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The Beta Kumaraswamy Exponential Model: Bayesian AndNon-Bayesian Estimation

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Abstract
This article addresses two estimation methods for the unknown parameters of beta Kumar-aswamy-exponential (BKw-E) distribution under complete samples. Also, we discussed the estimation of the reliability and hazard functions. A comparison between maximum likelihood method and Bayes method for two unknown parameters of BKw-E distribution is provided. Further, the Bayes estimators are studied under three types of loss functions; squared error, linear-exponential and general entropy, using importance sampling technique. Finally, a simulation study is presented to study the performance of the estimated parameters.

Keywords: *Beta Kumaraswamy exponential distribution; maximum likelihood estimator; Bayes estimators.2010 Mathematics Subject Classification: 62E10, 62Q05.*

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I. Introduction

A significant problem in statistical inference is the estimation of the unknown parameters, reliability, and hazard functions based on observed data. In recent decades many models have been developed using different statistical methods, and many researchers have been very interested in estimating these models. One of the commonly used methods of estimation is the maximum likelihood (ML) due to its simplicity and desirable properties. Further, another vital method of estimation is the Bayes method which is widely practiced in cases of having a brief knowledge about the parameters that we aim to estimate. In this article, those two methods will be used to estimate the unknownparameters of the proposed model.

In the last years, Kumaraswamy and its generalizations have been more convenient and handy to use since this distribution has a simple closed-form of its densities (see, (18)). Further, a new class of Kumaraswamy-H distribution with two extra positive parameters was suggested by (13). For any baseline cumulative distribution function CDF,H(x), (13) defined the CDF

where $a, b > 0$ are the shape parameters and $\lambda > 0$ is the scale parameter.

Many authors worked on Kumaraswamy distribution and its generalizations (see, (20), (14), (24)) also some of these works included different methods of estimations such as; the maximum likelihood and Bayes (see, (14), (2), (24), (12), (22), (17), (4), (1)).

Moreover, various researches covered the beta model and its generalizations. (15) introduced a class of generalized distribution known as beta-G(x), they considered a random variable X of the beta random variable and defined it as follows

$B(l, m)$ $_0$

where $l, m > 0$ are the shape parameters. The CDF F(x) is the beta-G(x) distribution and $B(.,.)$ is the beta function. The CDF, G(x), could be any baseline cumulative function. More researches of Beta generalizations and other statistical inferences are included in the references (see, (15), (23), (9), (7), (5), (8), (28)).

Following on, the beta Kumaraswamy-G family was introduced by (16). They obtained some of its characteri- zations including more vital properties, such as moment generating function, probability weighted moments, order statistics, and Rényi entropy. Further, two methods of estimation of the presented model were discussed. Moreover, the properties of beta generated Kumaraswamy-G family with the case of Burr type X as the baseline distributionwas presented by (21).

(3) introduced the beta Kumaraswamy-exponential distribution (BKw-E). Some of its characterizations were ob- tained such as, the probability density function, cumulative distribution function, reliability (survival) and hazard functions, other important proprieties including the moments, median, mode and more are studied. In this article, we investigated the estimation of BKw-E distribution using two methods of estimation.

In Section 2, we presented BKw-E distribution, including the reliability and the hazard functions. The maximum likelihood estimations is discussed in Section 3. Bayesian Estimation is derived in Section 4. Simulation study in Section presented 5. Finally, conclusions are discussed in Section 6.

II. The BKw-E Distribution

In this section, we introduce the five-prameter beta Kumaraswamy-exponential (BKw-E) distribution. By subtitling equation (2) as a baseline cumulative function in equation (3), the cdf of BKw-E is obtaind as follows

$x > 0,\ a, b, \lambda, l, m > 0,$

and the corresponding probability density function (pdf) of the BKw-E takes the form

$$f(x) = \frac{a\lambda b}{B(l,m)} e^{-\lambda x}(1-e^{-\lambda x})^{a-1}\left[1-(1-e^{-\lambda x})^{a}\right]^{mb-1}\left[1-\left[1-(1-e^{-\lambda x})^{a}\right]^{b}\right]^{l-1}, \quad (5)$$

$x > 0,\ a, b, \lambda, l, m > 0.$

III. Maximum likelihood estimation

In this section the maximum likelihood estimators (MLEs) of BKw-E distribution for the two unknown parameters, l and m, are discussed. Let $X_1, X_2, ..., X_n$ be a random sample of size n represents from $f(x_1, x_2, ..., x_n; \theta)$ which referred to as the likelihood function also known as the joint probability density function. The likelihood is a function in θ, usually denoted as, $L(\theta)$, (see, (6)).

The ML estimates of the two unknown parameters, say $\hat{l}$ and $\hat{m}$, of BKw-E will be found numerically by equating the derivatives in equations (13), (14), to zero and solve them using FindRoot function in Mathematica 11.0. We considered three cases, first when l is unknown while a, b, λ and m are known. Next, when m is unknown and a, b, λ, l are known. Finally, when both l and m are unknown where a, b, λ are known. Also, the estimates of $R(x_0)$ and

$h(x_0)$ denoted as $\hat{R}(x_0)$ and $\hat{h}(x_0)$ will be computed from equations (6) and (7) for given x_0 using the estimated parameters $\hat{l}$ and $\hat{m}$.

Bayesian Estimation

In Bayesian structure a parameter of interest is treated mathematically as a random variable that reflects a brief knowledge about the parameter unlike the maximum likelihood method. The parameter is assumed to be unknown. Therefore, the previous information we have about a parameter is called a prior distribution. For more details (see, (6)). The choice of prior is the most critical task, due to the previous information we might have about the parameter before the data is available. That what makes the Bayesian approach more superior in comparison to other procedures of estimation methods. A prior distribution is the PDF that represents the information of the uncertain parameter θ.

Further, if θ is a random variable follows prior distribution $\pi(\theta)$ and $f(\underline{x}|\theta)$ is the sampling distribution, then the conditional density of θ is called the posterior distribution, and is given by

$$\pi^*(\theta|\underline{x}) = \frac{f(x|\theta)\pi(\theta)}{\int f(x|\vartheta)\pi(\vartheta)d\vartheta}. \quad (15)$$

In the problem of statistical decision theory the loss involved to an event is called the loss function. Thus, in estimation method selecting a convenient estimator that minimizes the excepted loss is a priority task (see, (10)). Let T be an estimator of $\phi(\theta)$ then a loss function is described as a real-valued function of $S(t;\theta)$, (see, (6)), suchthat

$S(t;\theta) \geq 0,$ for every t

$S(t;\theta) = 0,$ when $t = \phi(\theta)$.

This paper considered three types of loss functions, squared error (SE) loss, linear-exponential (LINEX) loss and general entropy (GE) loss. SE loss function is a commonly used mainly for its mathematical convenience. It is classified as symmetric function and gives relatively more penalty for large discrepancies. SE loss function is givenby

$$S(t;\theta) = (\hat{\phi}(\theta) - \phi(\theta))^2. \quad (16)$$

Therefore, the Bayes estimate under the SE loss function is the mean of the posterior distribution which can bewritten as follow

$$\tilde{\phi}_{SE}(\theta) = E(\phi(\theta)|\underline{x}) = \int \phi(\theta)\pi^*(\theta|\underline{x})d\theta, \quad (17)$$

where $\pi^*(\theta|\underline{x})$ is the posterior distribution of θ.

Next is the LINEX loss, it was introduced by (Varian). Let $\tilde{\phi}(\theta)$ be the estimator of $\phi(\theta)$ then the LINEX loss

function is defined as follows

$$S(\Delta) = (e^{(\tau\Delta)} - \tau\Delta - 1), \quad (18)$$

where $\Delta = \tilde{\phi}(\theta) - \phi(\theta)$ and $\tau = 0$ is the shape parameter. LINEX loss is considered as asymmetric that is approximately exponential on one side of zero and approximately linear on the other side, the sign and the magnitude of the shape parameter determines the direction and the degree of asymmetric. So, when $\tau > 0$, the LINEX loss is approximately linear on the negative x-axis and approximately exponential on the positive x-axis and vice-versa when is $\tau < 0$. It is valid to use LINEX loss when an overestimate or underestimate might have serious consequences. Moreover, the LINEX loss function will be close to SE loss when τ is near to zero. The Bayes estimator of $\phi(\theta)$ under LINEX loss function is in the following form

$$\tilde{\phi}_{LINEX}(\theta) = -\frac{1}{\tau} \ln \left[E_{\theta}(e^{-\tau\phi(\theta)}|\underline{x}) \right]. \quad (19)$$

Where $E_{\theta}(e^{-\tau\phi(\theta)}|\underline{x})$ is the posterior expectation of the LINEX loss. Hence, it must be finite and existed (see, (10)). Finally, GE loss function was proposed by (11), it can be considered as an alternative to the modified LINEX

where $E_{\theta}(.)$ assumed to be finite and existed.

Note that for $q = 1$, the Bayes estimate of GE loss gives the same results with the Bayes estimate under squared error loss function. Also, the sign of the shape parameter q represents asymmetry direction and its magnitude expresses the degree of asymmetry, see (25) for more details. Moreover, to compute the estimates we used importance sampling technique. Importance sampling is a commonly used tool sampling for Monte Carlo methods, in which a statistical prediction about a target distribution is approximated by a weighted average of random drawings from another distribution. Important samples are expected to contribute more to the estimator, minimize variance and increase the convergence rate.

IV. Results and Remarks

The main remarks and findings of the two estimation methods from the above tables summed up as follows.

From Tables (1, 2, 3), the MSEs of ML estimates and Bayes estimates of l, m, $R(x_0)$ and $h(x_0)$ decrease as the sample size increases.

From Tables (1, 2, 3), the values of the MSEs of SE loss function, LINEX loss function with ($\tau = 0.001$), and GE loss function with ($q = -1$), are very similar.

In general, LINEX loss function gives better results of the estimates, biases, and MSEs as we increase the value of τ. Also, GE loss function mostly preforms better as we increase the valueof q.

ML method of estimation usually gives better estimates than Bayes estimation when n=10,

20. Thus, when n=50 Bayes estimation preforms better as it has the smallest MSEs.

The Bayes estimate of $\phi(m)$ often gives better estimates under LINIX and GE loss functions than ML when (n=10, 20) as shown in Table 2. However, in Table 3 the Bayes estimate of $\phi(m)$ preforms better under all loss functions for all samples size.

From Table 2 and 3 the Bayes estimate of $\phi(m)$ under LINEX loss function ($\tau = 7$, $\tau = 5$) when n=50 are the best estimate in comparison to their other corresponding estimates.

From Table 1 the Bayes estimate of $\phi(l)$ under LINEX loss function ($\tau = 5$) when n=50 is the best estimate in comparison to its other corresponding estimates. However, in Table 3 we could say the Bayes estimate of $\phi(l)$ under SE loss function is the best estimate in comparison to its other corresponding estimates.

In general we can assume that ML method gives better estimates in small samples size for the estimated parameters, then as we increase the sample size Bayesian estimation by using the importance sampling technique preforms better.

V. Conclusion

In this paper, we studied the estimation of the unknown parameters of BKw-E distribution under complete samples using two methods of estimation; Maximum likelihood and Bayesian. The estimators are studied when two parameters l and m are unknown while a, b, λ are known. Bayes estimates were obtained under three types of loss functions: SE, LINEX, GE loss functions, using importance sampling technique. A comparison of the two methods of estimation for the two param- eters l and m were provided. Three cases were considered, first, when l was unknown, next when the parameter m was unknown, finally, when both parameters l and m were unknown.

The importance sampling technique was carried according to the algorithm by (19). Thus, the Mathematica 11.0 was used to compute the estimates of the unknown parameters, the simulation study of the presented model was provided under different sample sizes. Furthermore, according to the MSEs of both methods of estimation we found that the estimators preform well and approach to the values of the estimated parameters as the sample size increase. We have established that in general ML method gives better estimates in small samples size while Bayesian estimation preforms better as we increase the sample size n. Also, the Bayes estimates under LINEX and GE loss functions mostly give better results of the estimates as we increase the values of τ and q. However, the Bayes estimates of both loss functions get closer to their corresponding MSEs of the Bayes estimates under SE loss function as ($\tau = 0.001$) and ($q = -1$).

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