstract
In the current work, we propose two modified Jacobi methods with two different preconditioners for M-matrices. For the superiority of the modified Jacobi methods, some comparison theorems and numerical examples have been introduced. The comparisons theorems and numerical experiments show that the proposed methods are better than the classical Jacobi method regarding memory requirement, time to converge and other necessary computation cost.
Keywords: *Preconditioning, Preconditioned linear systems, M-matrix, Spectral radius, Modified Jacobi method (MJM), Comparison theorem.*

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I. Introduction

Let us consider the Jacobi iterative method for the following preconditioned linear system with order n,

$$PAx = Pb$$

Here A is a nonsingular M-matrix, P is a preconditioner with positive real number, and x and b are vectors. In this paper, without loss of generality we can assume that A has a splitting of the form $A = I - L - U$, where I is the identity matrix, L and U are strictly lower and strictly upper triangular matrices of A, respectively.

The preconditioner P_s was introduced by Gunawardena et al. [1] in 1991 as follows:

$$P_s = I + S$$

Where S is defined by

$$S = (s_{ij}) = \begin{cases} -a_{ij}, & i = 1,2,\cdots\cdots,n-1; \quad j = i+1 \\ 0, & otherwise \end{cases}$$

In 2002 Kotakemori et al. [2] have introduced the preconditioner $P_{s_{max}}$ as follows:

$$P_{s_{max}} = I + S_{max}$$

Where

$$S_{max} = (s_{ij}^m) = \begin{cases} -a_{i,k_i}, & i = 1,2,\cdots\cdots,n-1; \quad j > i \\ 0, & otherwise \end{cases}$$

and

$$k_i = \min\left\{ \mathrm{j} : \max_{\mathrm{j}} |\mathrm{a_{ij}}|, \ \mathrm{i} < n \right\}.$$

In [3], Evans et al. proposed the following two preconditioners:

$P = I + P_1$ and $P = I + P_2$

Where

$$P_1 = (p_{nj}) = \begin{cases} -a_{nj}, & j = 1 \\ 0, & otherwise. \end{cases}$$

and

$$P_2 = (p_{in}) = \begin{cases} -a_{in}, & i = 1 \\ 0, & otherwise. \end{cases}$$

In [4], Jing-yu ZHAO et al. considered a preconditioner, $P = I + S + P_1$; Where S and P_1 are mentioned above and if $1 - a_{1n}a_{n1} > 0$ and $1 - a_{i,i+1}a_{i+1,i} > 0$, $i = 1,2,\cdots,n-1$; then in this case the preconditioned iteration matrix is defined.

In the present paper, we propose the Jacobi method with two preconditioners $P_m = I + S_{max} + P_1$ and $P_n = I + S + P_2$ and discuss their convergence property. Under some additional assumptions, the comparison theorems show that the new methods are preferable.

II. Proposed Method

We propose two preconditioned Jacobi methods with the preconditioners $P_m = I + S_{max} + P_1$ and $P_n = I + S + P_2$; where S, S_{max}, P_1 and P_2 are defined above.

Then the preconditioned system matrix for $P_m = I + S_{max} + P_1$ is

$$A_m = (I + S_{max} + P_1)A = M_m - N_m$$

$$= (I - D') - (L + E' - P_1 + P_1U + U - S_{max} + S_{max}U + F')$$

Where D', E' and F' are the diagonal, strictly lower and strictly upper triangular parts of $S_{max}L$ respectively and $P_1L = 0$. In this case, if $1 - a_{1n}a_{n1} > 0$ and $1 - a_{i,k_i}a_{k_i,i} > 0,\ i = 1,2,\cdots,n-1$; then the preconditioned iteration matrix $T_m = M_m^{-1}N_m$ can be defined by

$$T_m = M_m^{-1}N_m = (I - D')^{-1}(L + E' - P_1 + P_1U + U - S_{max} + S_{max}U + F').$$

Again the preconditioned system matrix for $P_n = I + S + P_2$ is

$$A_n = (I + S + P_2)A = M_n - N_n$$
$$= (I - D'' - D''') - (L + U'' - P_2 + U - S + SU + L''')$$

Where D'' and U'' are the diagonal and strictly upper triangular parts of P_2L respectively and $D''',\ L'''$ are the diagonal and strictly lower triangular parts of SL respectively. In this case, it can be easily seen that if $1 - a_{12}a_{21} - a_{1n}a_{n1} > 0$ and $1 - a_{i,i+1}a_{i+1,i} > 0,\ i = 1,2,\cdots,n-1$; then the preconditioned iteration matrix $T_n = M_n^{-1}N_n$ can be defined as follows:

$$T_n = M_n^{-1}N_n = (I - D'' - D''')^{-1}(L + U'' - P_2 + U - S + SU + L''').$$

We have organized the present paper as follows: In section 3, we present some definitions and preliminary results. In section 4, we prove the convergence of the proposed methods and some comparison theorems. In section 5, we present some numerical experiments to make sure our theoretical analysis. Finally in section 6, conclusion is drawn.

III. Preliminaries

Let $A = (a_{ij})_{nxn}$ and $B = (b_{ij})_{nxn}$, then we can write $A \geq B$ if $a_{ij} \geq b_{ij}$ holds for all $i, j = 1, 2, \cdots\cdots, n$ and $A \geq 0$ (called nonnegative) if $a_{ij} \geq 0$ holds for all $i, j = 1, 2, \cdots\cdots, n$; where 0 is an nxn zero matrix. For the $nx1$ vectors a, b; $a \geq b$ and $a \geq 0$ can be defined in the similar manner.

Definition 3.1. A matrix A is said to be a Z-matrix if $a_{ij} \leq 0$ for $i \neq j$.

Definition 3.2. The spectral radius of a matrix A is defined as the maximum of the module of the eigenvalues of A and is denoted by $\rho(A)$ i.e.

$$\rho(A) = \max\{|\lambda| : \lambda \textit{ is an eigenvalue of } A\}.$$

Definition 3.3. A Z-matrix A is called an M-matrx, if (a) all the diagonal entries of A are positive (b) all the real eigenvalues of A are positive and (c) the real part of any eigenvalue of A is positive.

Definition 3.4. [17] A matrix A is called an L-matrix if $a_{ii} > 0, 1 \leq i \leq n$ and $a_{ij} \leq 0;\ 1 \leq i \leq n, 1 \leq j \leq n$, $i \neq j$. A nonsingular L-matrix A is called a nonsingular M-matrix if $A^{-1} \geq 0$.

Definition 3.5. Let A be a real matrix. Then the representation $A = M - N$ is called a splitting of A if M is a nonsingular matrix. The splitting is called

(1) convergent if $\rho(M^{-1}N) < 1$;
(2) regular if $M^{-1} \geq 0$ and $N \geq 0$;
(3) weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$;
(4) nonnegative if $M^{-1}N \geq 0$;
(5) M-splitting if M is a nonsingular M-matrix and $N \geq 0$.

Definition 3.6. The splitting $A = M - N$ is called the Jacobi splitting of A if $M = I$ is nonsingular and $N = L + U$. In addition, the splitting is called

(1) Jacobi convergent if $\rho(M^{-1}N) < 1$;
(2) Jacobi regular if $M^{-1} = I^{-1} \geq 0$ and $N = (L + U) \geq 0$.

Lemma 3.7. [6] Let A be a nonnegative nxn nonzero matrix, then

(1) $\rho(A)$, the spectral radius of A, is an eigenvalue;
(2) A has a nonnegative eigenvector corresponding to $\rho(A)$;
(3) $\rho(A)$ is a simple eigenvalue of A;
(4) $\rho(A)$ increases when any entry of A increases.

Lemma 3.8. [7] Let $A = M - N$ be an M-splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is a nonsingular M-matrix.

Lemma 3.9. [8] Let A be a nonsingular M-matrix and let $A = M_1 - N_1 = M_2 - N_2$ be two convergence splitting, the first one weak regular and second one regular if $M_1^{-1} \geq M_2^{-1}$, then

$$\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2) < 1.$$

Lemma 3.10. [16] Let A be a nonsingular L-matrix. Then A is called a nonsingular M-matrix if and only if there exists a positive vector y such that $Ay > 0$.

IV. Convergence And Comparison Theorems

In this section, we shall prove that the two modified Jacobi iterative methods with the preconditioners P_m and P_n converge faster than the classical Jacobi method. To prove these, we require some results and owing to this we firstly prove that both the preconditioned system matrices A_m and A_n are regular and Jacobi convergent splittings.

On the preconditioner $P_m = I + S_{max} + P_1$

Theorem 4.1. Let A be a nonsingular M-matrix and let $0 \le a_{1n}a_{n1} < 1$ and $0 \le a_{i,k_i}a_{k_i,i} < 1$; $1 \le i \le n-1$, then $A_m = M_m - N_m$ is a regular and Jacobi convergent splitting.

Proof. We can easily observe that when $0 \le a_{1n}a_{n1} < 1$ and $0 \le a_{i,k_i}a_{k_i,i} < 1$, $1 \le i \le n-1$; the diagonal elements of A_m are positive and M_m^{-1} exists. We know by Lemma 3.10. ([16]) that, an L-matrix A is a nonsingular M-matrix if and only if there exists a positive vector y such that $Ay > 0$. By taking such y, the fact that $I + S_{max} + P_1 \ge 0$ implies $A_m y = (I + S_{max} + P_1)Ay > 0$ and consequently the L-matrix A_m is a nonsingular M-matrix, which means $A_m^{-1} \ge 0$. Therefore Lemma 3.8., implies that $\rho(M_m^{-1}N_m) < 1$.

Again, when $0 \le a_{1n}a_{n1} < 1$ and $0 \le a_{i,k_i}a_{k_i,i} < 1$; $1 \le i \le n-1$, we have $D' < I$ and so $(I - D') \ge 0$. Hence

$$M_m^{-1} = (I - D')^{-1}$$

$= I + D' + D'^2 + D'^3 + \cdots\cdots\cdots \ge 0;\ provided \quad \rho(D') < 1$

Since $L \ge P_1 \ge 0$ and $U \ge S_{max} \ge 0$ and so

$$N_m = L + E' - P_1 + P_1U + U - S_{max} + S_{max}U + F' \ge 0.$$

Thus $A_m = M_m - N_m$ is a regular and Jacobi convergent splitting by definition 3.6. and Lemma 3.8..

Theorem 4.2. Let A be a nonsingular M-matrix. Then under the assumptions of Theorem 4.1., the following inequality holds:

$$\rho(T_m) \le \rho(T) < 1.$$

Proof. For $M_m = I - D'$ and $N_m = L + E' - P_1 + P_1U + U - S_{max} + S_{max}U + F'$, by Theorem 4.1., we know that $A_m = P_mA = M_m - N_m$ is a regular Jacobi convergent splitting. Again, the iteration matrix of the classical Jacobi method for A is $T = I^{-1}(L + U) = I(L + U) = L + U$. Since A is a nonsingular M-matrix and so the classical Jacobi splitting $A = I - (L + U)$ of A is clearly regular and convergent.

To compare $\rho(T_m)$ with $\rho(T)$, we consider the following splitting of A:

$$A_m = P_mA = (I + S_{max} + P_1)A = M_m - N_m$$

Or, $$A = (I + S_{max} + P_1)^{-1}M_m - (I + S_{max} + P_1)^{-1}N_m$$

Suppose $M_1 = (I + S_{max} + P_1)^{-1}M_m$ and $N_1 = (I + S_{max} + P_1)^{-1}N_m$, then we can easily verify that $M_1^{-1}N_1 = M_m^{-1}N_m$ and so $\rho(M_1^{-1}N_1) < 1$.

Also we note that

$$M_1^{-1} = M_m^{-1}(I + S_{max} + P_1) = (I - D')^{-1}(I + S_{max} + P_1) \ge (I - D')^{-1} \ge I^{-1} = M^{-1}$$

i.e. $M_1^{-1} \ge M^{-1}$ and $A = M_1 - N_1 = M - N$ be two convergent and regular splittings.

Thus it follows from Lemma 3.9. that $\rho(M_1^{-1}N_1) \le \rho(M^{-1}N) < 1$. Hence

$\rho(M_m^{-1}N_m) \le \rho(M^{-1}N) < 1$

i.e. $$\rho(T_m) \le \rho(T) < 1.$$

On the preconditioner $P_n = I + S + P_2$

Theorem 4.3. Let A be a nonsingular M-matrix and let that $1 - a_{12}a_{21} - a_{1n}a_{n1} > 0$ and $1 - a_{i,i+1}a_{i+1,i} > 0$; $1 \le i \le n-1$, then $A_n = M_n - N_n$ is a regular and Jacobi convergent splitting.

Proof. When $1 - a_{12}a_{21} - a_{1n}a_{n1} > 0$ and $1 - a_{i,i+1}a_{i+1,i} > 0$, $1 \le i \le n-1$; then the diagonal elements of $A_n = M_n - N_n$ are positive and so M_n^{-1} exists. By Lemma 3.10. ([16]), an L-matrix A is a nonsingular M-matrix if and only if there exists a positive vector y such that $Ay > 0$. By taking such y, the fact that $I + S + P_2 \ge 0$ implies $A_ny = (I + S + P_2)Ay > 0$. Consequently, the L-matrix A_n is a nonsingular M-matrix i.e., $A_n^{-1} \ge 0$ and thus by Lemma 3.8., $\rho(M_n^{-1}N_n) < 1$.

Again, when $1 - a_{12}a_{21} - a_{1n}a_{n1} > 0$ and $1 - a_{i,i+1}a_{i+1,i} > 0, 1 \le i \le n-1$; we have $D'' + D''' < I$ and so $(I - D'' - D''') \ge 0$. Hence

$$M_n^{-1} = (I - D'' - D''')^{-1}$$

$= I + (D'' + D''') + (D'' + D''')^2 + (D'' + D''')^3 + \cdots\cdots\cdots \ge 0;\ provided\ \rho(D'' + D''') < 1.$

and $\quad N_n = (L + U'' - P_2 + U - S + SU + L''') \ge 0;$ Since $U \ge S + P_2 \ge 0$.

Therefore, $A_n = M_n - N_n$ is a regular and Jacobi convergent splitting by Lemma 3.8. and definition 3.6..

Theorem 4.4. Let A be a nonsingular M-matrix. Then under the assumptions of Theorem 4.3., the following inequality holds:

$$\rho(T_n) \le \rho(T) < 1.$$

Proof. In the classical Jacobi method, the iteration matrix for A is $T = L + U$. As A is a nonsingular M-matrix and so the classical Jacobi splitting $A = I - (L + U)$ of A is clearly regular and convergent. Also $M_n = I -$

$D'' - D'''$ and $N_n = L + U'' - P_2 + U - S + SU + L'''$, by Theorem 4.3., $A_n = P_n A = M_n - N_n$ is a regular Jacobi convergent splitting.

To compare $\rho(T_n)$ with $\rho(T)$, we consider the following splitting of A:

$$A_n = P_n A = M_n - N_n = (I + S + P_2)A$$

Or,
$$A = (I + S + P_2)^{-1} M_n - (I + S + P_2)^{-1} N_n$$

We take $M_2 = (I + S + P_2)^{-1} M_n$ and $N_2 = (I + S + P_2)^{-1} N_n$.

Then we can easily see that $M_2^{-1} N_2 = M_n^{-1} N_n$ and so $\rho(M_2^{-1} N_2) < 1$.

Again, $M_2^{-1} = M_n^{-1}(I + S + P_2) = (I - D'' - D''')^{-1}(I + S + P_2) \geq (I - D'' - D''')^{-1} \geq I^{-1} = M^{-1}$

i.e. $M_2^{-1} \geq M^{-1}$ and $A = M_2 - N_2 = M - N$ be two convergent and regular splittings.

Thus it follows from Lemma 3.9., that $\rho(M_2^{-1} N_2) \leq \rho(M^{-1} N) < 1$. Hence

$\rho(M_n^{-1} N_n) \leq \rho(M^{-1} N) < 1$

i.e.
$$\rho(T_n) \leq \rho(T) < 1.$$

V. Comparison Of Numerical Results

In this section, we give two examples to illustrate the theory in section 4.

Example 5.1. Let us consider the matrix A

$$A = \begin{pmatrix} 1 & -0.0 & -0.2 & -0.6 \\ -0.1 & 1 & -0.1 & -0.5 \\ -0.3 & -0.1 & 1 & -0.1 \\ -0.4 & -0.3 & -0.1 & 1 \end{pmatrix}$$

By using the preconditioners $I + S_{max} + P_1$ and $I + S + P_2$, we have the following preconditioned system matrices:

$$A_m = \begin{pmatrix} 0.76 & -0.18 & -0.26 & 0.00 \\ -0.30 & 0.85 & -0.15 & 0.00 \\ -0.34 & -0.13 & 0.99 & 0.00 \\ 0.00 & -0.30 & -0.18 & 0.76 \end{pmatrix}$$

and

$$A_n = \begin{pmatrix} 0.76 & -0.18 & -0.26 & 0.00 \\ -0.13 & 0.99 & 0.00 & -0.51 \\ -0.34 & -0.13 & 0.99 & 0.00 \\ -0.40 & -0.30 & -0.10 & 1 \end{pmatrix}$$

By computation using MATLAB R12, we have

$$\rho(T) = 0.7361, \quad \rho(T_m) = 0.5303, \quad \rho(T_n) = 0.6241$$

Clearly, $\rho(T_m) < \rho(T)$ and $\rho(T_n) < \rho(T)$ holds.

Example 5.2. We consider the following matrix

$$A = \begin{pmatrix} 1 & -0.2 & -0.3 & -0.1 & -0.2 \\ -0.1 & 1 & -0.1 & -0.3 & -0.1 \\ -0.2 & -0.1 & 1 & -0.1 & -0.2 \\ -0.2 & -0.1 & -0.1 & 1 & -0.3 \\ -0.1 & -0.2 & -0.2 & -0.1 & 1 \end{pmatrix}$$

The preconditioned system matrices for the preconditioners $I + S_{max} + P_1$ and $I + S + P_2$ are

$$A_m = \begin{pmatrix} 0.94 & -0.23 & 0.00 & -0.13 & -0.26 \\ -0.16 & 0.97 & -0.13 & 0.00 & -0.19 \\ -0.22 & -0.14 & 0.96 & -0.12 & 0.00 \\ -0.23 & -0.16 & -0.16 & 0.97 & 0.00 \\ 0.00 & -0.22 & -0.23 & -0.11 & 0.98 \end{pmatrix}$$

and

$$A_n = \begin{pmatrix} 0.96 & -0.04 & -0.36 & -0.18 & -0.02 \\ -0.12 & 0.99 & 0.00 & -0.31 & -0.12 \\ -0.22 & -0.11 & 0.99 & 0.00 & -0.23 \\ -0.23 & -0.16 & -0.16 & 0.97 & 0.00 \\ -0.10 & -0.20 & -0.20 & -0.10 & 1 \end{pmatrix}$$

After computation by MATLAB R12, we have

$$\rho(T) = 0.6551, \quad \rho(T_m) = 0.5578, \quad \rho(T_n) = 0.5832$$

Obviously, $\rho(T_m) < \rho(T)$ and $\rho(T_n) < \rho(T)$ holds.

VI. Conclusion

In this paper, we have proposed two preconditioners ($I + S_{max} + P_1$ and $I + S + P_2$) for modified Jacobi method and compared with classical Jacobi method. The comparison theorems and numerical experiments showed that the two modified Jacobi methods are superior to the classical Jacobi method.

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