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A Way Of Proving Fermat's Last Theorem And Beal Conjecture

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Abstract
The article gives a new proof of Fermat's last theorem. The proof is based on the study of the properties of natural numbers, an analysis of the constraints on the proposed solutions, and uses some general theorems on the roots of algebraic equations. The connection between Fermat's theorem and Beal conjecture is discussed. Beal conjecture is proved by induction using the same reasoning as in the proof of Fermat's theorem.

Keywords: *number theory, natural numbers, Fermat's theorem, Beal conjecture*

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I. Introduction

Great (last) Fermat's theorem was formulated over 300 years ago. In view of the significance of the problem in many areas of mathematic, large, but unsuccessful efforts have been made to prove it. Finally, a proof of the theorem was given in [1], based on the connection of the theorem with the properties of modular elliptic curves. The proof is too complicated, so attempts were made to find a simple proof of the theorem. The Beal conjecture is an open problem. We will prove both statements by induction, using well-known theorems about the roots of an algebraic equation. The theorem of P. Fermat, as is known, asserts that equation

$$x^p + y^p = z^p \tag{1}$$

has no positive integer solutions for $p > 2$. The purpose of this paper is to give a simple proof of Fermat's theorem within the framework of the elementary number theory. Define a few terms that we will use. The quantities x, y, z and their values are called bases of degree, or briefly bases. The bases from the first ten we call elementary bases. The set (combination) of three bases (x, y, z) we call a triplet or, accordingly, an elementary triplet. Thus, equation (1) depends on three bases and an exponent. For definiteness, let $x < y$, i.e. we assume that x is the base taking the least value. We put $y - x = u$, $z - x = v$, then equation (1) will depend on one base x, the exponent p, and two parameters u and v. It is easy to see that the parameters u and v do not change, if all three bases are increased by an arbitrary number a. Hereinafter (see sections III, IV, V) we use this technique, because it allows us to simplify the analysis of equation (1) and apply the Descartes' rule of signs to determine the position of the roots of the equation.

II. Restrictions on possible solutions of the equation and admissible transformations

Consider restrictions on the possible solutions of equation (1). Analysis of the restrictions allows us to establish the conditions that the bases and exponent must satisfy, so that they can be solutions of equation (1). Let us formulate the first restriction. Put for definiteness that $x < y$, i.e. x always means the smallest number on the left. Since the numbers x, y, z are all different, we have the following inequality:

$$x^p + y^p < (x+y)^p . \tag{2}$$

If a is a positive integer, then *a fortiori*

$$x^p + y^p < (x+y+a)^p . \tag{3}$$

From inequalities (2), (3) and the form of equation (1), *the first restriction* for numbers as possible solutions of equation (1) follows

$$\max(x, y) < z < (x+y) . \tag{4}$$

The second restriction is associated with the obvious requirement that the number $x^p + y^p$ ends in the same digit as the number z^p . In particular, it follows that the left and right sides of equation (1) must be of the same parity. Let us formulate *the third restriction*. If the following relation holds

$$x^2 + y^2 \le z^2 \ , \tag{5}$$

then x, y, z are not solutions of the basic equation (1). In this case, strict inequalities hold

$x^3 + y^3 < z^3$,

$x^4 + y^4 < z^4$,...

$x^p + y^p < z^p$. (6)

Indeed, multiplying (5) by z and using the left-hand side of inequality (4), we have $z^3 \geq zx^2 + zy^2 > x^3 + y^3$. Multiplying this inequality by z and using (4), we obtain $z^4 > z(x^3 + y^3) > x^4 + y^4$ etc. *The fourth restriction* is the equality of the exponents of all components in equation (1). The restrictions formulated are necessary conditions for numbers to be solutions of the basic equation (1). They are quite strong and allow us to select the proposed solutions for equation (1). *The second and fourth restrictions we call basic*, since their fulfillment is an unconditional requirement. The first and third restrictions are auxiliary and their implementation can be ensured through transformations (see below). Now consider the permissible transformations that keep safe the restrictions. As the starting point, we take the elementary bases, i.e. natural numbers from the first ten that satisfy the basic restrictions. Such transformations include:

1. Multiplication of all bases of degree in (1) by a positive integer q = 2, 3, 4, etc. Since the starting point is elementary bases, division is excluded.
2. An increase of one, two or all three bases by a number $a = 10l$ that is a multiple of 10, where l = 1, 2, 3, etc. Since the starting point is elementary bases, the first increase is 10.
3. If two triplets of numbers x, y, z and x', y', z' satisfy the second restriction, then the triplet of numbers (bases) { $x'' = x^p + x'^p$, $y'' = y^p + y'^p$, $z'' = z^p + z'^p$ }satisfies this restriction.

The first transformation is useful for obtaining from the known solution all solutions of the same class. For example, it can be used to obtain solutions of equation (1) for p = 2. This transformation does not change the "status" of the triplet, i.e. if the triplet is not a solution of (1), then after the transformation it will not be a solution of (1). Therefore, in proving the theorem without loss of generality, it is sufficient to consider only *prime* triplets, namely, those in which the bases do not have a common divisor different from 1. The second transformation is used to provide the first restriction (4) if it does not hold for elementary bases, but the second restriction holds. The third transformation can be applied only if some solutions are known, for example, when solving equation (1) for p = 2. Thus, the main "generator" of allowed combinations of numbers (triplets) is the second transformation; it allows us to go over all the numbers that are admissible by restrictions. We take a detailed look at the second restriction, for which we analyze the degrees of "elementary" numbers (bases) from 1 to 10, starting with degree 3. We have replaced 10 by 0, so as not to violate the uniformity of the representation (see below). The results are given in table 1. It follows from the data of table 1 that the repetition period of the last digit for bases 2, 3, 7 and 8 is 4, for bases 4 and 9 the period is 2, for bases 5, 6, 0 and 1, the period is 1. Now we consider combinations of powers of different bases, taking into account the basic restrictions. The analysis is performed in the following order. First, we consider combinations of numbers with period 4, i.e., powers of 2 are combined consistently with the degrees of the numbers 3, 4,..., 0, 1, then the number 3 is combined with the remaining, the number 7 – with the remaining, the number 8 – with the remaining. After that, the bases with period 2 are combined, i.e. number 4 – with the numbers 5, 6, 9, 0 and 1, then the number 9 – with 5, 6, 0 and 1. Lastly, the numbers with period 1 are combined, i.e. number 5 with the numbers 6, 0 and 1, the number 6 – with 0 and 1, the number 0 – with 1. Some of the combinations can be immediately excluded due to violation of the second restriction. In addition, we excluded some combinations that are not used in the preliminary analysis and proof of the lemmas (see below). For example, we have excluded combinations indifferent to the exponent of the form $0^p + 0^p = 0^p$, $1^p + 0^p = 1^p$, $2^p + 0^p = 2^p$ etc., $5^p + 5^p = 0^p$, as well as trivial combinations of the form $1^{5+4k} + 1^{5+4k} = 2^{5+4k}$, $2^{5+4k} + 2^{5+4k} = 4^{5+4k}$, $3^{5+4k} + 3^{5+4k} = 6^{5+4k}$, $4^{5+4k} + 4^{5+4k} = 8^{5+4k}$, $6^{5+4k} + 6^{5+4k} = 2^{5+4k}$, $7^{5+4k} + 7^{5+4k} = 4^{5+4k}$, $8^{5+4k} + 8^{5+4k} = 6^{5+4k}$, $9^{5+4k} + 9^{5+4k} = 8^{5+4k}$ and some others. Selected results are given in table 2. The data of tables 1, 2 are used to select the permissible combinations of bases (triplets) that satisfy the formulated restrictions.

III. Proof of auxiliary statements

We prove Fermat's theorem for the initial triplets (x, y, z) in which x takes values from 1 to 10 (Lemma 1), and for additional triplets in which x takes values from 11 to 14. We use these classes of triplets as particular case in the general proof of the theorem. Lemma 1. *Triplet (x, y, z), where x is an elementary base (x < y < z), y and z are arbitrary positive integers, cannot be a solution of equation (1) for p > 2.* The validity of the lemma is established by direct verification. The number of admissible triplets turns out to be finite and if we take into account the restrictions formulated above, then the verification procedure ends rather quickly. We must consistently consider triplets (x, y, z), for which x is an elementary base and $z-y=1$, $z-y=2$, $z-y=3$, etc. We call the triplet (x, y, z) boundary triplet if it satisfies equation (1) for p = 2, i.e. it is a solution of the quadratic Fermat equation. If equation $x^2+y^2=z^2$ is satisfied for the triplet (x, y, z), then the verification procedure ends. From (5) it is easy to obtain a relation that allows us to determine the number of admissible triplets. We have

$$x^2 > 2(x+u)m + m^2 \tag{5a}$$

Or

$$u < (x^2 - m^2)/2m - x, \tag{5b}$$

where x – elementary base, $u = y - x$, $m = z - y$; u = 1, 2, 3, etc.; m = 1, 2, 3, etc. Relation (5b) determines the upper limit of the variation of u. The equality of the left and right sides in (5b) corresponds to the boundary triplet. If x is an odd number, then only such triplets are allowed, in which y and z have different parity, which follows from the basic restrictions. Therefore, the difference $m = z - y$ can take only odd values, i.e. 1, 3, 5 and so on. If x is an even number, then only such triplets are allowed, in which y and z have the same parity, which follows from the basic restrictions. So, m can take only even values, i.e. 2, 4, 6 and so on. Now consider the procedure for obtaining admissible triplets. It is clear that triplets of the form (1, y, z) and (2, y, z) cannot be solutions of equation (1), since conditions (5) are certainly satisfied for them. Moreover, for triplets of the form (1, y, z) the relation $1 + y \le z$ holds. For example, 1 + 2 = 3, 1 + 12 < 15, etc. If y and z are both even, the triplets of the form (2, y, z) after dividing all the components by 2 reduce to triplets of the form (1, y, z), for example, (2, 4, 6) reduces to (1, 2, 3) etc. If y and z are both odd, then for the triplets (2, y, z) the relation $2 + y \le z$ holds, for example, 2 + 3 = 5, 2 + 15 < 19, etc. For triplets of the form (1, y, z) and (2, y, z), there are no boundary triplets. We start with the triplet (3, 4, 5). Since it is boundary triplet, it satisfies relation (5). Therefore, triplets of the form (3, y, z), where y > 4, z > 5, cannot be solutions of equation (1), so the verification procedure ends. Consider the following triplets of the form (4, y, z). For x = 4 and m = 2, we obtain from (5b) the inequality u < 0, so there are no such triplets. Note that although triplets of the form (1, y, z), (2, y, z), (3, y, z), (4, y, z) cannot be solutions of equation (1), but from them it is possible to obtain admissible triplets by successively increasing their bases simultaneously by a = 10l, where l = 1, 2, 3, etc. (see below). Consider the following triplets of the form (5, y, z) in more detail to show how restrictions are used. For x = 5 and m = 1, we obtain from (5b) the inequality u < 7, i.e. there are 6 such triplets. The value u = 7 corresponds to the boundary triplet (5, 12, 13). We write these triplets explicitly (5, 6, 7), (5, 7, 8), (5, 8, 9), (5, 9, 10), (5, 10, 11), (5, 11, 12). The triplets (5, 9, 10) and (5, 10, 11) can be excluded, since they do not satisfy the second (basic) restriction. Let m = 3. It follows from (5b) that u < 0, so there are no such triplets. In the future, when considering other bases, we will write down only the result of calculations. Consider the following triplets of the form (6, y, z). For x = 6 and m = 2, we obtain from (5b) the inequality u < 2, so there is only one such triplet (6, 7, 9). The triplet (6, 8, 10) = 2(3, 4, 5) is the boundary triplet. Consider the following triplets of the form (7, y, z). For x = 7 and m = 1, we obtain from (5b) the inequality u < 17, so there are 16 such triplets. The triplet (7, 24, 25) is the boundary triplet. For x = 7 and m = 3, we obtain from (5b) the inequality u < 0, so there are no such triplets. In fact, the number of permissible triplets is much smaller if we take into account the basic restrictions (see below). Consider the triplets of the form (9, y, z). For x = 9 and m = 1, we obtain from (5b) the inequality u < 31, so there are 30 such triplets. The triplet (9, 40, 41) is the boundary triplet. The number of triplets is rather large, but the analysis is simplified, because only triplets can be considered, in which y changes within the interval 10 ... 19, since the last digits are repeated in every ten. For the remaining triplets, the degrees allowed according to the basic restrictions will be repeated (see below). After exclusion, there remain twelve triplets. For x = 9 and m = 3, we obtain from (5b) the inequality u < 3, so there are 2 such triplets. One triplet is excluded due to the basic restrictions, so there is only one permissible triplet. The boundary triplet is (9, 12, 15) = 3(3, 4, 5). For x = 9 and m = 5, we obtain from (5b) the inequality u < 0, so there are no such triplets. Consider the triplets of the form (8, y, z). For x = 8 and m = 2, we obtain from (5b) the inequality u < 7, so there are 6 such triplets. Two triplets (8, 11, 13) and (8, 13, 15) can be excluded because of the basic restrictions. It should also be taken into account that if all bases in the triplet are even, then such triplet can be excluded, since after dividing all the components by 2 we obtain triplet considered earlier. For example, (8, 10, 12) reduces to (4, 5, 6), (8, 12, 14) to (4, 6, 7), etc. So there is only one permissible triplet (see below). The triplet (8, 15, 17) is the boundary triplet. For x = 8 and m = 4, we obtain from (5b) the inequality u < 0, so there are no such triplets. Consider the triplets of the form (10, y, z). For x = 10 and m = 2, we obtain from (5b) the inequality u < 14, so there are 13 such triplets. In view of the foregoing, after exclusion, there remain four triplets. The boundary triplet is (10, 24, 26) = 2(5, 12, 13). For x = 10 and m = 4, we obtain from (5b) the inequality u < 84/8 − 10, so there are no such triplets. The results are given in table 3. It is easy to verify by direct verification that the admissible triplets from table 3 are not solutions of equation (1). For example, for triplet (5, 6, 7) we have $5^2 + 6^2 > 7^2$, but $5^3 + 6^3 < 7^3$, therefore, all other powers of these bases will give the same result (zero is passed), which follows from relations (4), (6). For triplet (5, 7, 8) we have $5^2 + 7^2 > 8^2$, but $5^3 + 7^3 < 8^3$ (zero is passed). For triplet (5, 8, 9) we have $5^2 + 8^2 > 9^2$, but $5^3 + 8^3 < 9^3$ (zero is passed). For triplet (5, 11, 12) we have $5^2 + 11^2 > 12^2$, but $5^3 + 11^3 < 12^3$ (zero is passed). For triplet (6, 7, 9) we obtain $6^2 + 7^2 > 9^2$, but $6^3 + 7^3 < 9^3$ (zero is passed). For triplet (7, 9, 10) we obtain $7^2 + 9^2 > 10^2$, $7^3 + 9^3 > 10^3$, but $7^4 + 9^4 < 10^4$ (zero is passed). For triplet (7, 10, 11) we have $7^2 + 10^2 > 11^2$, $7^3 + 10^3 > 11^3$, but $7^4 + 10^4 < 11^4$ (zero is passed). For triplet (7, 20, 21) we have $7^2 + 20^2 > 21^2$, but $7^3 + 20^3 < 21^3$ (zero is passed). For triplet (8, 9, 11) we have $8^2 + 9^2 > 11^2$, but $8^3 + 9^3 < 11^3$ (zero is passed). For triplet (9, 10, 11) we obtain $9^2 + 10^2 > 11^2$, $9^3 + 10^3 > 11^3$, $9^4 + 10^4 > 11^4$, but $9^5 + 10^5 < 11^5$ (zero is passed). For triplet (9, 37, 38) we obtain $9^2 + 37^2 > 38^2$, but $9^3 + 37^3 < 38^3$ (zero is passed). For triplet (10, 11, 13) we have $10^2 + 11^2 > 13^2$, $10^3 + 11^3 > 13^3$, but $10^4 + 11^4 < 13^4$ (zero is passed). For triplet (10, 21, 23) we have $10^2 + 21^2 > 23^2$, but $10^3 + 21^3 < 23^3$. Similarly, the check is performed for the remaining triplets. Lemma 1 is proved. Hereinafter the triplets from table 3 will be called *initial triplets*. Triplets of the form (1, y, z), (2, y, z), (3, y, z) and (4, y, z) we have excluded, so as they do not satisfy the restrictions. However, if all the bases of these triplets are increased by 10, then we obtain permissible triplets that satisfy all the restrictions. Hereinafter these triplets will be called *additional triplets*. Prove Lemma 2. *Additional triplets cannot be solutions of equation (1) for $p > 2$.* The analysis of additional triplets was performed in the same way as for the initial triplets. We write them explicitly using the following notation: (p; x, y, z; p_{th}; u, v, m), where p is the allowed exponent, p_{th} is the threshold exponent, at which the difference between the left and the right sides of equation (1) changes sign from plus to minus; $u = y - x$, $v = z - x$, $m = z - y = v - u$. The number of permissible triplets of the form (11, y, z) is 51, of which 48 are triplets with m = 1 and 3 triplets with m = 3. For m = 1 we have triplets (5+4k; 11, 12, 13; 6; 1, 2, 1), (5+4k; 11, 13, 14; 5; 2, 3, 1), (3+2k; 11, 14, 15; 5; 3, 4, 1), (3+k; 11, 15, 16; 4; 4, 5, 1), (5+4k; 11, 16, 17; 4; 5, 6, 1), (5+4k; 11, 17, 18; 4; 6, 7, 1), (5+4k; 11, 18, 19; 4; 7, 8, 1), (3+2k; 11, 19, 20; 4; 8, 9, 1), (3+k; 11, 20, 21; 4; 9, 10, 1), (5+4k; 11, 21, 22; 3; 10, 11, 1), (5+4k; 11, 22, 23; 3; 11, 12, 1), (5+4k; 11, 23, 24; 3; 12, 13, 1), (3+2k; 11, 24, 25; 3; 13, 14, 1), (3+k; 11, 25, 26; 3; 14, 15, 1), (5+4k; 11, 26, 27; 3; 15, 16, 1), (5+4k; 11, 27, 28; 3; 16, 17, 1), (5+4k; 11, 28, 29; 3; 17, 18, 1), (3+2k; 11, 29, 30; 3; 18, 19, 1), (3+k; 11, 30, 31; 3; 19, 20, 1), (5+4k; 11, 31, 32; 3; 20, 21, 1), (5+4k; 11, 32, 33; 3; 21, 22, 1), (5+4k; 11, 33, 34; 3; 22, 23, 1), (3+2k; 11, 34, 35; 3; 23, 24, 1), (3+k; 11, 35, 36; 3; 24, 25, 1), (5+4k; 11, 36, 37; 3; 25, 26, 1), (5+4k; 11, 37, 38; 3; 26, 27, 1), (5+4k; 11, 38, 39; 3; 27, 28, 1), (3+2k; 11, 39, 40; 3; 28, 29, 1), (3+k; 11, 40, 41; 3; 29, 30, 1), (5+4k; 11, 41, 42; 3; 30, 31, 1), (5+4k; 11, 42, 43; 3; 31, 32, 1), (5+4k; 11, 43, 44; 3; 32, 33, 1), (3+2k; 11, 44, 45; 3; 33, 34, 1), (3+k; 11, 45, 46; 3; 34, 35, 1), (5+4k; 11, 46, 47; 3; 35, 36, 1), (5+4k; 11, 47, 48; 3; 36, 37, 1), (5+4k; 11, 48, 49; 3; 37, 38, 1), (3+2k; 11, 49, 50; 3; 38, 39, 1), (3+k; 11, 50, 51; 3; 39, 40, 1), (5+4k; 11, 51, 52; 3; 40, 41, 1), (5+4k; 11, 52, 53; 3; 41, 42, 1), (5+4k; 11, 53, 54; 3; 42, 43, 1), (3+2k; 11, 54, 55; 3; 43, 44, 1), (3+k; 11, 55, 56; 3; 44, 45, 1), (5+4k; 11, 56, 57; 3; 45, 46, 1), (5+4k; 11, 57, 58; 3; 46, 47, 1), (5+4k; 11, 58, 59; 3; 47, 48, 1), (3+2k; 11, 59, 60; 3; 48, 49, 1). The triplet (11, 60, 61) is the boundary triplet. For m = 3 we have triplets (6+4k: 11, 12, 15; 3; 1, 4, 3), (4+4k: 11, 15, 18; 3; 4, 7, 3), (6+4k: 11, 17, 20; 3; 6, 9, 3). The number of permissible triplets of the form (12, y, z) is 12, of which 11 are triplets with m = 2 and 1 triplet with m = 4. For m = 2 we have triplets (3+2k; 12, 13, 15; 4; 1, 3, 2), (3+k; 12, 15, 17; 4; 3, 5, 2), (5+4k; 12, 17, 19; 3; 5, 7, 2), (5+4k; 12, 19, 21; 3; 7, 9, 2), (5+4k; 12, 21, 23; 3; 9, 11, 2), (3+2k; 12, 23, 25; 3; 11, 13, 2), (3+k; 12, 25, 27; 3; 13, 15, 2), (5+4k; 12, 27, 29; 3; 15, 17, 2), (5+4k; 12, 29, 31; 3; 17, 19, 2), (5+4k; 12, 31, 33; 3; 19, 21, 2), (3+2k; 12, 33, 35; 3; 21, 23, 2). The triplet (12, 35, 37) is the boundary triplet. For m = 4 we have triplet (4+4k; 12, 15, 19; 3; 3, 7, 4). The triplet (12, 16, 20) = 4(3, 4, 5) is the boundary triplet. The number of permissible triplets of the form (13, y, z) is 69, of which 56 are triplets with m = 1 and 13 triplets with m = 3. For m = 1 we have triplets (6+4k; 13, 14, 15; 7; 1, 2, 1), (4+4k; 13, 15, 16; 6; 2, 3, 1), (3+4k; 13, 16, 17; 6; 3, 4, 1), (3+4k; 13, 18, 19; 5; 5, 6, 1), (6+4k; 13, 19, 20; 4; 6, 7, 1), (4+4k; 13, 20, 21; 4; 7, 8, 1), (3+4k; 13, 21, 22; 4; 8, 9, 1), (3+4k; 13, 23, 24; 4; 10, 11, 1), (6+4k; 13, 24, 25; 4; 11, 12, 1), (4+4k; 13, 25, 26; 4; 12, 13, 1), (3+4k; 13, 26, 27; 4; 13, 14, 1), (3+4k; 13, 28, 29; 3; 15, 16, 1), (6+4k; 13, 29, 30; 3; 16, 17, 1), (4+4k; 13, 30, 31; 3; 17, 18, 1), (3+4k; 13, 31, 32; 3; 18, 19, 1), (3+4k; 13, 33, 34; 3; 20, 21, 1), (6+4k; 13, 34, 35; 3; 21, 22, 1), (4+4k; 13, 35, 36; 3; 22, 23, 1), (3+4k; 13, 36, 37; 3; 23, 24, 1), (3+4k; 13, 38, 39; 3; 25, 26, 1), (6+4k; 13, 39, 40; 3; 26, 27 1), (4+4k; 13, 40, 41; 3; 27, 28, 1), (3+4k; 13, 41, 42; 3; 28, 29, 1), (3+4k; 13, 43, 44; 3; 30, 31, 1), (6+4k; 13, 44, 45; 3; 31, 32, 1), (4+4k; 13, 45, 46; 3; 32, 33, 1), (3+4k; 13, 46, 47; 3; 33, 34, 1), (3+4k; 13, 48, 49; 3; 35, 36, 1), (6+4k; 13, 49, 50; 3; 36, 37, 1), (4+4k; 13, 50, 51; 3; 37, 38, 1), (3+4k; 13, 51, 52; 3; 38, 39, 1), (3+4k; 13, 53, 54; 3; 40, 41, 1), (6+4k; 13, 54, 55; 3; 41, 42, 1), (4+4k; 13, 55, 56; 3; 42, 43, 1), (3+4k; 13, 56, 57; 3; 43, 44, 1), (3+4k; 13, 58, 59; 3; 45, 46, 1), (6+4k; 13, 59, 60; 3; 46, 47, 1), (4+4k; 13, 60, 61; 3; 47, 48, 1), (3+4k; 13, 61, 62; 3; 48, 49, 1), (3+4k; 13, 63, 64; 3; 50, 51, 1), (6+4k; 13, 64, 65; 3; 51, 52, 1), (4+4k; 13, 65, 66; 3; 52, 53, 1), (3+4k; 13, 66, 67; 3; 53, 54, 1), (3+4k; 13, 68, 69; 3; 55, 56, 1), (6+4k; 13, 69, 70; 3; 56, 57, 1), (4+4k; 13, 70, 71; 3; 57, 58, 1), (3+4k; 13, 71, 72; 3; 58, 59, 1), (3+4k; 13, 73, 74; 3; 60, 61, 1). (6+4k; 13, 74, 75; 3; 61, 62, 1), (4+4k; 13, 75, 76; 3; 62, 63, 1), (3+4k; 13, 76, 77; 3; 63, 64, 1), (3+4k; 13, 78, 79; 3; 65, 66, 1), (6+4k; 13, 79, 80; 3; 66, 67, 1), (4+4k; 13, 80, 81; 3; 67, 68, 1), (3+4k; 13, 81, 82; 3; 68, 69, 1), (3+4k; 13, 83, 84; 3; 70, 71, 1). The triplet (13, 84, 85) is the boundary triplet. For m = 3 we have triplets (5+4k; 13, 14, 17; 4; 1, 4, 3), (3+k; 13, 15, 18; 3; 2, 5, 3), (5+4k; 13, 16, 19; 3; 3, 6, 3), (3+2k; 13, 17, 20; 3; 4, 7, 3), (5+4k; 13, 18, 21; 3; 5, 8, 3), (5+4k; 13, 19, 22; 3; 6, 9, 3), (3+k; 13, 20, 23; 3; 7, 10, 3), (5+4k; 13, 21, 24; 3; 8, 11, 3), (3+2k; 13, 22, 25; 3; 9, 12, 3), (5+4k; 13, 23, 26; 3; 10, 13, 3), (5+4k; 13, 24, 27; 3; 11, 14, 3), (3+k; 13, 25, 28; 3; 12, 15, 3), (5+4k; 13, 26, 29; 3; 13, 16, 3). The number of permissible triplets of the form (14, y, z) is 10, of which 7 are triplets with m = 2 and 3 triplets with m = 4. For m = 2 we have triplets (4+4k; 14, 15, 17; 5; 1, 3, 2), (6+4k; 14, 23, 25; 3; 9, 11, 2), (4+4k; 14, 25, 27; 3; 11, 13, 2), (6+4k; 14, 33, 35; 3; 19, 21, 2), (4+4k; 14, 35, 37; 3; 21, 23, 2), (6+4k; 14, 43, 45; 3; 29, 31, 2), (4+4k; 14, 45, 47; 3; 31, 33, 2). The triplet (14, 48, 50) = 2(7, 24, 25) is the boundary triplet. For m = 4 we have triplets (3+4k; 14, 15, 19; 3; 1, 5, 4), (5+4k; 14, 17, 21; 3; 3, 7, 4), (3+4k; 14, 21, 25; 3; 7, 11, 4). Direct verification shows that the additional triples are not solutions of equation (1). So Lemma 2 is proved. We summarize the results of the analysis of the initial and additional triplets. For the initial triplets from table 3, the permissible exponent can take the values $p = 3 + 4k$, $4 + 2k$, $4 + 4k$ or $6 + 4k$; the smallest permissible exponent takes the values $p_{\min}$ = 3, 4 or 6; the repetition period of permissible ends takes the values b = 2 or 4; threshold exponent p_{th} = 3, 4 or 5. For additional triplets, we have $p = 3 + k$, $3 + 2k$, $3 + 4k$, $4 + 4k$, $5 + 4k$ or $6 + 4k$; $p_{\min}$ = 3, 4, 5 or 6; b = 1, 2 or 4; p_{th} = 3, 4, 5, 6 or 7. For most initial and additional triplets, $p_{\min} \geq p_{\text{th}}$. The exceptions are initial triplets (9, 10, 11), (9, 12, 13) and additional triplets (11, 12, 13), (11, 14, 15), (11, 19, 20), (11, 20, 21), (12, 13, 15), (12, 15, 17), (13, 14, 15), (13, 15, 16), (13, 16, 17), (13, 18, 19), (13, 21, 22), (13, 23, 24), (13, 26, 27), (14, 15, 17), for which the value of the exponent p_0 closest to $p_{\min}$ should be taken, taking into account the period, namely $p_0 = p_{\min} + b$, where b is the period (b = 1, 2 or 4). In this case the inequality $p_0 \geq p_{\text{th}}$ will be satisfied (see above). For example, for a triplet (9, 10, 11) $p_{\min} = 4$, $p_{\text{th}} = 5$, so we should take the exponent $p_0 = 4 + 2 = 6$, where 2 is period. For a triplet (9, 12, 13) $p_{\min} = 3$, and $p_{\text{th}} = 4$, therefore it is necessary to take $p_0 = 3 + 4 = 7$, where 4 is period (see table 3). The exponent for other such triplets is selected in the same way.

All admissible triplets can be obtained from the initial triplets of table 3 and additional triplets, successively increasing all bases of these triplets by $a = 10l$. Simple reasoning confirms this statement. Indeed. If the bases x and y do not change or only one of them y ($y > x$) increases by a, and simultaneously z increases by a, then either we obtain the already taken into account initial or additional triplets or condition (5) is satisfied for the obtained triplets, i.e. the third restriction is violated, since we go beyond the boundary triplet, and such triplets are excluded (see also (7), Section IV). If z does not change, and y ($y > x$) increases by a, then condition (4) is not satisfied, i.e. the first restriction is violated and such triplets are excluded. Therefore, so that the restrictions are not violated, all bases must increase by the same value a. In triplets obtained from the same initial or additional triplet, the bases have the same ends, the same permissible exponent p, and the same values of the parameters u and v. Therefore, the set of admissible triplets is divided into groups, in each of which the triplets have the same values of p, u and v.

IV. Study of the function $F(p; x, y, z) = x^p + y^p - z^p$

We study the properties of the function $F(p; x, y, z) = x^p + y^p - z^p$. On the set of natural numbers, it takes discrete values and the roots of this function are solutions of equation (1). From the previous analysis (see the proof of Lemmas 1 and 2), we have Corollary 1: *For initial and additional triplets $F > 0$ for $p < p_{th}$ and $F < 0$ for $p \geq p_{th}$. For triplets, which are exceptions, $F > 0$ for $p < p_0$ and $F < 0$ for $p \geq p_0$.* Therefore, the function takes negative values for all admissible exponent $p \geq p_{\min}$ (or $p \geq p_0$). The equality of the function to zero does not occur at natural values of x, y, z. To continue the analysis, we write a formal representation for the function $F(p; x + a, y + a, z + a)$. We have for arbitrary fixed p

$$F(p; x+a, y+a, z+a) \equiv (x+a)^p + (y+a)^p - (z+a)^p = (x^p + y^p - z^p) + C_p^1(x^{p-1} + y^{p-1} - z^{p-1})a + C_p^2(x^{p-2} + y^{p-2} - z^{p-2})a^2 + \ldots + C_p^{p-2}(x^2 + y^2 - z^2)a^{p-2} + , C_p^{p-1}(x + y - z)a^{p-1} + a^p \quad (6)$$

where $a = 10l$, l = 1, 2, 3, etc. Obviously, when $a = 0$, $F(p; x + a, y + a, z + a) = F(p; x, y, z)$. If only one base on the left side of equation (1) is increased by $a = 10l$, then (6) takes the form

$$(x+a)^p + y^p - (z+a)^p = (x^p + y^p - z^p) + C_p^1(x^{p-1} - z^{p-1})a + C_p^2(x^{p-2} - z^{p-2})a^2 + \ldots + C_p^{p-2}(x^2 - z^2)a^{p-2} + C_p^{p-1}(x - z)a^{p-1} , \quad (7)$$

where $y = x + u$, $z = x + v$. If $p \geq p_{\text{th}}$ (or $p \geq p_0$), all terms in the right-hand side of (7) are less than zero, therefore equality is impossible, and such bases can be ignored, since they cannot be solutions of equation (1). *A fortiori* this conclusion is valid if both bases x and y do not change. We represent the functions $F(p; x + a,$

$y + a$, $z + a$) and $F(p; x, y, z)$ in canonical form in decreasing powers of one base x, treating x as a variable, and u, v as parameters. Since u and v do not change with an increase in all bases on a, such a representation is convenient for analysis and allows us to apply the Descartes' rule of signs. We write the equation for finding the roots of the function $F(p; x + a, y + a, z + a)$

$$F(p;x+a,y+a,z+a) \equiv (x+a)^p + (y+a)^p - (z+a)^p = F(p;x+a,u,v) = x^p +$$
$$x^{p-1}[C_p^1 a + C_p^1 (u-v)] + x^{p-2}[C_p^2 a^2 + C_p^1 C_{p-1}^1 a(u-v) + C_p^2 (u^2 - v^2)] +$$
$$x^{p-3}[C_p^3 a^3 + C_p^1 C_{p-1}^2 a^2 (u-v) + C_p^2 C_{p-2}^1 a(u^2 - v^2) + C_p^3 (u^3 - v^3)] + \ldots +$$
$$x[C_p^{p-1} a^{p-1} + C_p^1 C_{p-1}^{p-2} a^{p-2} (u-v) + C_p^2 C_{p-2}^{p-3} a^{p-3} (u^2 - v^2) +$$
$$C_p^3 C_{p-3}^{p-4} a^{p-4} (u^3 - v^3) + \ldots + C_p^{p-1} (u^{p-1} - v^{p-1})] + [a^p + C_p^1 a^{p-1} (u-v) +$$
$$C_p^2 a^{p-2} (u^2 - v^2) + \ldots + C_p^{p-1} a (u^{p-1} - v^{p-1}) + (u^p - v^p)] \quad (6a)$$

If in (6a) we put $a = 0$, then we obtain the equation for finding the roots of the initial function $F(p; x, y, z)$. We have

$$F(p;x,y,z) \equiv x^p + y^p - z^p = F(p;x,u,v) = x^p + C_p^1 x^{p-1}(u-v) + C_p^2 x^{p-2}(u^2 - v^2) + \ldots$$
$$+ C_p^{p-1} x (u^{p-1} - v^{p-1}) + (u^p - v^p) = 0 \quad (6b)$$

Equations (6a) and (6b) are equivalent to equation (1) for different values of bases. Find out what happens if we simultaneously increase all the bases of the triplet by the value of $a = 10l$. This increase must be simultaneous and by a multiple of 10 so that the restrictions stated above are not violated. With a simultaneous increase in the bases of the triplet, if the function F was positive, i.e. $F > 0$, then its sign does not change; if it was negative, i.e. $F < 0$, its sign changes to the opposite, i.e. will be $F > 0$. Let us explain this with an example. Take the triplet (5, 6, 7). From table 3 it follows that $5^4 + 6^4 - 7^4 < 0$ (here $p = 4$ is the smallest allowed exponent, $p_{th} = 3$ is the threshold exponent). If we increase all bases by 10, we get $15^4 + 16^4 - 17^4 > 0$, and the transition to the negative region occurs only when $p_{th} = 8$ (we have $5^8 + 6^8 - 7^8 < 0$. If only one or two bases are increased by 10, then the restrictions formulated above are violated. In our example, we have triplets (5, 6, 17), (5, 16, 17), (6, 15, 17), (15, 16, 7). It is easy to verify that these triplets do not satisfy the first or the third restriction, so they can be ignored. A similar result is observed for other triplets. The larger a, the greater must be the exponent p_{th}, at which F changes the sign (becomes negative). So, for arbitrary admissible fixed p, increasing the bases of initial and additional triplets by $a = 10l$, we can observe a change in the sign of the function $F(p, x, y, z)$ from minus to plus due to the fact that, as follows from (6), the quantity a^p will predominate over the rest of members. On the other hand, for arbitrary fixed base (or, which is the same, for arbitrary $a = 10l$), increasing the exponent p, we can observe a change in the sign of the function $F(p, x, y, z)$ from plus to minus due to the fact that, as follows from (6), the quantity z^p will predominate over the rest of members (as $z > \max(x, y)$). Below we show that the equality $F(p, x, y, z)$ to zero on the set of natural numbers is impossible for any fixed $a = 10l$ and for any fixed $p > 2$. We now consider the equation $F(p; x, y, z) = 0$, where the function $F(p; x, y, z)$ is given by expression (6b), in the field of real numbers; x is a variable that changes continuously. The characteristic (essential) parameters of this equation are p, u and v. In other words, in order to maintain succession (continuity) with the Fermat equation, it is necessary to consider different equations (6b) for different admissible values of p, u and v. The coefficients of equation (6b) have one change of sign; therefore, according to the Descartes' rule of signs, this equation has one positive real root. In our case, this conclusion does not depend on the parity of the number p, since the first and the last coefficients of the equation have different signs, namely, $a_0 = 1 > 0$, $a_n = (u^p - v^p) < 0$. From the previous analysis (see Lemmas 1, 2) it follows that for the initial and additional triplets, this root is not a natural number. It also cannot be a rational number, since otherwise it follows from the theory that it would be an integer [2], which is not the case. From (6) it follows that if $p < p_{th}$, then all terms in (6) are positive and the increase in a does not change the sign of the function. If $p = p_{th}$, then only the first term is negative, and the rest are positive. If $p = p_{th} + 1$, then the first two terms are negative. If $p = p_{th} + 2$, then the first three members are negative, and so on. The sign of the function depends on the value of a, which, in turn, depends on the value of u for a given p and m. So, when $p \geq p_{th}$, for all initial and additional triplets $F(p; x, u, v) < 0$, and $F(p; x + a, u, v) > 0$. Therefore, according to the well-known Weierstrass theorem the root of equation (6b) is located between x and $x + a$. It cannot be a natural or rational number, since in the interval from x to $x + a$ there are no admissible values of the bases or that the same there are no admissible values of x, u and v. We give the illustrative example. The triplet (5, 6, 7) from table 3 corresponds to three numbers $p = 4 + 4k$, $u = 1$, $v = 2$. Let $p = p_{min} = 4$, then this triplet is described by the function $F(4;\ x\ , 1, 2)$. Calculations by (6b) show that $F(4;\ 5\ , 1, 2) < 0$ and for any admissible p, it remains negative, since $p = 4 + 4k > p_{th} = 3$. Increase the base x by 10, then the calculations by (6a) give that $F(4;\ 15, 1, 2) > 0$.

V. Determining the position of real positive root of the function $F(p;\ x\ , y, z)$

The position of root can be determined more accurately using the Descartes' rule of signs, which can be useful for large a. We apply this rule to equation (6b), which is equivalent to (1), but depends on one variable x (u and v are parameters). The Descartes' rule allows us to determine the number of roots of equation (6b), exceeding a certain number a. In our case, $a = 10l$ (l = 1, 2, 3, etc.). Consider equation (6b), in which the change of variables is made $x \rightarrow x + a$. Equation (6b) transforms into (6a). In this case, as is known, all the roots of the initial equation (6b) are reduced by the same value a. It turns out that for all triplets (in particular, initial and additional), only two cases are possible, all the coefficients of equation (6a) are positive or they have one change of sign (from plus to minus). According to the Descartes' rule, in the first case all the roots of the initial equation (6b) are less than a, and in the second case there is one root larger than a. From the analysis of equation (6a) it follows that it is sufficient to determine the sign of the free term in (6a). In the first case, the sign is positive, and in the second, the sign is negative. We carried out calculations of the position of the roots for all permissible values of p, u and v by equation (6a) and additionally checked the results on the table of powers of numbers. Since the values of u and v remain constant when the bases simultaneously increase by a, the results have a general applicability. The sign of the free term depends on p and m, and for identical p and m on the value of u. We prove Lemma 3. *For initial and additional triplets, the real positive root of equation (6b) is located between $x + a - 10$ and $x + a$.* The validity of Lemma 3 is established by direct verification. We give the results of calculation for the initial and additional triplets. For initial triplets, value of a varies from 10 to 30. For all triplets of the form (5, y, z) with $m = 1$, $p_{min} = 4 > p_{th} = 3$ we obtain that $a = 10$. Moreover, it is easy to verify using (6a) that for triplets (5, 6, 7), (5, 7, 8) and (5, 8, 9) the root of (6b) is less than 10 and is located between 5 and 10. For triplet (5, 11, 12) the root is greater than 10 and is located between 10 and 15. For triplet (6, 7, 9) with $m = 2$, $p_{min} = 3 = p_{th}$ we get that $a = 10$ and the root of (6b) is less than 10 and is located between 6 and 10. For triplet (7, 9, 10) with $m = 1$, $p_{min} = 6 > p_{th} = 4$ we get that $a = 10$ and the root is located between 10 and 17. For triplet (7, 10, 11) with $m = 1$, $p_{min} = 4 = p_{th}$ we have $a = 10$ and the root is located between 7 and 10. For triplet (7, 14, 15) with $m = 1$, $p_{min} = 6 > p_{th} = 3$ we have $a = 20$ and the root is located between 17 and 27. For triplet (7, 15, 16) with $m = 1$, $p_{min} = 4 = p_{th}$ we have $a = 10$ and the root is located between 10 and 17. For triplet (7, 19, 20) with $m = 1$, $p_{min} = 6 > p_{th} = 3$ we have $a = 30$ and the root is located between 27 and 37. For triplet (7, 20, 21) with $m = 1$, $p_{min} = 4 > p_{th} = 3$ we have $a = 20$ and the root is located between 17 and 27. For triplet (8, 9, 11) with $m = 2$, $p_{min} = 3 = p_{th}$ we have $a = 10$ and the root is located between 8 and 10. For triplet (9, 10, 11) with $m = 1$, $p_0 = 6 > p_{th} = 5$ we have $a = 10$ and the root is located between 9 and 19. For triplet (9, 12, 13) with $m = 1$, $p_0 = 7 > p_{th} = 4$ we have $a = 20$ and the root is located between 19 and 29. For triplet (9, 15, 16) with $m = 1$, $p_{min} = 4 = p_{th}$ we have $a = 10$ and the root is located between 9 and 19. For triplet (9, 20, 21) with $m = 1$, $p_{min} = 4 > p_{th} = 3$ we have $a = 10$ and the root is located between 9 and 19. For triplets (9, 25, 26), (9, 30, 31), (9, 35, 36), with $m = 1$, $p_{min} = 4 > p_{th} = 3$ we have $a = 20$ and the root is located between 19 and 29. For triplets (9, 17, 18), (9, 22, 23), (9, 27, 28), (9, 32, 33), (9, 37, 38) with $m = 1$, $p_{min} = 3 = p_{th}$ we have $a = 10$ and the root is located between 9 and 19. For triplet (9, 10, 13) with $m = 3$, $p_{min} = 4 > p_{th} = 3$ we have $a = 10$ and the root is located between 9 and 19. For triplet (10, 11, 13) with $m = 2$, $p_{min} = 4 = p_{th}$ we have $a = 10$ and the root is located between 10 and 20. For triplets (10, 17, 19), (10, 19, 21), (10, 21, 23), with $m = 2$, $p_{min} = 4 > p_{th} = 3$ we have $a = 20$ and the root is located between 20 and 30. The regularity (behavior) is obvious. If the values of p and m increase, then a increases; with the same p and m, if u is increased by 10, then a, as a rule, increases by 10. So, the root of equation (6b) for initial triplets is between x and 10, if $p = 4$ and ($u = 1$, $v = 2$), ($u = 2$, $v = 3$), ($u = 3$, $v = 4$) or $p = 3$ and ($u = 1$, $v = 3$). The root of equation (6b) is located between 10 and ($x + 10$) if $p = 6$ and ($u = 2$, $v = 3$) or $p = 4$ and ($u = 8$, $v = 9$). For the remaining initial triplets, the root of equation (6b) is located between $x + a - 10$ and $x + a$. Thus, Lemma 3 is valid for initial triplets. For additional triplets calculations are carried out similarly. Below are the results of calculations. The dependence of a on the parameters p, m and u for additional triplets is the same as for the initial ones. For additional triplets, a varies from 10 to 80. For triplets of the form (11, y, z) for $m = 1$, the following results are obtained. If $p = 5+4k$, then $a = 10$, when $p_{th} = 6$ and $u = 1$; a =10, when $p_{th} = 5$ and $u = 2$; a =10, when $p_{th} = 4$ and u = 5, 6, 7; a = 20, when $p_{th} = 3$ and u =10, 11, 12, 15, 16, 17; a = 30, when $p_{th} = 3$ and u =20, 21, 22, 25, 26, 27; a =40, when $p_{th} = 3$ and u = 30, 31, 32, 35, 36, 37; a =50, when $p_{th} = 3$ and u = 40, 41, 42, 45, 46, 47. If $p = 3+2k$, then $a = 10$, when $p_{th} = 5$ and $u = 3$; $a = 10$, when $p_{th} = 4$ and $u = 8$; $a = 10$, when $p_{th} = 3$ and u = 13, 18, 23, 28, 33; $a = 20$, when $p_{th} = 3$ and u = 38, 43, 48. If $p = 3+k$, then $a = 10$, when $p_{th} = 4$ and u = 4, 9; $a = 10$, when $p_{th} = 3$ and u = 14, 19, 24, 29, 34; $a = 20$, when $p_{th} = 3$ and u = 39, 44. For $m = 3$, the following results are obtained. If $p = 6+4k$, then $a = 20$, when $p_{th} = 3$ and $u = 3$; $a = 40$, when $p_{th} = 3$ and $u = 6$. If $p = 4+4k$, then $a = 20$, when $p_{th} = 3$ and $u = 4$. For triplets of the form (12, y, z) for $m = 2$, the following results are obtained. If $p = 3+2k$, then $a = 10$, when $p_{th} = 4$

and $u = 1$; $a = 10$, when $p_{th} = 3$ and $u = 11$; $a = 20$, when $p_{th} = 3$ and $u = 21$. If $p = 3+k$, then $a = 10$, when $p_{th} = 4$ and $u = 3$; $a = 10$, when $p_{th} = 3$ and $u = 13$. If $p = 5+4k$, then $a = 20$, when $p_{th} = 3$ and $u = 5, 7, 9$; $a = 30$, when $p_{th} = 3$ and $u = 15, 17, 19$. For $m = 4$ we have the following results. If $p = 4+4k$, then $a = 20$, when $p_{th} = 3$ and $u = 3$. For triplets of the form $(13, y, z)$ for $m = 1$, the following results are obtained. If $p = 6+4k$, then $a = 10$, when $p_{th} = 7$ and $u = 1$; $a = 10$, when $p_{th} = 4$ and $u = 6$; $a = 20$, when $p_{th} = 4$ and $u = 11$; $a = 30$, when $p_{th} = 3$ and $u = 16$; $a = 40$, when $p_{th} = 3$ and $u = 21, 26$; $a = 50$, when $p_{th} = 3$ and $u = 31, 36$; $a = 60$, when $p_{th} = 3$ and $u = 41, 46$; $a = 70$, when $p_{th} = 3$ and $u = 51, 56$; $a = 80$, when $p_{th} = 3$ and $u = 61, 66$. If $p = 4+4k$, then $a = 10$, when $p_{th} = 6$ and $u = 2$; $a = 10$, when $p_{th} = 4$ and $u = 7, 12$; $a = 10$, when $p_{th} = 3$ and $u = 17$; $a = 20$, when $p_{th} = 3$ and $u = 22, 27, 32$; $a = 30$, when $p_{th} = 3$ and $u = 37, 42, 47$; $a = 40$, when $p_{th} = 3$ and $u = 52, 57, 62$; $a = 50$, when $p_{th} = 3$ and $u = 67$. If $p = 3+4k$, then $a = 10$, when $p_{th} = 6$ and $u = 3$; $a = 20$, when $p_{th} = 5$ and $u = 5$; $a = 20$, when $p_{th} = 4$ and $u = 8$; $a = 30$, when $p_{th} = 4$ and $u = 10, 13$; $a = 10$, when $p_{th} = 3$ and $u = 15, 18, 20, 23, 25, 28, 30, 33, 35, 38, 40$; $a = 20$, when $p_{th} = 3$ and $u = 43, 45, 48, 50$; $a = 30$, when $p_{th} = 3$ and $u = 53, 55, 58, 60$; $a = 40$, when $p_{th} = 3$ and $u = 63, 65, 68, 70$. For $m = 3$ we have the following results. If $p = 5+4k$, then $a = 10$, when $p_{th} = 4$ and $u = 1$; $a = 20$, when $p_{th} = 3$ and $u = 3, 5$; $a = 30$, when $p_{th} = 3$ and $u = 6, 8, 10, 11$; $a = 40$, when $p_{th} = 3$ and $u = 13$. If $p = 3+k$, then $a = 10$, when $p_{th} = 3$ and $u = 2, 7, 12$. If $p = 3+2k$, then $a = 10$, when $p_{th} = 3$ and $u = 4, 9$. For triplets of the form $(14, y, z)$ for $m = 2$, the following results are obtained. If $p = 4+4k$, then $a = 10$, when $p_{th} = 5$ and $u = 1$; $a = 20$, when $p_{th} = 3$ and $u = 11$; $a = 30$, when $p_{th} = 3$ and $u = 21$; $a = 40$, when $p_{th} = 3$ and $u = 31$. If $p = 6+4k$, then $a = 30$, when $p_{th} = 3$ and $u = 9$; $a = 50$, when $p_{th} = 3$ and $u = 19$; $a = 60$, when $p_{th} = 3$ and $u = 29$. For $m = 4$ we have the following results. If $p = 3+4k$, then $a = 10$, when $p_{th} = 3$ and $u = 1, 7$. If $p = 5+4k$, then $a = 30$, when $p_{th} = 3$ and $u = 3$. A direct verification shows that for additional triplets the root is located between $x + a - 10$ and $x + a$. So, Lemma 3 is completely proved. In our case, a more accurate determination of the position of the root is not required. The main conclusion from the analysis performed is that increasing the base of a triplet by the number $a = 10l$ does not change the class (type) of solutions of equation (6b) or that the same of equation (1), so that they are not natural (rational) numbers. Since an arbitrary triplet can be obtained from initial or additional triplets by increasing the bases of the triplets by the number $a = 10l$, this conclusion is valid in the general case.

VI. Proof of the theorem

We now prove Fermat's theorem by the method of induction with respect to the parameters a and p. According to Lemmas 1 and 2, the theorem is valid for all initial and additional triplets. We prove the induction transition. The proof of the theorem consists of two parts: the proof for all permissible triplets (induction on a) and the proof for all permissible exponents (induction on p). It follows from the previous analysis that as the parameters p and a increase alternately, the function $F(p, a) \equiv F(p; x + a, u, v)$ changes sign. If $F(p, a) > 0$, then we use induction on a for fixed p; if $F(p, a) < 0$, then we use induction by p for fixed a. Otherwise, the proof fails. *Induction on a*. The value of p is fixed, although it is chosen arbitrarily. We prove that Fermat's theorem is valid for any permissible triplet. It is enough to prove that the theorem is valid for any base x when it increases by an arbitrary number a. Let $a = 0$. It follows from Corollary 1 that for all initial and additional triplets with an arbitrary permissible exponent p, we have $F(p; x, y, z) = F(p; x, u, v) > 0$ if $p < p_{th}$ or $F(p; x, u, v) < 0$ if $p \geq p_{th}$. The transition of the function $F(p; x, u, v)$ through 0 does not occur for a natural or rational value of x, so equation (6b) has a real positive root. If we successively increase the base x by 10, then this procedure, on the one hand, allows us to obtain permissible triplets, and, on the other hand, to determine the position of the root of the initial equation (6b) using the Descartes' rule. Assume that for $a \leq a_l = 10l$, the theorem is valid. Then the equation (6a), namely, $F(p; x + a_l, u, v) = 0$ does not turn into 0 for the natural value of x, u and v. Equation (6a) corresponds to equation (6b), in which the change of variable is made $x \rightarrow x + a_l$; therefore, equation (6b) has a real positive root. From the definition of a_l, it follows that we can choose the value of l so that a_l is the smallest number for which the inequality $F(p; x + a_l, u, v) > 0$ holds. This value of a_l, of course, depends on the given p. We designate it $a_{th}(p)$. Then the root of the function $F(p; x, u, v)$ or that the same of equation (6b) is located between $x + a_{th} - 10$ and $x + a_{th}$, and it is not a natural (rational) number, since, by assumption, the theorem is valid for $a \leq a_{th}$. We put $a_{l+1} = 10(l + 1) = a_{th} + 10$, which corresponds to the change of variables $x \rightarrow x + a_{l+1}$ or, that the same, $x + a_l \rightarrow x + a_{l+1}$. Then *a fortiori* $F(p; x + a_{l+1}, u, v) > 0$. So, the real positive root of the function $F(p; x, u, v)$ does not change its position, namely, it is located between $x + a_l - 10$ and $x + a_l$, and it remains real positive, that is, cannot be a natural number. Therefore, there are no permissible bases (natural numbers), which are solutions of equation (6b) or equivalent equation (1). Induction transition on the parameter a is proved. *Induction on p*. The value of a is fixed, although it is chosen arbitrarily. We prove that Fermat's theorem is valid for any admissible exponent p. We put $p = p_{min}$, where p_{min} can take the values 3, 4, 5, or 6 (see above). To avoid confusion, we denote x_0 – the base of the initial or additional triplet; x_0 takes the values 5, 6, ..., 14; u and v take the corresponding permissible values (see above). Then, for a fixed a, an arbitrary base is represented as $x = x_0 + a$. When $a = 0$, for most initial and additional triplets $p_{min} \geq p_{th}$, then $F(p_{min}; x_0, u, v) < 0$. For triplets that are exceptions $p_{min} < p_{th}$, then $F(p_{min}; x_0, u, v) > 0$ and for arbitrary a, we have $F(p_{min}; x_0 + a, u, v) > 0$. For these triplets, if we put $p = p_0 = p_{\min} + b$, where b is period (b = 1, 2 or 4), then $F(p_0; x_0, u, v) < 0$. It follows from Lemmas 1 and 2 that for all initial and additional triplets, the root of the function $F(p_{\min}|p_0; x_0, u, v)$ is not a natural number. It follows from Lemma 3 that for all initial and additional triplets, the root of equation (6b), in which $p = p_{\min}$ or $p = p_0 = p_{\min} + b$, is located between $x_0 + a - 10$ and $x_0 + a$, where a = 10 ... 80. (Of course, a is different for different triplets as well as $p_{\min}$ and p_0). A more accurate determination of the position of the root is not required. Therefore, $F(p_{\min}|p_0; x_0 + a - 10, u, v) < 0$, but $F(p_{\min}|p_0; x_0 + a, u, v) > 0$. Since in the given interval between $x_0 + a - 10$ and $x_0 + a$ there are no permissible triplets (permissible values of x_0, u, v), the root of the function $F(p_{\min}|p_0; x_0, u, v)$ is a real positive and cannot be a natural number. *A fortiori* this is true for $a > 80$, since in this case for an arbitrary a we have $F(p_{\min}|p_0; x_0 + a, u, v) > 0$. Thus, the Fermat theorem is valid for $p = p_{\min}$ and $p = p_0$. If we successively increase the exponent p by period b_k, then this procedure, on the one hand, allows us to obtain all permissible exponents, and on the other hand, to determine groups of triplets described by equation (6b) of the same (given) degree. Assume that for $p = p_k \leq p_{\min} + b_k$, where $b_k = k$, $2k$ or $4k$ (see above), the theorem is valid. Then, for given a, the equation (6b), namely $F(p_k; x, u, v) = 0$ has not natural solution, so its root is a real positive number. We use $p_{\min}$, since $p_0 = p_{\min} + b$ and therefore there is no need to consider p_0 separately. From the definition of p_k, it follows that we can choose the value of k so that p_k is the smallest number for which the inequality $F(p_k; x, u, v) < 0$ holds. This value of p_k depends on the given a. We designate it, as above, $p_{\text{th}}(a)$. We put $p = p_{\text{th}} - b$, where b = 1, 2 or 4, then $F(p_{\text{th}} - b; x, u, v) > 0$, since, by assumption, the theorem is valid for $p \leq p_{\text{th}}$. We put $p = p_{k+1} = p_k + b = p_{\text{th}} + b$, where b = 1, 2 or 4. Then *a fortiori* $F(p_{k+1}; x, u, v) < 0$. Therefore, the root of the equation $F(p_{k+1}; x, u, v) = 0$ is not a natural number. Therefore, there is no equation (6b) or that the same equation (1) with permissible exponent, the root of which is a natural number. Induction transition on the parameter p is proved. Thus, a change in the base and the exponent does not change the class of solutions of equation (6b) or, it is the same, of equation (1). The root of the equation remains real positive number and is not a natural number. This implies the validity of the Fermat theorem.

VII. Discussion of results

The function $F(p; a)$ alternately changes sign when the exponent p and the parameter a change. The threshold value of the exponent at which the function changes sign depends on a, and the threshold value of the parameter a depends on the exponent. Our proof is based on the following reasoning. If $F(p; a) > 0$, then this is possible under one of the conditions: $p < p_{\text{th}}$ or $a \geq a_{\text{th}}$. These conditions are indistinguishable, since they lead to the same sign of the function F. We assume that $a \geq a_{\text{th}}$ for a fixed p, i.e. the zero of the function is passed in the previous step, since the theorem is valid by assumption, which allows us to prove the inductive transition (induction by a). If we assume that $p < p_{\text{th}}$, then the proof fails. In this case, we cannot specify the value of p for which F passes through zero, since we do not know the value of a. Similarly, if $F(p; a) < 0$, then this is possible under one of the conditions: $p \geq p_{\text{th}}$ or $a < a_{\text{th}}$. These conditions are indistinguishable, since they lead to the same sign of the function F. We assume that $p \geq p_{\text{th}}$ for a fixed a, i.e. the zero of the function is passed, since the theorem is valid by assumption, which allows us to prove the inductive transition (induction by p). If we assume that $a < a_{\text{th}}$, then the proof fails. In this case, we cannot specify the value of a for which F passes through zero, since we do not know the value of p. The above reasoning uses the standard techniques of the method of induction and does not contain the logical error *petitio principii*.

VIII. Connection of Fermat's theorem with Beal conjecture

Beal conjecture consists in the statement that the equation $x^p + y^q = z^r$ has no solution in positive integers x, y, z, p, q and r with p, q and r at least 3 and x, y, and z coprime. If $p = q = r$, then this equation turns into equation (1) and then, of course, the validity of Beal conjecture follows from Fermat's theorem, but the opposite is not true. We show that in this case the validity of the Beal conjecture follows from our method of proving Fermat's theorem. Indeed. For all initial and additional triplets, the Beal conjecture is true, since the bases of these triplets (natural numbers) are coprime. In addition, in each initial or additional triplet, one base is an even number, and the other two bases are odd numbers. The difference of the parameters v and u, namely $m = v - u$, takes values 1, 2 or 3 for the initial triplets (see table 3), 1 or 3 for additional triplets of the form (11, y, z) and (13, y, z) and 2 or 4 for additional triplets of the form (12, y, z) and (14, y, z). All permissible triplets are obtained from the initial and additional triplets by increasing all the bases of the triplet (initial or additional) simultaneously by $a = 10l$. If we simultaneously increase the bases of the triplet by $a = 10l$, then the properties noted above are remain unchanged (saved), namely, the parity of the bases does not change and the parameters v, u and $m = v - u$ do not change their values. Therefore, the bases of triplets cannot have an even number as a common divisor. The common divisor also cannot be the number 3, although 3 can divide the two bases. Since $m \leq 4$, other divisors may not be considered. It follows that the permissible bases remain mutually prime (coprime) numbers, which proves the Beal conjecture. In the general case, when p, q and r are different, the analysis technique used in the proof of Fermat's theorem can be used to prove Beal conjecture. In particular, the second restriction on permissible solutions established for equation (1) remains valid for the Beal equation. We give an example. We can assume, without loss of generality, that $x < y < z$. If $p < q < r$, then the Beal equation obviously has no solutions not only for coprime numbers x, y and z, but, also for numbers having a common divisor. If $p > q > r$, then the permissible values of exponents p, q and r and bases x, y and z are determined using the second restriction. In particular, it follows from the second restriction that three cases are possible: 1) x, y and z are even numbers; 2) x and y are odd, and z is an even number; 3) x and y have different parity, and z is odd. The first case, after canceling all bases by a common factor, reduces to the second or third case. Table 1 can be used to select permissible combinations of endings and exponents. Preliminary analysis shows that the number of admissible initial triplets should not be large.

Using table 1 consider the Beal equation under the conditions $x < y < z$ and $p > q > r$. The values x, y, z are elementary bases from the first ten or, if necessary, from the second ten, so that the condition $x < y < z$ is not violated. We will show that in this case the Beale equation has no solutions for coprime x, y, z. To reduce the number of admissible solutions, we will use, along with the indicated conditions, the main restriction, which consists in the coincidence of the last digit of the numbers on the left and right sides of the equation. Admissible solutions are triples of numbers (triplets) of the form (x^p, y^q, z^r). We write them in the form $(x^{l+wk_1}, y^{m+sk_2}, z^{n+tk_3})$, where w, s, t are the periods of bases x, y and z, respectively, which can take the values 4, 2 or 1 depending on the value of the base; l, m, n are initial exponents, which take values depending on the value of the base. If the base is 2, 3, 7 or 8, then w, s, t are 4, and l, m, n can be 3, 4, 5, or 6 (see Table 1). If the base is 4 or 9, then w, s, t are 2, and l, m, n can be 3 or 4. If the base is 5, 6, 10, or 11, then w, s, t are 1, and l, m, n can only take the value 3. The indices k_1, k_2, k_3 take the values 0, 1, 2, 3, and so on. For a given x, the values k_1, k_2, k_3, w, s, t are chosen so that the condition $p > q > r$ is satisfied, and the values y, z are chosen so that the condition $x < y < z$ is satisfied.

We consider the minimum sufficient values of the exponents: k_1 is chosen to be the minimum sufficient to satisfy the condition $l + wk_1 \geq m + sk_2 + 1$; k_2 is chosen to be the minimum sufficient to satisfy the condition $m + sk_2 \geq n + tk_3 + 1$. Then k_3 can take the values 0, 1, etc., up to $k_{3\max}$, where $k_{3\max}$ is the largest possible value of index k_3 satisfying the relation $m + sk_2 \geq n + tk_{3\max} + 1$. Here and below, we will always assume $k_3 = 0$, i.e., take the smallest possible value of k_3.

Put $x = 2$, then the triple of numbers has the form $(2^{l+4k_1}, y^{m+sk_2}, z^{n+tk_3})$, where l can take the values 3, 4, 5 or 6; k_1 = 0, 1 or 2, k_2 = 0, 1, 2, 3 or 4; $k_{3\max}$ can take the values 0, 1, 2 or 3 depending on the value of the bases; y, z cannot equal 2. Taking into account the main restriction, 92 combinations are allowed, consisting of triples (2, 3, 5), (2, 3, 9), (2, 3, 11), (2, 3, 7), (2, 5, 7), (2, 5, 13), (2, 5, 9), (2, 5, 11), (2, 7, 11), (2, 7, 9), (2, 7, 13), (2, 7, 17), (2, 7, 15), (2, 9, 13), (2, 9, 17), (2, 9, 19), (2, 9, 15), (2 , 9, 11), (2, 11, 13), (2, 11, 17), (2, 11, 19), (2, 11, 15) with different allowable exponents of bases. In 58 combinations $k_1 = 1$, $k_1 = 2$ in 19 combinations, $k_1 = 0$ in 15 combinations; $k_2 = 1$ in 41 combinations, $k_2 = 0$ in 36 combinations, $k_2 = 2$ in 7 combinations, the values $k_2 = 3$ and $k_2 = 4$ are in four combinations each; $k_{3\max} = 0$ in most combinations ($k_{3\max} > 0$ in 17 combinations and $k_{3\max}$ can be 3 in four combinations). We do not give these combinations explicitly, so as not to clutter up the text, since they are easily determined from Table 1. Triples, in which all bases are even numbers, we excluded from consideration, since after reduction by a common factor (some power of the number 2), we get a triple of numbers that does not satisfy the main restriction due to the condition $p > q > r$.

Put $x = 3$, then the triple of numbers has the form $(3^{l+4k_1}, y^{m+sk_2}, z^{n+tk_3})$, where l can take the values 3, 4, 5 or 6; k_1 = 0, 1 or 2, k_2 = 0, 1, 2, 3 or 4; $k_{3\max}$ can take the values 0, 1, 2 or 3 depending on the value of the bases; y, z cannot equal 2 or 3. Taking into account the main restriction, 121 combinations are allowed, consisting of triples (3, 4, 11), (3, 4, 9), (3, 4, 13), (3, 4, 5), (3, 4, 7), (3, 5, 12), (3 , 5, 8), (3, 5, 14), (3, 6, 7), (3, 6, 13), (3, 7, 10), (3, 7, 12), (3, 7 , 14), (3, 7, 8), (3, 7, 16), (3, 8, 9), (3, 8, 13), (3, 8, 17), (3, 8, 15), (3, 8, 11), (3, 9, 10), (3, 9, 14), (3, 9, 16), (3, 10, 13), (3, 10, 11), (3, 10, 19), (3, 11, 14), (3, 11, 12), (3, 11, 18), (3, 11, 20) with different allowable exponents of bases. In 89 combinations $k_1 = 1$, $k_1 = 0$ in 17 combinations, $k_1 = 2$ in 15 combinations; $k_2 = 1$ in 57 combinations, $k_2 = 0$ in 43 combinations, $k_2 = 2$ in 8 combinations, $k_2 = 4$ in 9 combinations, $k_2 = 3$ in 4 combinations; $k_{3\max} = 0$ in most combinations ($k_{3\max} > 0$ in 15 combinations and $k_{3\max}$ can be 3 in three combinations). We do not take into account triples (3, 6, 9), (3, 6, 15), (3, 9, 12), (3, 9, 18) with different exponents of bases. These triples are excluded from further consideration, since after reducing the bases by a common factor (some power of 3), we get a triple of numbers that does not satisfy the main restriction due to the condition $p > q > r$.

Put $x = 4$, then the triple of numbers has the form $(4^{l+2k_1}, y^{m+sk_2}, z^{n+tk_3})$, where l can take the values 3 or 4; k_1 = 1, 2 or 3, k_2 = 0, 1, 2 or 3; $k_{3\max}$ can take the values 0, 1 or 2 depending on the value of the bases; y, z cannot equal 2, 3 or 4. Taking into account the main restriction, 28 combinations are allowed, consisting of triples (4, 5, 9), (4, 5, 11), (4, 7, 9), (4, 7, 13), (4, 7, 17), (4, 7, 15), (4 , 7, 11), (4, 9, 13), (4, 9, 17), (4, 9, 15), (4, 11, 15), (4, 11, 13), (4, 11, 17) with different allowable exponents of bases. In 13 combinations $k_1 = 2$, $k_1 = 1$ in 10 combinations, $k_1 = 3$ in 5 combinations; $k_2 = 1$ in 14 combinations, $k_2 = 0$ in 12 combinations, $k_2 = 2$ in 1 combination, $k_2 = 3$ in 1 combination; $k_{3\max} = 0$ in most combinations ($k_{3\max} > 0$ in 3 combinations and $k_{3\max}$ can be 2 in one combination).

Put $x = 5$, then the triple of numbers has the form $(5^{l+k_1}, y^{m+sk_2}, z^{n+tk_3})$, where l can take the value 3; $k_1 = 2, 3, 4, 5, 6, 7$ or 8, $k_2 = 0, 1$ or 2 depending on the value of the bases; $k_3 = 0$; y, z cannot equal 2, 3, 4 or 5. Taking into account the main restriction, 37 combinations are allowed, consisting of triples (5, 6, 11), (5, 6, 9), (5, 6, 13), (5, 6, 7), (5, 7, 12), (5, 7, 8), (5 , 7, 16), (5, 7, 14), (5, 8, 17), (5, 8, 13), (5, 8, 11), (5, 8, 9), (5, 9 , 12), (5, 9, 18), (5, 9, 14), (5, 9, 16), (5, 11, 16), (5, 11, 14), (5, 11, 12), (5, 11, 18) with different allowable exponents of bases. In 9 combinations $k_1 = 3$, $k_1 = 5$ in 6 combinations, $k_1 = 6$ in 6 combinations, $k_1 = 2$ in 5 combinations, $k_1 = 4$ in 5 combinations, $k_1 = 8$ in 4 combinations, $k_1 = 7$ in 2 combinations; $k_2 = 1$ in 22 combinations, $k_2 = 2$ in 8 combination, $k_2 = 0$ in 7 combinations; $k_{3\max} = 0$ in all combinations. We excluded the triple (5, 10, 15) from further consideration, since after reducing the bases by a common factor (some power of 5), we get a triple of numbers that does not satisfy the main restriction due to the condition $p > q > r$.

Put $x = 6$, then the triple of numbers has the form $(6^{l+k_1}, y^{m+sk_2}, z^{n+tk_3})$, where l can take the value 3; $k_1 = 2, 3, 4, 5, 6$ or 7, $k_2 = 0, 1$ or 3 depending on the value of the bases; $k_3 = 0$; y, z cannot equal 2, 3, 4, 5 or 6. Taking into account the main restriction, 12 combinations are allowed, consisting of triples (6, 7, 9), (6, 7, 13), (6, 7, 17), (6, 7, 15), (6, 9, 13), (6, 9, 17), (6, 11, 17), (6, 11, 13) with different allowable exponents of bases. In 3 combinations $k_1 = 5$, $k_1 = 2$ in 3 combinations, $k_1 = 4$ in 3 combinations; the values $k_1 = 3$, $k_1 = 6$ and $k_1 = 7$ are in one combination each; $k_2 = 1$ in 7 combinations, $k_2 = 0$ in 4 combination, $k_2 = 3$ in 1 combination; $k_{3\max} = 0$ in all combinations. We excluded the triple (6, 9, 15) from further consideration, since after reducing the bases by a common factor (some power of 3), we get a triple of numbers that does not satisfy the main restriction due to the condition $p > q > r$.

Put $x = 7$, then the triple of numbers has the form $(7^{l+4k_1}, y^{m+sk_2}, z^{n+tk_3})$, where l can take the values 3, 4, 5 or 6; $k_1 = 0, 1$ or 2, $k_2 = 0, 1, 2, 3$ or 4; $k_{3\max}$ can take the values 0, 1, 2 or 3 depending on the value of the bases; y, z cannot equal 2, 3, 4, 5, 6 or 7. Taking into account the main restriction, 72 combinations are allowed, consisting of triples (7, 8, 15), (7, 8, 13), (7, 8, 17), (7, 8, 9), (7, 8, 11), (7, 8, 15), (7 , 9, 12), (7, 9, 18), (7, 9, 14), (7, 9, 10), (7, 10, 13), (7, 10, 17), (7, 10 , 19), (7, 10, 11), (7, 11, 12), (7, 11, 14), (7, 11, 18), (7, 11, 20) with different allowable exponents of bases. In 51 combinations $k_1 = 1$, $k_1 = 2$ in 11 combinations, $k_1 = 0$ in 10 combinations; $k_2 = 1$ in 33 combinations, $k_2 = 0$ in 24 combinations, $k_2 = 2$ in 7 combinations, values $k_2 = 3$ and $k_2 = 4$ are available in 4 combinations each; $k_{3\max} = 0$ in most combinations ($k_{3\max} > 0$ in 5 combinations and $k_{3\max}$ can be 3 in two combinations).

Put $x = 8$, then the triple of numbers has the form $(8^{l+4k_1}, y^{m+sk_2}, z^{n+tk_3})$, where l can take the values 3, 4, 5 or 6; $k_1 = 0$ or 1, $k_2 = 0, 1, 2, 3$ or 4; $k_3 = 0$ or 1 depending on the value of the bases; y, z cannot equal 2, 3, 4, 5, 6, 7 or 8. Taking into account the main restriction, 25 combinations are allowed, consisting of triples (8, 9, 11), (8, 9, 13), (8, 9, 17), (8, 9, 19), (8, 9, 15), (8, 11, 13), (8, 11, 17), (8, 11, 19), (8, 11, 15) with different allowable exponents of bases. In 20 combinations $k_1 = 1$, $k_1 = 0$ in 5 combinations; $k_2 = 1$ in 13 combinations, $k_2 = 0$ in 4 combinations, $k_2 = 2$ in 4 combinations, the values $k_2 = 3$ and $k_2 = 4$ are in two combinations each; $k_{3\max} = 0$ in all combinations, except for two, in which $k_{3\max}$ can be equal to 1.

Put $x = 9$, then the triple of numbers has the form $(9^{l+2k_1}, y^{m+sk_2}, z^{n+tk_3})$, where l can take the values 3 or 4; $k_1 = 1, 2$ or 3, $k_2 = 1, 2, 3$ or 4 depending on the value of the bases; $k_3 = 0$; y, z cannot equal 2, 3, 4, 5, 6, 7, 8 or 9. Taking into account the main restriction, 9 combinations are allowed, consisting of triples (9, 10, 19), (9, 10, 13), (9, 10, 17), (9, 10, 11), (9, 11, 12) (9, 11, 18) (9, 11, 20) with different allowable exponents of bases. In 6 combinations $k_1 = 1$, $k_1 = 3$ in 2 combinations, $k_1 = 2$ in 1 combination; $k_2 = 1$ in 5 combinations, $k_2 = 4$ in 2 combinations, $k_2 = 2$ in 1 combination, $k_2 = 3$ in 1 combination; $k_{3\max} = 0$ in all combinations.

Put $x = 10$, then the triple of numbers has the form $(10^{l+k_1}, y^{m+sk_2}, z^{n+tk_3})$, where l can take the value 3; $k_1 = 2$ or 3, $k_2 = 1$ or 2 depending on the value of the bases; $k_3 = 0$; y, z cannot equal 2, 3, 4, 5, 6, 7, 8 9, or 10. Taking into account the main restriction, 4 combinations are allowed, consisting of triples (10, 11, 13), (10, 11, 17), (10, 11, 19), (10, 11, 21) with different allowable exponents of bases. In the first, second and third combinations $k_1 = 3$, $k_2 = 2$; in the fourth combination $k_1 = 2$, $k_2 = 1$; $k_{3\max} = 0$ in all combinations. We do not give detailed results for additional combinations involving triplets (11, y, z) and (10, y, z) for y > 11; these triples also consist of pairwise primes, and the corresponding combinations are not solutions of the Beale equation.

It follows from the above results that all bases in triplets are coprime and we can directly verify that the combinations of elementary bases are not solutions of the Beale equation. Thus, Beal's conjecture is valid for all elementary bases.

Define the function $F(x, y, z; p, q, r) = x^p + y^q - z^r$. Its roots coincide with the solutions of the Beal equation. For most of the combinations considered above, $F(x, y, z; p, q, r) > 0$. In particular, at $x = 2$, $F < 0$ for

31 combinations; at x = 3, $F < 0$ for 29 combinations; at $x = 4$, $F < 0$ for 5 combinations; at $x = 5$, $F < 0$ for 2 combinations; at $x = 6$, $F < 0$ for 2 combinations; at $x = 7$, $F < 0$ for 1 combination; in all other combinations $F > 0$. In triples for which $F(x, y, z; p, q, r) < 0$, simultaneously increasing the bases by 10 (or a multiple of 10), we can always make the inequality $F(x, y, z; p, q, r) > 0$. In this case, the transition of the function through zero does not occur for natural values of the bases. In some cases, when $k_{3\max} > 0$, the sign of the function changes from plus (at $k_3 = 0$) to minus (at $k_3 = k_{3\max}$) at constant k_1 and k_2. Then we should first put $k_3 = k_{3\max}$, and then simultaneously increase the bases by 10 (or a multiple of 10). In all the considered combinations, it is enough to increase the bases by 10 for the function to become positive. Therefore, we assume that in the initial combinations with elementary bases $F(x, y, z; p, q, r) > 0$.

If $F(x, y, z; p, q, r) < 0$, then for fixed bases, the simultaneous increase in all exponents by the corresponding periods does not change the sign of the function. If we successively increase the exponents only at the base x or at the same time at the bases x and y, then the sign of the function changes from minus to plus, but zero crossing does not occur for natural values of the arguments. If $F(x, y, z; p, q, r) > 0$, then for fixed exponents, the simultaneous increase of all bases by 10 or a multiple of 10 does not change the sign of the function. If only the base z is successively increased, then the sign of the function changes from plus to minus, but the transition through zero does not occur for natural values of the arguments.

Thus, if $F(x, y, z; p, q, r) < 0$, then this is possible under one of the following conditions: a) the exponent of the base z is more than some threshold value or the exponents of the bases x and (or) y are less than some threshold values; b) the base z is more than some threshold value, or the base x and (or) y are less than some threshold values. In general case, these conditions are not distinguishable, since the threshold values of the exponents depend on the value of the bases, and the threshold values of the bases depend on the exponents.

Similarly, if $F(x, y, z; p, q, r) > 0$, then this is possible under one of the following conditions: a) the exponent of the base z is less than some threshold value, or the exponents of the bases x and (or) y are more than some threshold values; b) the base z is less than some threshold value, or the base x and (or) y are more than some threshold values. In general case, these conditions are not distinguishable, since the threshold values of the exponents depend on the value of the bases, and the threshold values of the bases depend on the exponents.

We use the minimum sufficiency condition, which does not restrict the generality and at the same time preserves the clarity of the problem statement. Therefore, all valid triplets can be obtained from elementary triplets by simultaneously increasing all bases by 10 or a multiple of 10.

Consider an arbitrary triplet $((x + a)^p, (y + a)^q, (z + a)^r)$ obtained from the elementary triplet (x^p, y^q, z^r), where $a = 10j$, $j = 1, 2, 3$ etc. The quantities u, v and m do not change their values, where, as above, $y = x + u$, $z = x + v$, $m = z - y = u - v$. We show that the bases in this triplet are coprime numbers. From the analysis of elementary triplets, it follows that if x is an even number, i.e. 2, 4, 6, 8, or 10, then m can be 2, 4, 6, 8, or 10; if x is an odd number, i.e. 3, 5, 7, 9, or 11, then m can be 1, 3, 5, 7, or 9. When the bases are increased by a, the parity of the numbers does not change, and since the elementary bases are not simultaneously even, the bases of an arbitrary triplet cannot have an even number as a common divisor. It is easy to see that the numbers 3, 5, 7 or 9 also cannot be a common divisor of the bases included in the triplet. Assume the opposite, namely that the bases of the triplet have a common divisor b, where b can be 3, 5, 7, or 9. Since x, y, and z must be divisible by b in this case, u and m must also be divisible by b. But this is impossible, because among the elementary triplets considered above, there is no one in which u and m are simultaneously divisible by b. It follows from this that the numbers included in an arbitrary triplet obtained by the above method are coprime. It is not necessary to consider other odd divisors, since m does not exceed 9.

In general, Beal conjecture can be proved by induction using the same reasoning as in the proof of Fermat's theorem. For fixed exponents, the hypothesis is proved by induction on the bases, or, which is the same, on the parameter a, and for fixed bases, by induction on the exponents. *Induction on the parameter a*. The exponents have fixed values, although they can be chosen arbitrarily. We denote them (p_0, q_0, r_0). Assume that the hypothesis holds for $a \leq a_j(p_0, q_0, r_0)$. We choose j so that a_j is the smallest number for which the inequality $F(x_0 + a_j, u, v; p_0, q_0, r_0) > 0$ holds, where x_0 is the elementary base from which the considered triplet is obtained. Designate it $a_{\text{th}}(p_0, q_0, r_0)$. We have $F(x_0 + a_{\text{th}}, u, v; p_0, q_0, r_0) > 0$, but $F(x_0 + a_{\text{th}} - 10, u, v; p_0, q_0, r_0) < 0$. Put $a_{j+1} = a_{\text{th}} + 10$. Then a fortiori $F(x_0 + a_{\text{th}} + 10, u, v; p_0, q_0, r_0) > 0$. The induction transition is proved. *Induction on exponents*. It suffices to consider induction on the exponent r, since the exponents p and q are uniquely determined from r by the minimum sufficiency condition. The parameter a has a fixed value, although it can be chosen arbitrarily. Since the induction on a has been proved, we can choose it equal to 0, which corresponds to elementary bases. But for elementary bases, as shown above, the hypothesis is valid for all admissible exponents, which is verified by direct calculation. It follows that the conjecture is valid in the general case. It is possible to prove the induction transition without using the condition $a = 0$. The parameter a has an arbitrary but fixed value. Denote it $a = a_0$. Assume that the hypothesis is valid for $r \leq r_j(a_0)$. We choose j so that r_j is the smallest number for which the inequality $F(r_j) < 0$ holds. To simplify the notation, we have omitted the other arguments under the function sign. Denote it $r_{\text{th}}(a_0)$. We have $F(r_{\text{th}}) < 0$, but $F(r_{\text{th}} - t) > 0$, where t is the period, which can take the values 4, 2, or 1 (see above). Put $r_{j+1} = r_j + t$. Then a fortiori $F(r_{\text{th}} + t) < 0$. The induction transition is proved.

IX. Conclusion

1. The above proof of the theorem uses only the characteristic properties of natural numbers and some general theorems on the roots of algebraic equations.
2. Fermat's theorem has an obvious geometric interpretation. For $p = 1$, equation (1) always has a solution, i.e. the sum of two integer segments is always an integer segment. For $p = 2$, equation (1) has a solution only in some cases, i.e. the sum of the areas of two squares with integer sides is only sometimes equal to the area of the square with integer sides. For $p = 3$, equation (1) has no solution, i.e. the volume of a cube with integer sides is never the sum of the volumes of two cubes with integer sides. This is true *a fortiori* for hypercube.
3. The analysis technique used in the proof of Fermat's theorem can be used to prove Beal conjecture. When $p = q = r$, Beal conjecture is one of the consequences of Fermat's theorem. In general case, Beal conjecture is proved by induction using the same reasoning as in the proof of Fermat's theorem.

References

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Table 1
Admissible ends of powers for elementary bases

Number	Last digit of number	Number	Last digit of number	Number	Last digit of number	Number	Last digit of number
2^3	8	3^3	7	4^3	4	5^3	5
2^4	6	3^4	1	4^4	6	5^4	5 (repeat)
2^5	2	3^5	3	4^5	4 (repeat)		
2^6	4	3^6	9				
2^7	8 (repeat)	3^7	7 (repeat)				
Number	Last digit of number	Number	Last digit of number	Number	Last digit of number	Number	Last digit of number
6^3	6	7^3	3	8^3	2	9^3	9
6^4	6 (repeat)	7^4	1	8^4	6	9^4	1
		7^5	7	8^5	8	9^5	9 (repeat)
		7^6	9	8^6	4		
		7^7	3 (repeat)	8^7	2 (repeat)		
Number	Last digit of number	Number	Last digit of number				
0^3	0	1^3	1				
0^4	0 (repeat)	1^4	1(repeat)				

Table 2
Combination of ends for elementary bases permissible under basic restrictions

$2^{3+2k}+3^{3+2k}=5^{3+2k}$	$3^{3+4k}+4^{3+4k}=1^{3+4k}$	$7^{3+4k}+4^{3+4k}=3^{3+4k}$	$8^{3+4k}+4^{3+4k}=6^{3+4k}$
$2^{4+2k}+4^{4+2k}=0^{4+2k}$	$3^{5+4k}+4^{5+4k}=7^{5+4k}$	$7^{5+4k}+4^{5+4k}=1^{5+4k}$	$8^{5+4k}+4^{5+4k}=2^{5+4k}$
$2^{5+4k}+4^{5+4k}=6^{5+4k}$	$3^{6+4k}+4^{6+4k}=5^{6+4k}$	$7^{6+4k}+4^{6+4k}=5^{6+4k}$	$8^{6+4k}+4^{6+4k}=0^{6+4k}$
$2^{3+k}+5^{3+k}=7^{3+k}$	$3^{3+k}+5^{3+k}=8^{3+k}$	$7^{3+k}+5^{3+k}=2^{3+k}$	$8^{3+4k}+5^{3+4k}=3^{3+4k}$
$2^{4+4k}+5^{4+4k}=1^{4+4k}$	$3^{4+4k}+5^{4+4k}=6^{4+4k}$	$7^{4+4k}+5^{4+4k}=6^{4+4k}$	$8^{4+4k}+5^{4+4k}=1^{4+4k}$
$2^{6+4k}+5^{6+4k}=3^{6+4k}$	$3^{6+4k}+5^{6+4k}=2^{6+4k}$	$7^{4+4k}+5^{4+4k}=4^{4+4k}$	$8^{4+2k}+5^{4+2k}=7^{4+2k}$

$2^{3+4k}+6^{3+4k}=4^{3+4k}$	$3^{3+4k}+6^{3+4k}=7^{3+4k}$	$7^{4+2k}+5^{4+2k}=8^{4+2k}$	$8^{3+k}+5^{3+k}=3^{3+k}$
$2^{5+4k}+6^{5+4k}=8^{5+4k}$	$3^{5+4k}+6^{5+4k}=9^{5+4k}$	$7^{3+4k}+6^{3+4k}=9^{3+4k}$	$8^{6+4k}+5^{6+4k}=7^{6+4k}$
$2^{6+4k}+6^{6+4k}=0^{6+4k}$	$3^{6+4k}+6^{6+4k}=5^{6+4k}$	$7^{5+4k}+6^{5+4k}=3^{5+4k}$	$8^{3+4k}+6^{3+4k}=2^{3+4k}$
$2^{3+4k}+7^{3+4k}=1^{3+4k}$	$3^{3+2k}+7^{3+2k}=0^{3+2k}$	$7^{6+4k}+6^{6+4k}=5^{6+4k}$	$8^{5+4k}+6^{5+4k}=4^{5+4k}$
$2^{5+4k}+7^{5+4k}=9^{5+4k}$	$3^{3+4k}+8^{3+4k}=9^{3+4k}$	$7^{3+2k}+8^{3+2k}=5^{3+2k}$	$8^{6+4k}+6^{6+4k}=0^{6+4k}$
$2^{3+2k}+8^{3+2k}=0^{3+2k}$	$3^{5+4k}+8^{5+4k}=1^{5+4k}$	$7^{3+4k}+9^{3+4k}=8^{3+4k}$	$8^{3+4k}+9^{3+4k}=1^{3+4k}$
$2^{3+4k}+9^{3+4k}=3^{3+4k}$	$3^{3+4k}+9^{3+4k}=6^{3+4k}$	$7^{5+4k}+9^{5+4k}=6^{5+4k}$	$8^{5+4k}+9^{5+4k}=7^{5+4k}$
$2^{5+4k}+9^{5+4k}=1^{5+4k}$	$3^{5+4k}+9^{5+4k}=2^{5+4k}$	$7^{6+4k}+9^{6+4k}=0^{6+4k}$	$8^{6+4k}+9^{6+4k}=5^{6+4k}$
$2^{6+4k}+9^{6+4k}=5^{6+4k}$	$3^{6+4k}+9^{6+4k}=2^{6+4k}$	$7^{4+4k}+0^{4+4k}=1^{4+4k}$	$8^{4+2k}+0^{4+2k}=2^{4+2k}$
$2^{3+4k}+1^{3+4k}=9^{3+4k}$	$3^{6+4k}+9^{6+4k}=0^{6+4k}$	$7^{6+4k}+0^{6+4k}=3^{6+4k}$	$8^{4+4k}+0^{4+4k}=6^{4+4k}$
$2^{4+4k}+0^{4+4k}=4^{4+4k}$	$3^{4+2k}+0^{4+2k}=7^{4+2k}$	$7^{4+4k}+0^{4+4k}=9^{4+4k}$	$8^{3+4k}+1^{3+4k}=7^{3+4k}$
$2^{4+2k}+0^{4+2k}=8^{4+2k}$	$3^{4+4k}+0^{4+4k}=9^{4+4k}$	$7^{6+4k}+4^{6+4k}=5^{6+4k}$	$8^{5+4k}+1^{5+4k}=9^{5+4k}$
$2^{5+4k}+1^{5+4k}=3^{5+4k}$	$3^{3+4k}+1^{3+4k}=2^{3+4k}$	$7^{6+4k}+1^{6+4k}=0^{6+4k}$	$8^{6+4k}+1^{6+4k}=5^{6+4k}$
$2^{6+4k}+1^{6+4k}=5^{6+4k}$	$3^{5+4k}+1^{5+4k}=4^{5+4k}$	$7^{5+4k}+1^{5+4k}=8^{5+4k}$	$8^{4+4k}+5^{4+4k}=9^{4+4k}$
$\times$	$3^{6+4k}+1^{6+4k}=0^{6+4k}$	$\times$	$\times$
$4^{3+4k}+5^{3+4k}=9^{3+4k}$	$3^{3+4k}+3^{3+4k}=4^{3+4k}$	$5^{4+2k}+6^{4+2k}=9^{4+2k}$	$6^{3+4k}+1^{3+4k}=3^{3+4k}$
$4^{4+4k}+5^{4+4k}=3^{4+4k}$	$3^{4+4k}+0^{4+4k}=1^{4+4k}$	$5^{3+k}+6^{3+k}=1^{3+k}$	$6^{5+4k}+1^{5+4k}=7^{5+4k}$
$4^{4+4k}+5^{4+4k}=7^{4+4k}$	$\times$	$5^{4+4k}+1^{4+4k}=2^{4+4k}$	$\times$
$4^{3+4k}+6^{3+4k}=0^{3+4k}$	$9^{3+k}+5^{3+k}=4^{3+k}$	$5^{3+k}+1^{3+k}=6^{3+k}$	$0^{4+4k}+1^{4+4k}=3^{4+4k}$
$4^{3+4k}+9^{3+4k}=7^{3+4k}$	$9^{4+2k}+5^{4+2k}=6^{4+2k}$	$5^{4+4k}+6^{4+4k}=3^{4+4k}$	$0^{4+4k}+1^{4+4k}=7^{4+4k}$
$4^{4+4k}+0^{4+4k}=2^{4+4k}$	$9^{4+4k}+5^{4+4k}=2^{4+4k}$	$5^{4+4k}+6^{4+4k}=7^{4+4k}$	$0^{4+2k}+1^{4+2k}=9^{4+2k}$
$4^{4+4k}+0^{4+4k}=8^{4+4k}$	$9^{4+4k}+5^{4+4k}=8^{4+4k}$	$5^{4+4k}+1^{4+4k}=8^{4+4k}$	$\times$
$4^{3+4k}+1^{3+4k}=5^{3+4k}$	$9^{3+2k}+6^{3+2k}=5^{3+2k}$	$\times$	
$\times$	$9^{4+2k}+0^{4+2k}=1^{4+2k}$		
	$9^{4+4k}+0^{4+4k}=3^{4+4k}$		
	$9^{4+4k}+0^{4+4k}=7^{4+4k}$		
	$9^{3+2k}+1^{3+2k}=0^{3+2k}$		

Note. The relationships given in table 2 do not mean actual equality and they are symbolic notation representing the fulfillment of the basic restrictions necessary for equation (1), namely, the coincidence of the exponents of all components and the coincidence of the last digit, to which the left and right sides of equation (1) end.

Table 3

Permissible triplets of the form (x, y, z), in which x is an elementary base

Form of triplet	Permissible triplets of this form	Degree, permissible by basic restrictions, p	The number of permissible triplets	Boundary triplet	p_{th}	u	v	m
(1, y, z)	-	-	0	-				
(2, y, z)	-	-	0	-				
(3, y, z)	-	-	0	(3, 4, 5)				
(4, y, z)	-	-	0	-				
(5, y, z)	(5, 6, 7)	4+4k	4	(5, 12, 13)	3	1	2	1
	(5, 7, 8)	4+2k			3	2	3	1
	(5, 8, 9)	4+4k			3	3	4	1
	(5, 11, 12)	4+4k			3	6	7	1
(6, y, z)	(6, 7, 9)	3 + 4k	1	(6, 8, 10)	3	1	3	2
(7, y, z)	(7, 9, 10)	6+4k	6	(7, 24, 25)	4	2	3	1
	(7, 10, 11)	4+4k			4	3	4	1
	(7, 14, 15)	6+4k			3	7	8	1
	(7, 15, 16)	4+4k			3	8	9	1

	(7, 19, 20)	6+4*k*			3	12	13	1
	(7, 20, 21)	4+4*k*			3	13	14	1
(8, *y*, *z*)	(8, 9, 11)	3 + 4*k*	1	(8, 15, 17)	3	1	3	2
(9, *y*, *z*)	(9, 10, 11)	4+2*k*	13	(9, 40, 41)	5	1	2	1
	(9, 12, 13)	3 + 4*k*			4	3	4	1
	(9, 15, 16)	4+2*k*			4	6	7	1
	(9, 17, 18)	3 + 4*k*			3	8	9	1
	(9, 20, 21)	4+2*k*			3	11	12	1
	(9, 22, 23)	3 + 4*k*			3	13	14	1
	(9, 25, 26)	4+2*k*			3	16	17	1
	(9, 27, 28)	3 + 4*k*			3	18	19	1
	(9, 30, 31)	4+2*k*			3	21	22	1
	(9, 32, 33)	3 + 4*k*			3	23	24	1
	(9, 35, 36)	4+2*k*			3	26	27	1
	(9, 37, 38)	3 + 4*k*			3	28	29	1
	(9, 10, 13)	4+4*k*		(9, 12, 15)	3	1	4	3
(10, *y*, *z*)	(10, 11, 13)	4+4*k*	4	(10, 24, 26)	4	1	3	2
	(10, 17, 19)	4+4*k*			3	7	9	2
	(10, 19, 21)	4+2*k*			3	9	11	2
	(10, 21, 23)	4+4*k*			3	11	13	2

Note. To determine the degree p, we used tables 1, 2. Threshold exponent p_{th} is the value of the exponent p, at which the difference between the left and right sides of equation (1) changes sign from plus to minus.