IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-0661,p-ISSN: 2278-8727, Volume 19, Issue 6, Ser. 2 (Nov. – Dec. 2023), PP 50-65
www.iosrjournals.org

The Dynamic Instability of A Periodically Loaded Simple Model Structure

Onuoha N.O. and Vincent Ele Asor
Department of Mathematics, Imo State University, Owerri, Imo State, Nigeria
Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Abia State, Nigeria

ABSTRACT
Structures not subjected to any kind of load are rare to be seen in real life. This makes it necessary to consider the factors that can affect the stability of structures when subjected to different loadings. There are different loading histories for instance; step load, impulse load, periodic load, and moving load. This research investigated the effect of two factors; viscous damping and geometric imperfections on the dynamic buckling load of a model structure lying on a nonlinear cubic foundation trapped by a periodic load. Two-timing regular perturbation method and asymptotic expansions are applied to the model representing the structure. The results obtained showed that viscous damping and geometric imperfections indeed affect the dynamic buckling load of a structure subjected to a particular loading history, periodic load, thereby altering the stability of the structure. As damping increases, the dynamic buckling load of the structure increases and it reduces instability. Increase in the imperfections, decreases the dynamic buckling load which causes it to buckle easily. Structures become unstable when the buckle.

Key words: *Instability, Buckling load, Periodic load, Structure, Geometric imperfections, Damping*

Date of Submission: 09-12-2023 Date of Acceptance: 18-12-2023

I. Introduction

Modern structural engineering occasionally demands the loading of some structures at various loading conditions and loading durations. Such structures are normally inhibited by a series of imperfections caused by some geometrical irregularities and nonlinearities that could have been inadvertently introduced into the structure during the manufacturing process. These irregularities have the tendency of reducing the elastic stability of structures below the stability level of the perfect structure. Damping of a structure is to enhance the dynamic stability of the structure

The subject of dynamic buckling of structures whether at the elastic or plastic stage is a current area of investigation embarked upon by applied mathematicians, civil engineers, mechanical engineers, structural engineers, all in an attempt to determine the best designs for structures under various loading histories. The subject matter originally developed from the initial exploratory investigation by Budiansky [9], Budiansky and Hutchinson [10], and Hutchinson and Budiansky [26] and has since received international and global appeal and acceptance. However, it must be emphasized that most of the investigation so far initiated, have mainly concentrated on the dynamic buckling of structures at the elastic range, Ette [22]. It was noticed that before, research in this area was dominated by Harvard school of engineers pioneered by Budiansky and Hutchinson and there seemed to have been skepticism by the rest of other research communities in probing into this emerging research interest Ette [17,18,19,20]. Later the pre-eminence of dynamic buckling became fully established and soon peaked up to a very high crescendo. Thus, Tamura and Babcock [42] published an appetizing finding on the dynamic stability of cylindrical shells under step loading while much earlier on, Budiansky and Ruth [11], had investigated the axisymmetric dynamic buckling of damped shallow spherical shells. Their findings are still relevant in our modern dispensation because they permit computer application. In the same vein, Roth and Klosner [34] investigated the nonlinear response of cylindrical shells subjected to dynamic axial loads while Chitra et al [13], looked at the dynamic buckling of composite cylindrical shells subjected to axial impulse. It must be stressed for emphases that some loading histories are time independent while some are strictly time dependent. Amazigo and Ette [4] investigated a two-small parameter nonlinear differential equation with application to dynamic buckling while Svalbonas and Kalnus [41] explored the dynamic buckling of shells. Simitses [38] introduced a new type of loading in which a structure would be pre-statically loaded to a level below the static buckling load only to be trapped by a dynamic step load of finite duration. Birman [8] investigated the problem of dynamic buckling of antisymmetric rectangular laminates and also solved the problem of a pre-statically loaded plate superposed on a step load. The exact solution and dynamic buckling of a beam-column system having the elliptic type of loading were introduced by Artem and

Aydin [5]. Nima and Kai-Uwe [35] studied the dynamic buckling of crash boxes under an impact load. Ahmed Naif et al [3] studied the improvement of dynamic buckling behavior of intermediate aluminized stainless steel columns. Song-Hak et al [40], researched on the dynamic buckling of composite structures subjected to impulse loads using the Lyapunov exponent. Most dynamic buckling problems have a high level of nonlinearity and their solutions can involve a high level of complexities and formable computation. In fact, almost all the investigations by Ette [17,18,19,20,21,22] are of this type. However, the method of solution depends on the loading history as well as on the level of nonlinearity. Crocco [14] applied coordinate perturbation and multiple scales to solve problems on gas dynamics. Relatively recent investigations into the dynamic buckling of elastic structures have been rewarding and insightful. Mention in this regard must be made of Sahu and Datta [36], Bazant [6], Onuoha [32], and Karagiozova [27]. The last mentioned actually investigated the dynamic plastic and dynamic progressive buckling of elastic-plastic circular shell. More so, Ahmed and Gareth [2] investigated lateral buckling of offshore pipeline as a result of high temperature and pressure on the pipelines. Onuoha and Ette [31] determined the dynamic stability of a viscously damped elastic model structure subjected to step load. Gladden et al [24] came up with the verification that buckling leads to fragmentation of rods. Special area of interest on triply coupled vibrations of axially loaded thin-walled composite beams were investigated by Thuc et al [43]. The investigation by Enrico et al [16] on the dynamic buckling of impulsively loaded prismatic cores was particularly stimulating while Adhikari and woodhouse [1] studied identification of damping on structures. Ferri et al [23] gave a brief recipe on the buckling of impulsive loaded prismatic cores. Capiez-Lernout et al [12] came up with post-buckling dynamics of a cylindrical shell subjected to a horizontal seismic excitation. The effect of damping on dynamic buckling was similarly investigated by Sapsis et al [37]. Belyaer et al [7] studied the stability of transverse vibration of rod under longitudinal step-wise loading. Lei et al [28], in their research work, investigated the vibration of nonlocal kelvin-voigtviscoelstic damped Timoshenko beams. An excellent treaty by Slim et al [39] discussed the buckling of a thin-layer coquette flow. The dynamic buckling of an inclined struct was investigated by Mcshane et al [30].
Problems with cubic nonlinearity appears to have been first studied by Hansen and Roorda [25] though it is quadratic-cubic while Elishakoff [15] and Ette [17] later made similar investigation on quadratic-cubic nonlinearities. Udo-Akpan and Ette [44], applied two-timing perturbation procedure on the dynamic buckling load of a model structure with quadratic nonlinearities struck by a step load and superposed on a quasi-static load. Osuji et al [33] employed the phase plane using asymptotic expansions of various variables to determine the static buckling analysis of a quadratic-cubic model structure.

1. Buckling Load of the Simple Model Structure

Onuoha and Ette [31], investigated an elastic model structure under step load. This research extends their work to study the dynamic instability of a simple model structure trapped by a periodic load. The model of the simple structure is given as

$$\frac{d^2z}{dt^2}+2\delta\frac{dz}{dt}+\left(1-\lambda\right)z-z^3=\varepsilon\lambda\cos\alpha t \tag{1}$$

$$z\left(0\right)=\frac{dz\left(0\right)}{dt}=0 \tag{2}$$

We shall obtain the classical buckling load, the static buckling load and the dynamic buckling load. Thereafter, know the effect of geometric imperfections and damping on the structure's buckling load.

2. Classical Buckling load λ_c

This is defined as the value of λ at which the perfect structure buckles. λ_c is obtained by neglecting the nonlinear term in (1), and both the inertia and damping term, setting $\cos\alpha t=1,\ \varepsilon=0$, we get

$$\left(1-\lambda\right)z=0 \tag{3}$$

Finally, we obtain λ_c from the condition (Bundisky and Hucthson [9])

$$\frac{d\lambda_c}{dz}=0 \tag{4a}$$

to get

$$\lambda_c=1 \tag{4b}$$

3. Static Buckling load λ_s

This is the load at which the imperfect structure buckles statically. It is obtained from (1) by neglecting the derivative terms and setting $\cos\alpha t = 1$ to get

$$(1-\lambda)z - z^3 = \lambda\varepsilon \tag{5}$$

The condition for λ_s is the same as (4a), and we get

$$(1-\lambda_s) = 3z_s^2 \tag{6}$$

where z_s is the value of z at $\lambda = \lambda_s$.
From equation (6)

$$z_s = \pm\sqrt{\frac{1-\lambda_s}{3}} \tag{7}$$

Determining λ_s from equation (5), we get

$$(1-\lambda_s)^{\frac{3}{2}} = \frac{3\sqrt{3}}{2}\lambda_s\varepsilon \tag{8}$$

4. Dynamic buckling load of a model structure under a periodic load λ_D

We intend to derive the dynamic buckling load, λ_D, of a simple model structure. Equation (1) shall be solved using two-timing regular perturbation and asymptotic expansions.
We let

$$\tau = \delta t \tag{9}$$

$$z(t) = w(t,\tau) \tag{10a}$$

Here, we note that

$$\frac{dz}{dt} = w_{,t} + \delta w_{,\tau} \tag{10b}$$

$$\frac{d^2z}{dt^2} = w_{,tt} + 2\delta w_{,t\tau} + \delta^2 w_{,\tau\tau} \tag{10c}$$

Using equations (10a), (10b) and (10c), equation (1) becomes

$$w_{,tt} + 2\delta w_{,t\tau} + \delta^2 w_{,\tau\tau} + \delta w_{,t} + \delta^2 w_{,\tau} + (1-\lambda)w - w^3 = \varepsilon\lambda\cos\alpha t \tag{11a}$$

$$w(0,0) = \frac{\partial w(0,0)}{\partial t} = 0 \tag{11b}$$

We let

$$w(t,\tau) = \sum_{\substack{i=1 \\ j=0}}^{\infty} w^{(ij)}\varepsilon^i\delta^j \tag{12}$$

On substituting equation (12) in to equation (11a), andequating the coefficients of powers of $\varepsilon^i\delta^j, i = 1,2,3...;\ j = 0,1,2,...$, we get

$$(\varepsilon.\delta^0):\ w_{,tt}^{(10)} + (1-\lambda)w^{(10)} = \lambda\cos\alpha t \tag{13}$$

$$(\varepsilon.\delta):\ w_{,tt}^{(11)} + 2w_{,t\tau}^{(10)} + w_{,t}^{(10)} + (1-\lambda)w^{(11)} = 0 \tag{14}$$

$$(\varepsilon.\delta^2):\ w_{,tt}^{(12)} + 2w_{,t\tau}^{(11)} + w_{,t}^{(11)} + w_{,\tau\tau}^{(10)} + w_{,\tau}^{(10)} + (1-\lambda)w^{(12)} = 0 \tag{15}$$

$$(\varepsilon^2.\delta^0):\ w_{,tt}^{(20)} + (1-\lambda)w^{(20)} = 0 \tag{16}$$

$$(\varepsilon^2.\delta):\ w_{,tt}^{(21)} + 2w_{,t\tau}^{(20)} + w_{,t}^{(20)} + (1-\lambda)w^{(21)} = 0 \tag{17}$$

$$(\varepsilon^2.\delta^2):\ w_{,tt}^{(22)} + 2w_{,t\tau}^{(21)} + w_{,t}^{(21)} + w_{,\tau\tau}^{(20)} + w_{,\tau}^{(20)} + (1-\lambda)w^{(22)} = 0 \tag{18}$$

$$(\varepsilon^3.\delta^0):\ w_{,tt}^{(30)} + (1-\lambda)w^{(30)} - \left(w^{(10)}\right)^3 = 0 \tag{19}$$

$$(\varepsilon^3.\delta):\ w_{,tt}^{(31)} + 2w_{,t\tau}^{(30)} + w_{,t}^{(30)} + (1-\lambda)w^{(31)} - \left(w^{(1)}\right)^2 w^{(11)} = 0 \tag{20}$$

$$(\varepsilon^3.\delta^2):\ w_{,tt}^{(32)} + 2w_{,t\tau}^{(31)} + w_{,\tau\tau}^{(30)} + w_{,t}^{(31)} + w_{,\tau}^{(30)} + (1-\lambda)w^{(32)} - \left(w^{(10)}\right)^2 w^{(12)} - w^{(10)}\left(w^{(11)}\right)^2 = 0 \tag{21}$$

The corresponding initial conditions are:

$$w^{(ij)}(0,0)=0 \tag{22}$$

$$w_{,t}^{(10)}(0,0)=0 \tag{23}$$

$$w_{,t}^{(11)}(0,0)+w_{,\tau}^{(10)}(0,0)=0 \tag{24}$$

$$w_{,t}^{(12)}(0,0)+w_{,\tau}^{(11)}(0,0)=0 \tag{25}$$

$$w_{,t}^{(20)}(0,0)=0 \tag{26}$$

$$w_{,t}^{(21)}(0,0)+w_{,\tau}^{(20)}(0,0)=0 \tag{27}$$

$$w_{,t}^{(22)}(0,0)+w_{,\tau}^{(21)}(0,0)=0 \tag{28}$$

$$w_{,t}^{(30)}(0,0)=0 \tag{29}$$

$$w_{,t}^{(31)}(0,0)+w_{,\tau}^{(30)}(0,0)=0 \tag{30}$$

$$w_{,t}^{(32)}(0,0)+w_{,\tau}^{(31)}(0,0)=0 \tag{31}$$

Solution to equation of order $\left(\varepsilon.\delta^{0}\right)$

$$w_{,tt}^{(10)}+(1-\lambda)w^{(10)}=\lambda\cos\alpha t \tag{32}$$

Solving equation (32), we get

$$w^{(10)}(t,\tau)=A_{10}(\tau)\cos\varphi t+B_{10}(\tau)\sin\varphi t+\frac{\lambda}{\varphi^{2}-\alpha^{2}}\cos\alpha t \tag{33}$$

where $1-\lambda=\varphi^{2}$

On imposing the initial conditions, equations (22) and (23) on equation (33), we have

$$A_{10}(0)=-\frac{\lambda}{\varphi^{2}-\alpha^{2}} \tag{34a}$$

$$B_{10}(0)=0 \tag{34b}$$

Solution to equation of order $(\varepsilon.\delta)$

$$w_{,tt}^{(11)}+\varphi^{2}w^{(11)}=-2w_{,t\tau}^{(10)}-w_{,t}^{(10)} \tag{35a}$$

Equation (35a) can be written as

$$w_{,tt}^{(11)}+\varphi^{2}w^{(11)}=-2\left(-\varphi A_{10}'(\tau)\sin\varphi t+\varphi B_{10}'(\tau)\cos\varphi t\right)-\left(-\varphi A_{10}(\tau)\sin\varphi t+\varphi B_{10}(\tau)\cos\varphi t\right) \tag{35b}$$

To ensure a bounded solution in t, we equate to zero the coefficients of $\sin\varphi t$ and $\cos\varphi t$ respectively.

For $\sin\varphi t$, we get

$$A_{10}(\tau)=k_{1}e^{-\frac{1}{2}\tau} \tag{36a}$$

From equation (34a),

$$A_{10}(0)=-\frac{\lambda}{\varphi^{2}-\alpha^{2}}\Rightarrow k_{1}=-\frac{\lambda}{\varphi^{2}-\alpha^{2}} \tag{36b}$$

Hence,

$$A_{10}(\tau)=-\frac{\lambda}{\varphi^{2}-\alpha^{2}}e^{-\frac{1}{2}\tau} \tag{36c}$$

For $\cos\varphi t$, we get

$$B_{10}(\tau)=k_{2}e^{-\frac{1}{2}\tau} \tag{37a}$$

From equation (34b)

$$B_{10}(0)=0\Rightarrow k_{2}=0 \tag{37b}$$

Hence,

$$B_{10}(\tau)=0 \tag{37c}$$

The remaining part of equation (35b) is solved to get

$$w^{(11)}(t,\tau)=A_{11}(\tau)\cos\varphi t+B_{11}(\tau)\sin\varphi t \tag{38}$$

On imposing the initial conditions, equations (22) and (24) on equation (38), we get

$$A_{11}(0)=0 \quad (39a)$$

$$B_{11}(0)=-\frac{\lambda}{\varphi(\varphi^2-\alpha^2)} \quad (39b)$$

Solution to equation of order $(\varepsilon.\delta^2)$

$$w_{,tt}^{(12)}+\varphi^2 w^{(12)}=-2w_{,t\tau}^{(11)}-w_{,t}^{(11)}-w_{,\tau\tau}^{(10)}-w_{,\tau}^{(10)} \quad (40a)$$

Substituting for $w_{,t\tau}^{(11)}, w_{,t}^{(11)}, w_{,\tau\tau}^{(10)}, w_{,\tau}^{(10)}$, equation (40a) becomes

$$\begin{aligned} w_{,tt}^{(12)}+\varphi^2 w^{(12)}=&-2\left\{-\varphi A_{11}'(\tau)\sin\varphi t+\varphi B_{11}'(\tau)\cos\varphi t\right\}-\left\{-\varphi A_{11}(\tau)\sin\varphi t+\varphi B_{11}(\tau)\cos\varphi t\right\}- \\ &\left\{A_{10}''(\tau)\cos\varphi t+B_{10}''(\tau)\sin\varphi t\right\}-\left\{A_{10}'(\tau)\cos\varphi t+B_{10}'(\tau)\sin\varphi t\right\} \end{aligned} \quad (40b)$$

To ensure a bounded solution in t, we equate to zero the coefficients of $\sin\varphi t$ and $\cos\varphi t$ respectively.
For $\sin\varphi t$, we get

$$A_{11}(\tau)=k_3 e^{-\frac{1}{2}\tau} \quad (41a)$$

From equation (39a)

$$A_{11}(0)=0 \Rightarrow k_3=0 \quad (41b)$$

Hence,

$$A_{11}(\tau)=0 \quad (41c)$$

For $\cos\varphi t$, we get

$$B_{11}(\tau)=k_4 e^{-\frac{1}{2}\tau} \quad (42a)$$

From equation (39b)

$$B_{11}(0)=-\frac{\lambda}{\varphi(\varphi^2-\alpha^2)} \Rightarrow k_4=-\frac{\lambda}{\varphi(\varphi^2-\alpha^2)} \quad (42b)$$

Hence,

$$B_{11}(\tau)=-\frac{\lambda}{\varphi(\varphi^2-\alpha^2)}e^{-\frac{1}{2}\tau} \quad (42c)$$

The remaining part of equation (40b) is solved to get

$$w^{(12)}(t,\tau)=A_{12}(\tau)\cos\varphi t+B_{12}(\tau)\sin\varphi t \quad (43)$$

On imposing the initial conditions, equations (22) and (25) on equation (43), we get

$$A_{12}(0)=0 \quad (44a)$$

$$B_{12}(0)=0 \quad (44b)$$

Solution to equation of order $(\varepsilon^2.\delta^0)$

$$w_{,tt}^{(20)}+\varphi^2 w^{(20)}=0 \quad (45)$$

Solving equation (45), we get

$$w^{(20)}(t,\tau)=A_{20}(\tau)\cos\varphi t+B_{20}(\tau)\sin\varphi t \quad (46)$$

On imposing the initial conditions, equations (22) and (26) on equation (46), we get

$$A_{20}(0)=0 \quad (47a)$$

$$B_{20}(0)=0 \quad (47b)$$

Solution to equation of order $(\varepsilon^2.\delta)$

$$w_{,tt}^{(21)}+\varphi^2 w^{(21)}=-2w_{,t\tau}^{(20)}-w_{,t}^{(20)} \quad (48a)$$

Substituting for the terms on the right hand side of equation (48a), equation (48a) becomes

$$w_{,tt}^{(21)}+\varphi^2 w^{(21)}=-2\left\{-\varphi A_{20}'(\tau)\sin\varphi t+\varphi B_{20}'(\tau)\cos\varphi t\right\}-\left\{-\varphi A_{20}(\tau)\sin\varphi t+\varphi B_{20}(\tau)\cos\varphi t\right\} \quad (48b)$$

To ensure a bounded solution in t, we equate to zero the coefficients of $\sin\varphi t$ and $\cos\varphi t$ respectively.
For $\sin\varphi t$, we get

$$A_{20}(\tau) = k_5 e^{-\frac{1}{2}\tau} \quad (49a)$$

From equation (47a)

$$A_{20}(0) = 0 \Rightarrow k_5 = 0 \quad (49b)$$

Hence,

$$A_{20}(\tau) = 0 \quad (49c)$$

For $\cos\varphi t$, we get

$$B_{20}(\tau) = k_6 e^{-\frac{1}{2}\tau} \quad (50a)$$

From equation (47b)

$$B_{20}(0) = 0 \Rightarrow k_6 = 0 \quad (50b)$$

Hence,

$$B_{20}(\tau) = 0 \quad (50c)$$

The remaining part of equation (48b) is solved to get

$$w^{(21)}(t,\tau) = A_{21}(\tau)\cos\varphi t + B_{21}(\tau)\sin\varphi t \quad (51)$$

On imposing the initial conditions equation (22) and (27) on equation (51), we have

$$A_{21}(0) = B_{21}(0) = 0 \quad (52)$$

Solution to equation of order $\left(\varepsilon^2.\delta^2\right)$

$$w^{(22)}_{,tt} + \varphi^2 w^{(22)} = -2w^{(21)}_{,t\tau} - w^{(21)}_{,t} - w^{(20)}_{,\tau\tau} - w^{(20)}_{,\tau} \quad (53a)$$

Substituting for the terms on the right hand side of equation (53a), we get

$$w^{(22)}_{,tt} + \varphi^2 w^{(22)} = -2\left\{-\varphi A'_{21}(\tau)\sin\varphi t + \varphi B'_{21}(\tau)\cos\varphi t\right\} - \left\{-\varphi A_{21}(\tau)\sin\varphi t + \varphi B_{21}(\tau)\cos\varphi t\right\} \quad (53b)$$

To ensure a bounded solution in t , we equate to zero the coefficients of $\sin\varphi t$ and $\cos\varphi t$ respectively.

For $\sin\varphi t$, we get

$$A_{21}(\tau) = k_7 e^{-\frac{1}{2}\tau} \quad (54a)$$

From equation (52a)

$$A_{21}(0) = 0 \Rightarrow k_7 = 0 \quad (54b)$$

Hence,

$$A_{21}(\tau) = 0 \quad (54c)$$

For $\cos\varphi t$, we get

$$B_{21}(\tau) = k_8 e^{-\frac{1}{2}\tau} \quad (55a)$$

From equation (52b)

$$B_{21}(0) = 0 \Rightarrow k_8 = 0 \quad (55b)$$

Hence,

$$B_{21}(\tau) = 0 \quad (55c)$$

The remaining part of equation (53b) is solved to get

$$w^{(22)}(t,\tau) = A_{22}(\tau)\cos\varphi t + B_{22}(\tau)\sin\varphi t \quad (56)$$

On imposing the initial conditions, equation (22) and (28) on equation (56), we get

$$A_{22}(0) = 0 \quad (57a)$$

$$B_{22}(0) = 0 \quad (57b)$$

Solution to equation of order $\left(\varepsilon^3.\delta^0\right)$

$$w^{(30)}_{,tt} + \varphi^2 w^{(30)} = \left(w^{(10)}\right)^3 \quad (58)$$

Expanding $\left(w^{(10)}\right)^3$ and substituting into equation (58), equation (58) becomes

$$w_{,tt}^{(30)} + \varphi^2 w^{(30)} = \left\{ \frac{3}{4}(A_{10})^3 + \frac{3}{2}A_{10}\left(\frac{\lambda}{\varphi^2-\alpha^2}\right)^2 \right\}\cos\varphi t + \frac{1}{4}(A_{10})^3\cos 3\varphi t +$$
$$\frac{3}{4}(A_{10})^2\left(\frac{\lambda}{\varphi^2-\alpha^2}\right)\cos(\alpha+2\beta)t + \frac{3}{4}(A_{10})^2\left(\frac{\lambda}{\varphi^2-\alpha^2}\right)\cos(\alpha-2\beta)t +$$
$$\frac{3}{4}A_{10}\left(\frac{\lambda}{\varphi^2-\alpha^2}\right)^2\cos(\beta+2\alpha)t + \frac{3}{4}A_{10}\left(\frac{\lambda}{\varphi^2-\alpha^2}\right)^2\cos(\beta-2\alpha)t + \tag{59}$$
$$\left\{ \frac{3}{4}\left(\frac{\lambda}{\varphi^2-\alpha^2}\right)^3 + \frac{3}{2}(A_{10})^2\frac{\lambda}{\varphi^2-\alpha^2} \right\}\cos\alpha t + \frac{1}{4}\left(\frac{\lambda}{\varphi^2-\alpha^2}\right)^3\cos 3\alpha t$$

We write equation (59) in the form

$$w_{,tt}^{(30)} + \varphi^2 w^{(30)} = Q_1\cos\varphi t + Q_2\cos 3\varphi t + Q_3\cos(\alpha+2\beta)t + Q_4\cos(\alpha-2\beta)t +$$
$$Q_5\cos(\beta+2\alpha)t + Q_6\cos(\beta-2\alpha)t + Q_7\cos\alpha t + Q_8\cos 3\alpha t \tag{60}$$

where

$$Q_1 = Q_1(\tau) = \left\{ \frac{3}{4}(A_{10})^3 + \frac{3}{2}A_{10}\left(\frac{\lambda}{\varphi^2-\alpha^2}\right)^2 \right\} \tag{61a}$$

$$Q_2 = Q_2(\tau) = \frac{1}{4}(A_{10})^3 \tag{62b}$$

$$Q_3 = Q_3(\tau) = Q_4 = Q_4(\tau) = \frac{3}{4}(A_{10})^2\left(\frac{\lambda}{\varphi^2-\alpha^2}\right) \tag{62c}$$

$$Q_5 = Q_5(\tau) = Q_6 = Q_6(\tau) = \frac{3}{4}A_{10}\left(\frac{\lambda}{\varphi^2-\alpha^2}\right)^2 \tag{62d}$$

$$Q_7 = Q_7(\tau) = \left\{ \frac{3\lambda}{2(\varphi^2-\alpha^2)}(A_{10})^2 + \frac{3}{4}\left(\frac{\lambda}{\varphi^2-\alpha^2}\right)^3 \right\} \tag{62e}$$

$$Q_8 = Q_8(\tau) = \frac{1}{4}\left(\frac{\lambda}{\varphi^2-\alpha^2}\right)^3 \tag{62f}$$

To ensure a bounded solution in t, we equate to zero the coefficient of $\cos\varphi t$ in equation (60) and get

$$Q_1(\tau) = 0 \tag{63}$$

The remaining part of equation (60) is solved to get

$$w^{(30)}(t,\tau) = A_{30}(\tau)\cos\varphi t + B_{30}(\tau)\sin\varphi t - \frac{Q_2}{8\varphi^2}\cos 3\varphi t + \frac{Q_3}{\varphi^2-(\alpha+2\varphi)^2}\cos(\alpha+2\varphi)t +$$
$$\frac{Q_4}{\varphi^2-(\alpha-2\varphi)^2}\cos(\alpha-2\varphi)t + \frac{Q_5}{\varphi^2-(\varphi+2\alpha)^2}\cos(\varphi+2\alpha)t + \tag{64}$$
$$\frac{Q_6}{\varphi^2-(\varphi-2\alpha)^2}\cos(\varphi-2\alpha)t + \frac{Q_7}{\varphi^2-\alpha^2}\cos\alpha t + \frac{Q_8}{\varphi^2-9\alpha^2}\cos 3\alpha t$$

On imposing the initial conditions, equation (22) and (29) on equation (64), we get

$$A_{30}(0) = -\frac{Q_2(0)}{8\varphi^2} - \frac{Q_3(0)}{\varphi^2-(\alpha+2\varphi)^2} - \frac{Q_4(0)}{\varphi^2-(\alpha-2\varphi)^2} - \frac{Q_5(0)}{\varphi^2-(\varphi+2\alpha)^2} - \frac{Q_6(0)}{\varphi^2-(\varphi-2\alpha)^2} -$$
$$\frac{Q_7(0)}{\varphi^2-\alpha^2} - \frac{Q_8(0)}{\varphi^2-9\alpha^2} \tag{65a}$$

$$B_{30}(0) = 0 \tag{65b}$$

Solution to equation of order $(\varepsilon^3.\delta)$

$$w_{,tt}^{(31)} + \varphi^2 w^{(31)} = -2w_{,t\tau}^{(30)} - w_{,t}^{(30)} - \left(w^{(10)}\right)^2 w^{(11)}$$
$$= -2\left\{ -\varphi A_{30}'(\tau)\sin\varphi t + \varphi B_{30}'(\tau)\cos\varphi t - \frac{3\varphi Q_2'}{8\varphi^2}\sin 3\varphi t - \frac{(\alpha+2\varphi)Q_3'}{\varphi^2-(\alpha+2\varphi)^2}\sin(\alpha+2\varphi)t - \right.$$
$$\frac{(\alpha-2\varphi)Q_4'}{\varphi^2-(\alpha-2\varphi)^2}\sin(\alpha-2\varphi)t - \frac{(\varphi+2\alpha)Q_5'}{\varphi^2-(\varphi+2\alpha)^2}\sin(\varphi+2\alpha)t - \frac{(\varphi-2\alpha)Q_6'}{\varphi^2-(\varphi-2\alpha)^2}\sin(\varphi-2\alpha)t -$$
$$\left.\frac{\alpha Q_7'}{\varphi^2-\alpha^2}\sin\alpha t - \frac{3\alpha Q_8'}{\varphi^2-9\alpha^2}\sin 3\alpha t\right\} - \left\{-\varphi A_{30}(\tau)\sin\varphi t + \varphi B_{30}(\tau)\cos\varphi t - \frac{3\varphi Q_2}{8\varphi^2}\sin 3\varphi t - \right.$$
$$\frac{(\alpha+2\varphi)Q_3}{\varphi^2-(\alpha+2\varphi)^2}\sin(\alpha+2\varphi)t - \frac{(\alpha-2\varphi)Q_4}{\varphi^2-(\alpha-2\varphi)^2}\sin(\alpha-2\varphi)t - \frac{(\varphi+2\alpha)Q_5}{\varphi^2-(\varphi+2\alpha)^2}\sin(\varphi+2\alpha)t -$$
$$\left.\frac{(\varphi-2\alpha)Q_6}{\varphi^2-(\varphi-2\alpha)^2}\sin(\varphi-2\alpha)t - \frac{\alpha Q_7}{\varphi^2-\alpha^2}\sin\alpha t - \frac{3\alpha Q_8}{\varphi^2-9\alpha^2}\sin 3\alpha t\right\} + \frac{1}{4}(A_{10})^2 B_{11}\sin\varphi t +$$
$$\frac{1}{4}(A_{10})^2 B_{11}\sin 3\varphi t - \frac{\lambda}{2(\varphi^2-\alpha^2)}A_{10}B_{11}\sin(2\varphi+\alpha)t + \frac{\lambda}{2(\varphi^2-\alpha^2)}A_{10}B_{11}\sin(2\varphi-\alpha)t +$$
$$\frac{\lambda^2}{2(\varphi^2-\alpha^2)^2}B_{11}\sin\varphi t + \frac{\lambda^2}{4(\varphi^2-\alpha^2)^2}B_{11}\sin(\varphi^2+2\alpha)t + \frac{\lambda^2}{4(\varphi^2-\alpha^2)^2}B_{11}\sin(\varphi^2-2\alpha)t \quad (66)$$

To ensure a bounded solution in t, we equate to zero the coefficients of $\sin\varphi t$ and $\cos\varphi t$ respectively.
For $\sin\varphi t$, we get

$$A_{30}(\tau) = e^{-\frac{1}{2}\tau}\left\{\int_0^\tau H_1(\tau)e^{\frac{1}{2}\tau}d\tau + A_{30}(0)\right\} \quad (67a)$$

where

$$H_1(\tau) = -\frac{1}{4\varphi^2}(A_{10})^2 B_{11} + \frac{1}{8\varphi^2}(A_{10})^2 B_{11} - \frac{\lambda^2}{4\varphi^2(\varphi^2-\alpha^2)^2}B_{11} \quad (67b)$$

For $\cos\varphi t$, we get

$$B_{30}(\tau) = k_9 e^{-\frac{1}{2}\tau} \quad (68a)$$

From equation (65b)

$$B_{30}(0) = 0 \Rightarrow k_9 = 0 \quad (68b)$$

Hence,

$$B_{30}(\tau) = 0 \quad (68c)$$

The remaining part of equation (66) is solved to get

$$w^{(31)}(t,\tau) = A_{31}(\tau)\cos\varphi t + B_{31}(\tau)\sin\varphi t - \frac{Q_9}{8\varphi^2}\sin 3\varphi t - \frac{Q_{10}}{\varphi^2-(2\varphi+\alpha)^2}\sin(2\varphi+\alpha)t +$$
$$\frac{Q_{11}}{\varphi^2-(2\varphi-\alpha)^2}\sin(2\varphi-\alpha)t - \frac{Q_{12}}{\varphi^2-(\varphi+2\alpha)^2}\sin(\varphi+2\alpha)t + \quad (69)$$
$$\frac{Q_{13}}{\varphi^2-(\varphi-2\alpha)^2}\sin(\varphi-2\alpha)t - \frac{Q_{14}}{\varphi^2-\alpha^2}\sin\alpha t - \frac{Q_{15}}{\varphi^2-9\alpha^2}\sin 3\alpha t$$

where

$$Q_9(\tau) = -\frac{3Q_2'}{4\varphi} - \frac{3Q_2}{8\varphi} - \frac{1}{4}(A_{10})^2 B_{11} \quad (70a)$$

$$Q_{10}(\tau) = \frac{2(2\varphi+\alpha)Q_3'}{\varphi^2-(2\beta+\alpha)^2} + \frac{(2\varphi+\alpha)Q_3}{\varphi^2-(2\varphi+\alpha)^2} \quad (70b)$$

$$Q_{11}(\tau)=\frac{2(2\varphi-\alpha)Q_4'}{\varphi^2-(2\varphi-\alpha)^2}+\frac{(2\varphi-\alpha)Q_4}{\varphi^2-(2\varphi-\alpha)^2}+\frac{\lambda A_{10}B_{11}}{2(\varphi^2-\alpha^2)} \tag{70c}$$

$$Q_{12}(\tau)=\frac{2(\varphi+2\alpha)Q_5'}{\varphi^2-(\varphi+2\alpha)^2}+\frac{(\varphi+2\alpha)Q_5}{\varphi^2-(\varphi+2\alpha)^2}-\frac{\lambda^2 B_{11}}{4(\varphi^2-\alpha^2)^2} \tag{70d}$$

$$Q_{13}(\tau)=\frac{2(\varphi-2\alpha)Q_6'}{\varphi^2-(\varphi-2\alpha)^2}+\frac{(\varphi-2\alpha)Q_6}{\varphi^2-(\varphi-2\alpha)^2}-\frac{\lambda^2 B_{11}}{4(\varphi^2-\alpha^2)^2} \tag{70e}$$

$$Q_{14}(\tau)=\frac{\alpha}{\varphi^2-\alpha}(2Q_7'+Q_7) \tag{70f}$$

$$Q_{15}(\tau)=\frac{3\alpha}{\varphi^2-9\alpha^2}(2Q_8'+Q_8) \tag{70g}$$

On imposing the initial conditions, equation (22) and (30) on equation (69), we get

$$A_{31}(0)=0 \tag{71a}$$

$$\begin{aligned}B_{31}(0)=\frac{1}{\varphi}\Bigg\{&\frac{3Q_9(0)}{8\varphi}+\frac{(2\varphi+\alpha)Q_{10}(0)}{\varphi^2-(2\varphi+\alpha)^2}+\frac{(2\varphi-\alpha)Q_{11}(0)}{\varphi^2-(2\varphi-\alpha)^2}+\frac{(\varphi+2\alpha)Q_{12}(0)}{\varphi^2-(\varphi+2\alpha)^2}+\\&\frac{(\varphi-2\alpha)Q_{13}(0)}{\varphi^2-(\varphi-2\alpha)^2}+\frac{\alpha Q_{14}(0)}{\varphi^2-\alpha^2}+\frac{3\alpha Q_{15}(0)}{\varphi^2-9\alpha^2}-A_{30}'(0)+\frac{Q_2'(0)}{8\varphi}-\frac{Q_3'(0)}{\varphi^2-(2\varphi+\alpha)^2}+\\&\frac{Q_4'(0)}{\varphi^2-(2\varphi-\alpha)^2}-\frac{Q_5'(0)}{\varphi^2-(\varphi+2\alpha)^2}-\frac{Q_6'(0)}{\varphi^2-(\varphi-2\alpha)^2}-\frac{Q_7'(0)}{\varphi^2-\alpha^2}-\frac{Q_8'(0)}{\varphi^2-9\alpha^2}\Bigg\}\end{aligned} \tag{71b}$$

Solution to equation of order $(\varepsilon^3.\delta^2)$

$$w_{,tt}^{(32)}+\varphi^2 w^{(32)}=-2w_{,t\tau}^{(31)}-w_{,t}^{(31)}-w_{,\tau\tau}^{(30)}-w_{,\tau}^{(30)}+\left(w^{(10)}\right)^2 w^{(12)}+w^{(10)}\left(w^{(11)}\right)^2 \tag{72}$$

Expanding the terms on the right hand side of equation (72) and substituting all into the same equation, we get

$$w_{,tt}^{(32)}+\varphi^2 w^{(32)}=-2\Bigg\{-\varphi A_{31}'\sin\varphi t+\varphi B_{31}'\cos\varphi t-\frac{3Q_9'}{8\varphi}\cos 3\varphi t-\frac{(2\varphi+\alpha)Q_{10}'}{\varphi^2-(2\varphi+\alpha)^2}\cos(2\varphi+\alpha)t-\frac{(2\varphi-\alpha)Q_{11}'}{\varphi^2-(2\varphi-\alpha)^2}\cos(2\varphi-\alpha)t-$$
$$\frac{(\varphi+2\alpha)Q_{12}'}{\varphi^2-(\varphi+2\alpha)^2}\cos(\varphi+2\alpha)t-\frac{(\varphi-2\alpha)Q_{13}'}{\varphi^2-(\varphi-2\alpha)^2}3\cos(\varphi-2\alpha)t-\frac{\alpha Q_{14}'}{\varphi^2-\alpha^2}\cos\alpha t-\frac{3\alpha Q_{15}'}{\varphi^2-9\alpha^2}\cos 3\alpha t\Bigg\}-$$
$$\Bigg\{-\varphi A_{31}\sin\varphi t+\varphi B_{31}\cos\varphi t-\frac{3Q_9}{8\varphi}\cos 3\varphi t-\frac{(2\varphi+\alpha)Q_{10}}{\varphi^2-(2\varphi+\alpha)^2}\cos(2\varphi+\alpha)t-\frac{(2\varphi-\alpha)Q_{11}}{\varphi^2-(2\varphi-\alpha)^2}\cos(2\varphi-\alpha)t-$$
$$\frac{(\varphi+2\alpha)Q_{12}}{\varphi^2-(\varphi+2\alpha)^2}\cos(\varphi+2\alpha)-\frac{(\varphi-2\alpha)Q_{13}}{\varphi^2-(\varphi-2\alpha)^2}\cos(\varphi-2\alpha)t-\frac{\alpha Q_{14}}{\varphi^2-\alpha^2}\cos\alpha t-\frac{3\alpha Q_{15}}{\varphi^2-9\alpha^2}\cos 3\alpha t\Bigg\}-$$
$$\Bigg\{A_{30}''\cos\varphi t-\frac{Q_2''}{8\varphi}\cos 3\varphi t+\frac{Q_3''}{\varphi^2-(2\varphi+\alpha)^2}\cos(2\varphi+\alpha)t+\frac{Q_4''}{\varphi^2-(2\varphi-\alpha)^2}\cos(2\varphi-\alpha)t+$$
$$\frac{Q_5''}{\varphi^2-(\varphi+2\alpha)^2}\cos(\varphi+2\alpha)t+\frac{Q_6''}{\varphi^2-(\varphi-2\alpha)^2}\cos(\varphi-2\alpha)t+\frac{Q_7''}{\varphi^2-\alpha^2}\cos\alpha t+\frac{Q_8''}{\varphi^2-9\alpha^2}\cos 3\alpha t\Bigg\}-$$
$$\Bigg\{A_{30}'\cos\varphi t-\frac{Q_2'}{8\varphi}\cos 3\varphi t+\frac{Q_3'}{\varphi^2-(2\varphi+\alpha)^2}\cos(2\varphi+\alpha)t+\frac{Q_4'}{\varphi^2-(2\varphi-\alpha)^2}\cos(2\varphi-\alpha)t$$
$$\frac{Q_5'}{\varphi^2-(\varphi+2\alpha)^2}\cos(\varphi+2\alpha)t+\frac{Q_6'}{\varphi^2-(\varphi-2\alpha)^2}\cos(\varphi-2\alpha)t+\frac{Q_7'}{\varphi^2-\alpha^2}\cos\alpha t+\frac{Q_8'}{\varphi^2-9\alpha^2}\cos 3\alpha t\Bigg\}-$$
$$\frac{3}{4}(A_{10})^2A_{12}\cos\varphi t+\frac{1}{4}(A_{10})^2A_{12}\cos 3\varphi t+\frac{\lambda A_{10}A_{12}}{\varphi^2-\alpha^2}\cos\alpha t+\frac{\lambda A_{10}A_{12}}{2(\varphi^2-\alpha^2)}\big(\cos(2\varphi+\alpha)t+\cos(2\varphi-\alpha)t\big)+$$
$$\frac{\lambda^2A_{12}}{2(\varphi^2-\alpha^2)^2}\cos\varphi t+\frac{\lambda^2A_{12}}{4(\varphi^2-\alpha^2)^2}\big(\cos(\varphi+2\alpha)t+\cos(\varphi-2\alpha)t\big)+\frac{1}{2}(A_{10})^2B_{12}\sin\varphi t+$$
$$\frac{1}{4}(A_{10})^2B_{12}(\sin 3\varphi t-\sin\varphi t)+\frac{\lambda A_{10}B_{12}}{4(\varphi^2-\alpha^2)}\big(\sin(2\varphi+\alpha)t+\sin(2\varphi-\alpha)t\big)+\frac{\lambda^2B_{12}}{2(\varphi^2-\alpha^2)^2}\sin\varphi t+$$
$$\frac{\lambda^2B_{12}}{4(\varphi^2-\alpha^2)^2}\big(\sin(\varphi+2\alpha)t+\sin(\varphi-2\alpha)t\big)+\frac{A_{10}(B_{11})^2}{4}\cos\varphi t-\frac{A_{10}(B_{11})^2}{4}\cos 3\varphi t+\frac{\lambda(B_{11})^2}{2(\varphi^2-\alpha^2)}\cos\alpha t-$$
$$\frac{\lambda(B_{11})^2}{2(\varphi^2-\alpha^2)}\big(\cos(2\varphi+\alpha)t+\cos(2\varphi-\alpha)t\big) \tag{73}$$

To ensure a bounded solution in t, we equate to zero coefficients of $\sin\varphi t$ and $\cos\varphi t$ respectively.

For $\sin\varphi t$, we get

$$A_{31}(\tau)=e^{\frac{1}{2}\tau}\left\{\int_0^\tau H_2(\tau)e^{\frac{1}{2}\tau}d\tau\right\} \tag{74a}$$

where

$$H_2(\tau)=\frac{1}{8\varphi}(A_{10})^2B_{12}-\frac{\lambda^2}{4\varphi(\varphi^2-\alpha^2)^2}B_{12} \tag{74b}$$

For $\cos\varphi t$, we get

$$B_{31}(\tau)=e^{-\frac{1}{2}\tau}\left\{\int_0^\tau H_3(\tau)e^{\frac{1}{2}\tau}+B_{31}(0)\right\} \tag{75a}$$

where

$$H_3(\tau)=\frac{1}{2\varphi}A_{30}''+\frac{1}{2\varphi}A_{30}'-\frac{3}{8\varphi}(A_{10})^2A_{12}+\frac{\lambda^2A_{12}}{4\varphi(\varphi^2-\alpha^2)^2}+\frac{1}{8\varphi}A_{10}(B_{11})^2 \tag{75b}$$

The remaining part of equation (73) is solved to get

$$w^{(32)}(t,\tau)=A_{32}(\tau)\cos\varphi t+B_{32}(\tau)\sin\varphi t-\frac{Q_{16}}{\varphi^2-9\alpha^2}\cos 3\varphi t-\frac{Q_{17}}{\varphi^2-(2\varphi+\alpha)^2}\cos(2\varphi+\alpha)t-$$
$$\frac{Q_{18}}{\varphi^2-(2\varphi-\alpha)^2}\cos(2\varphi-\alpha)t-\frac{Q_{19}}{\varphi^2-(\varphi+2\alpha)^2}\cos(\varphi+2\alpha)t-$$
$$\frac{Q_{20}}{\varphi^2-(\varphi-2\alpha)^2}\cos(\varphi-2\alpha)t-\frac{Q_{21}}{\varphi^2-\alpha^2}\cos\alpha t-\frac{Q_{22}}{\varphi^2-9\alpha^2}\cos 3\alpha t+\frac{Q_{23}}{8\varphi^2}\sin 3\varphi t- \quad (76)$$
$$\frac{Q_{24}}{\varphi^2-(\varphi+2\alpha)^2}\sin(\varphi+2\alpha)t-\frac{Q_{25}}{\varphi^2-(\varphi-2\alpha)^2}\sin(\varphi-2\alpha)t-$$
$$\frac{Q_{26}}{\varphi^2-(2\varphi+\alpha)^2}\sin(2\varphi+\alpha)t-\frac{Q_{27}}{\varphi^2-(2\varphi-\alpha)^2}\sin(2\varphi-\alpha)t$$

where

$$Q_{16}(\tau)=\frac{3Q_9'}{4\varphi}+\frac{3Q_9}{8\varphi}+\frac{Q_2''}{8\varphi}+\frac{Q_2'}{8\varphi}+\frac{(A_{10})^2A_{12}}{4}+\frac{\lambda^2A_{12}}{2(\varphi^2-\alpha^2)}-\frac{A_{10}(B_{11})^2}{4} \quad (77a)$$

$$Q_{17}(\tau)=\frac{2(2\varphi+\alpha)Q_{10}'}{\varphi^2-(2\varphi-\alpha)^2}+\frac{(2\varphi+\alpha)Q_{10}}{\varphi^2-(2\varphi-\alpha)^2}-\frac{Q_3''}{\varphi^2-(2\varphi+\alpha)^2}-\frac{Q_3'}{\varphi^2-(2\varphi+\alpha)^2}+\frac{\lambda A_{10}A_{12}}{2(\varphi^2-\alpha^2)}-\frac{\lambda(B_{11})^2}{2(\varphi^2-\alpha^2)} \quad (77b)$$

$$Q_{18}(\tau)=\frac{2(2\varphi-\alpha)Q_{11}'}{\varphi^2-(2\varphi-\alpha)^2}+\frac{(2\varphi-\alpha)Q_{11}}{\varphi^2-(2\varphi-\alpha)^2}-\frac{Q_4''}{\varphi^2-(2\varphi-\alpha)^2}-\frac{Q_4'}{\varphi^2-(2\varphi-\alpha)^2}+$$
$$\frac{\lambda A_{10}A_{12}}{2(\varphi^2-\alpha^2)}-\frac{\lambda(B_{11})^2}{2(\varphi^2-\alpha^2)} \quad (77c)$$

$$Q_{19}(\tau)=\frac{2(\varphi+2\alpha)Q_{12}'}{\varphi^2-(\varphi+2\alpha)^2}+\frac{(\varphi+2\alpha)Q_{12}}{\varphi^2-(\varphi+2\alpha)^2}-\frac{Q_5''}{\varphi^2-(\varphi+2\alpha)^2}-\frac{Q_5'}{\varphi^2-(\varphi+2\alpha)^2}+\frac{\lambda^2A_{12}}{4(\varphi^2-\alpha^2)^2} \quad (77d)$$

$$Q_{20}(\tau)=\frac{2(\varphi-2\alpha)Q_{13}'}{\varphi^2-(\varphi-2\alpha)^2}+\frac{(\varphi-2\alpha)Q_{13}}{\varphi^2-(\varphi-2\alpha)^2}-\frac{Q_6''}{\varphi^2-(\varphi-2\alpha)^2}-\frac{Q_6'}{\varphi^2-(\varphi-2\alpha)^2}+$$
$$\frac{\lambda^2A_{12}}{4(\varphi^2-\alpha^2)^2}-\frac{\lambda(B_{11})^2}{2(\varphi^2-\alpha^2)} \quad (77e)$$

$$Q_{21}(\tau)=\frac{2\alpha Q_{14}'}{(\varphi^2-\alpha^2)}+\frac{\alpha Q_{14}}{(\varphi^2-\alpha^2)}-\frac{Q_7''}{(\varphi^2-\alpha^2)}-\frac{Q_7'}{(\varphi^2-\alpha^2)}+\frac{\lambda A_{10}A_{12}}{(\varphi^2-\alpha^2)}-\frac{\lambda(B_{11})^2}{2(\varphi^2-\alpha^2)} \quad (77f)$$

$$Q_{22}(\tau)=\frac{6\alpha Q_{15}'}{\varphi^2-9\alpha^2}+\frac{3\alpha Q_{15}}{\varphi^2-9\alpha^2}-\frac{Q_8''}{\varphi^2-9\alpha^2}-\frac{Q_8'}{\varphi^2-9\alpha^2} \quad (77g)$$

$$Q_{23}(\tau)=\frac{(A_{10})^2B_{12}}{4} \quad (77h)$$

$$Q_{24}(\tau)=Q_{25}(\tau)=\frac{\lambda^2B_{12}}{4(\varphi^2-\alpha^2)^2} \quad (77i)$$

$$Q_{26}(\tau)=Q_{27}(\tau)=\frac{\lambda A_{10}B_{12}}{4(\varphi^2-\alpha^2)} \quad (77j)$$

On imposing the initial conditions, equation (22) and (31) on equation (76), we get

$$A_{32}(0)=\frac{Q_{16}(0)}{\varphi^2-9\alpha^2}-\frac{Q_{17}(0)}{\varphi^2-(2\varphi+\alpha)^2}-\frac{Q_{18}(0)}{\varphi^2-(2\varphi-\alpha)^2}-\frac{Q_{19}(0)}{\varphi^2-(\varphi+2\alpha)^2}-\frac{Q_{20}(0)}{\varphi^2-(\varphi-2\alpha)^2}-$$
$$\frac{Q_{21}(0)}{\varphi^2-\alpha^2}-\frac{Q_{22}(0)}{\varphi^2-9\alpha^2} \quad (78a)$$

$$B_{32}(\tau)=\frac{3Q_{23}(0)}{8\varphi}-\frac{(\varphi+2\alpha)Q_{24}(0)}{\varphi^2-(\varphi+2\alpha)^2}-\frac{(\varphi-2\alpha)Q_{25}(0)}{\varphi^2-(\varphi-2\alpha)^2}-\frac{(2\varphi+\alpha)Q_{26}(0)}{\varphi^2-(2\varphi+\alpha)^2}-\frac{(2\varphi-\alpha)Q_{27}(0)}{\varphi^2-(2\varphi-\alpha)^2}-A'_{31}(0) \quad (78b)$$

Now the displacement $w(t,\tau)$ becomes

$$w(t,\tau)=\varepsilon\left\{w^{(10)}+\delta w^{(11)}\right\}+\varepsilon^3\left\{w^{(30)}+\delta w^{(31)}+\delta^2 w^{(32)}\right\} \quad (79)$$

5. Maximum Displacement

Let the maximum displacement be denoted by $w_a(t_a,\tau_a)$. Then,

$$w_a(t_a,\tau_a)=\varepsilon\left\{w^{(10)}(t_a,\tau_a)+\delta w^{(11)}(t_a,\tau_a)\right\}+\varepsilon^3\left\{w^{(30)}(t_a,\tau_a)+\delta w^{(31)}(t_a,\tau_a)+\delta^2 w(t_a,\tau_a)\right\} \quad (80)$$

where t_a and τ_a are the critical values of the associated time variables at maximum displacement.
We shall now determine the maximum displacement. In determining the maximum displacement, we shall assume the following asymptotic expansions as in Onuoha and Ette [31].

$$t_a=t_0+\delta t_{01}+\delta^2 t_{02}+...+\varepsilon\left(t_{10}+\delta t_{11}+\delta^2 t_{12}+...+\right)+\varepsilon^2\left(t_{20}+\delta t_{21}+\delta^2 t_{22}+...+\right)+... \quad (81a)$$

$$\tau_a=\delta\left\{t_0+\delta t_{01}+\delta^2 t_{02}+...+\varepsilon\left(t_{10}+\delta t_{11}+\delta^2 t_{12}+...+\right)+\varepsilon^2\left(t_{20}+\delta t_{21}+\delta^2 t_{22}+...+\right)+...\right\} \quad (81b)$$

Originally, the condition for maximum displacement is

$$\frac{dz}{dt}=0 \quad (82)$$

This translates through (10b) to

$$\sum_{\substack{i=1\\ j=0}}^{\infty} w_{,t}^{(ij)}+\delta\sum_{\substack{i=1\\ j=0}}^{\infty} w_{,\tau}^{(ij)}=0 \quad (83)$$

We evaluate equation (83) at the critical values t_a and τ_a. Expanding the terms of $w(t_a,\tau_a)$ in Taylor's series and equating to zero the coefficients of powers of $\varepsilon^i\delta^j$; $i=1,2,...;\ j=0,1,2,...$, we get

$$(\varepsilon.\delta^0): w_{,t}^{(10)}(t_0,0)=0 \quad (84a)$$

$$(\varepsilon.\delta^1): w_{,tt}^{(10)}t_{01}+w_{,t\tau}^{(10)}t_0+w_{,t}^{(11)}(t_0,0)+w_{,\tau}^{(10)}(t_0,0)=0 \quad (84b)$$

$$(\varepsilon.\delta^2): w_{,tt}^{(10)}t_{02}+w_{,t\tau}^{(10)}t_{01}+w_{,tt}^{(11)}t_{01}+w_{,t\tau}^{(11)}t_{01}=0 \quad (84c)$$

$$(\varepsilon^2.\delta^0): w_{,tt}^{(10)}t_{10}=0 \quad (84d)$$

$$(\varepsilon^2.\delta^1): w_{,tt}^{(10)}t_{11}+w_{,t\tau}^{(10)}t_{10}+w_{,tt}^{(11)}t_{10}=0 \quad (84e)$$

$$(\varepsilon^2.\delta^2): w_{,tt}^{(10)}t_{12}+w_{,t\tau}^{(10)}t_{11}+w_{,tt}^{(11)}t_{11}=0 \quad (84f)$$

$$(\varepsilon^3.\delta^0): w_{,tt}^{(10)}t_{20}+w_{,t}^{(30)}=0 \quad (84g)$$

$$(\varepsilon^3.\delta^1): w_{,tt}^{(10)}t_{21}+w_{,t\tau}^{(10)}t_{20}+w_{,t}^{(11)}t_{20}+w_{,tt}^{(30)}t_{01}=0 \quad (84h)$$

$$(\varepsilon^3.\delta^2): w_{,tt}^{(10)}t_{22}+w_{,t\tau}^{(10)}t_{21}+w_{,tt}^{(11)}t_{21}=0 \quad (84i)$$

Solving equation (84a) – (84i) respectively, we get
$t_0=n\pi,\ n=0,1,2,...$
We need the least non-trivial value of t_0 and so we set $n=1$ and get

$$t_0=\pi \quad (85a)$$

$$t_{01}=\frac{1}{2\left(\varphi^2-\alpha^2\right)} \quad (85b)$$

$$t_{02}=\frac{1}{\varphi^2-\alpha^2} \quad (85c)$$

$$t_{10}=0 \quad (85d)$$

$$t_{11}=0 \quad (85e)$$

$$t_{12} = 0 \quad (85f)$$

$$t_{20} = 0 \quad (85g)$$

$$t_{21} = -\frac{1}{2\lambda\left(\varphi^2 - \alpha^2\right)} w^{(30)}\left(t_0, 0\right) \quad (85h)$$

$$t_{22} = 0 \quad (85i)$$

Equation (80) becomes

$$w_a\left(t_a, \tau_a\right) = \varepsilon\left[w^{(10)} + \delta\left(w_{,\tau}^{(10)} t_0 + w^{(11)} t_0\right) + \delta^2\left(w_{,\tau}^{(10)} t_{01} + w_{,t}^{(11)} t_{01}\right)\right] + \varepsilon^3\left[w^{(30)} + \delta\left(w_{,\tau}^{(10)} t_{20} + w_{,\tau}^{(30)} t_0\right)\right] \quad (86)$$

Further simplification of equation (86) gives

$$w_a\left(t_a, \tau_a\right) = N_1\varepsilon\left\{\pi + \frac{\delta}{\varphi^2 - \alpha^2}\right\} + N_2\varepsilon^3\left(2 + 3\delta\pi\right) \quad (87)$$

where

$$N_1 = \frac{\delta\lambda}{\varphi^2 - \alpha^2}, \; N_2 = \frac{\lambda^3}{36\varphi^2\left(\varphi^2 - \alpha^2\right)^3} \quad (88)$$

For ease of further analysis, as in Onuoha and Ette [31], we let

$$w_a\left(t_a, \tau_a\right) = \varepsilon c_1 + \varepsilon^2 c_2 + \varepsilon^3 c_3 + \ldots \quad (89)$$

where

$$c_1 = N_1\left(\pi + \frac{\delta}{\varphi^2 - \alpha^2}\right)$$

$$c_2 = 0 \quad (90)$$

$$c_3 = N_2\left(2 + 3\delta\pi\right)$$

As in Budiansky and Hutchinson [10], the condition for dynamic buckling is

$$\frac{d\lambda}{dw_a} \quad (91)$$

As in Ette [21, 22] we first reverse the series in the form

$$\varepsilon = d_1 w_a + d_2 w_a^2 + d_3 w_a^3 + \ldots \quad (92)$$

By substituting for w_a in equation (92) from (90) and equating the coefficients of powers of ε , we get

$$O(\varepsilon): d_1 c_1 = 1$$

$$d_1 = \frac{1}{c_1} \quad (93a)$$

$$O\left(\varepsilon^2\right): d_1 c_2 + d_2 c_1^2 = 0$$

$$d_2 = 0 \quad (93b)$$

$$O\left(\varepsilon^3\right): d_1 c_3 + d_3 c_1^3 = 0$$

$$d_3 = -\frac{c_3}{c_1^4} \quad (94c)$$

where d_i depends on λ for $i = i, 2, 3, \ldots$

The maximization equation (91) is now accomplished through (92) to give

$$\frac{d\lambda}{dw_a} = d_1 + 3d_3 w_a^2 = 0 \quad (95)$$

which is evaluated at $\lambda = \lambda_D$.

On solving for w_a , we get

$$w_a^2 = -\frac{d_1}{3d_3} = \pm\sqrt{\frac{c_1^3}{3c_3}} \quad (96)$$

6. Dynamic Buckling Load λ_D

To determine the dynamic buckling load λ_D , we evaluate equation (92) at $\lambda = \lambda_D$

$$\varepsilon = d_{1D} w_a + d_{3D} w_a^3 \tag{97}$$

Further simplification of equation (97), we have

$$\varepsilon = \frac{2}{3}\sqrt{\frac{c_1}{3c_3}} \tag{98}$$

On simplifying equation (98) at $\lambda = \lambda_D$, we get

$$\varepsilon = \frac{2}{3}\sqrt{\frac{N_1\left(\pi + \frac{\delta}{\varphi^2 - \alpha^2}\right)}{3N_2\left(2 + 3\delta\pi\right)}} \tag{99}$$

Further simplification of equation (99) gives

$$\lambda_D = \frac{4\left(1-\lambda_D\right)}{\varepsilon}\left\{\frac{\left\{\delta\left(1-\lambda_D\right)^2 - \alpha^2\right\}\left[\pi\left\{\left(1-\lambda_D\right)^2 - \alpha^2\right\} + \delta\right]}{3\left(2+3\delta\pi\right)}\right\}^{\frac{1}{2}} \tag{100}$$

The dynamic buckling load, λ_D of the simple model structure is computed from equation (100). Numerically computed values of the dynamic buckling load for various values of the parameters, δ and ε are summarized in tables I and Fig. I

Table I: Computed Values of Buckling Load λ_D at Various Values of Damping δ and Imperfection ε

δ	λ_D at $\varepsilon = 0.01$	λ_D at $\varepsilon = 0.02$	λ_D at $\varepsilon = 0.03$	λ_D at $\varepsilon = 0.04$	λ_D at $\varepsilon = 0.05$	λ_D at $\varepsilon = 0.06$	λ_D at $\varepsilon = 0.07$	λ_D at $\varepsilon = 0.08$	λ_D at $\varepsilon = 0.09$	λ_D at $\varepsilon = 0.10$
0.00	0.797190	0.743390	0.701617	0.666700	0.636454	0.609657	0.585558	0.563646	0.543552	0.513900
0.01	0.798325	0.744360	0.703751	0.660000	0.638725	0.612638	0.588723	0.566900	0.547900	0.519877
0.02	0.799527	0.746000	0.706017	0.671700	0.642115	0.615700	0.592095	0.570000	0.551000	0.523242
0.03	0.799999	0.748842	0.708428	0.674500	0.645227	0.619000	0.595699	0.574325	0.554600	0.526354
0.04	0.800000	0.750010	0.709999	0.677582	0.648557	0.622787	0.599563	0.578400	0.559671	0.530000
0.05	0.803026	0.753000	0.713755	0.680000	0.651220	0.626674	0.599067	0.582788	0.563554	0.530968
0.06	0.806750	0.755446	0.716712	0.684227	0.653500	0.630000	0.599936	0.587531	0.569872	0.538762
0.07	0.810056	0.757000	0.718004	0.687000	0.658061	0.635400	0.599987	0.587784	0.570111	0.540019
0.08	0.818735	0.760000	0.723351	0.690000	0.664660	0.638856	0.608974	0.598296	0.578764	0.546708
0.09	0.820486	0.763674	0.725000	0.696376	0.669000	0.640356	0.613865	0.600087	0.581003	0.549762
0.10	0.827623	0.766000	0.731215	0.700000	0.670189	0.646089	0.617423	0.601687	0.583768	0.550000

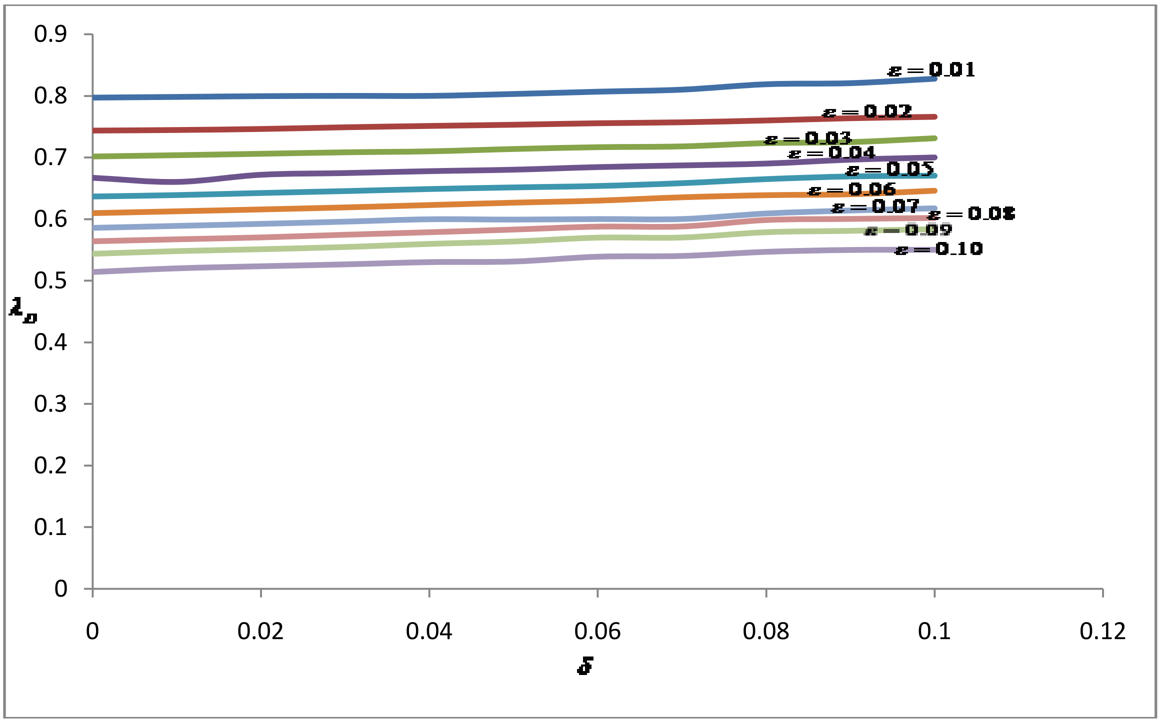

Fig I: Variation of the dynamic buckling load, λ_D with damping, δ at various values of geometric imperfection ε

II. Conclusion

We have carried out an asymptotic determination of the dynamic buckling load of a cubic elastic model structure from perturbation procedures. The unique feature here is that our analysis contains two small mathematically unrelated parameters and upon which asymptotic series expansions are executed in two-timing regular perturbation analysis. Our results showed that increase in viscous damping increases the dynamic buckling load of periodically loaded simple model structure and the presences of imperfections in the structure decreases its dynamic buckling load.

References

[1]. Adhikari, S. and Woodhouse, J. (2000). Identification of damping: part 1, viscous damping, Journal of Sound and Vibrations. 243(1), 43-61.
[2]. Ahmed, M. Reda and Gareth, L. Forbes (2012). Investigation into the dynamic effects of lateral buckling of high temperature/high pressure offshore pipelines. Proceedings of Acoustics.83, 21-2.
[3]. Ahmed, Naif Al-Khazraji, Samir, Ali Al-Rabii and Hameed, Shamkhi Al-Khazaali (2017). Improvement of dynamic buckling behavior of intermediate aluminized stainless steel column. Al-Khwarizmi Engineering Journal. 13(1), 26-41.
[4]. Amazigo, J.C. and Ette, A.M. (1987). On a two-small parameter nonlinear differential equation with application to dynamic buckling. Journal of Nigerian Math-Soc. 6, 91-102.
[5]. Artem, H.S. and Aydin, L. (2010). Exact solution and dynamic buckling analysis of a beam-column system having the elliptic type loading. Appl, Math, Mech Engl. Ed. 31(10), 1317-1324.
[6]. Bazant, Z.P. (2000). Stability of elastic, inelastic and disintegrating structures conspectus of main results.ZAMM. Z. Angew. Math. Mech. 80, 709-732.
[7]. Belyaer, A.K., Uni, D.N. and Morozov, N.F. (2013). Stability of transverse vibrations of under longitudinal step-wise loading. Journal of Physics. 45(1), 1-6.
[8]. Birman, V. (1989).Problems of dynamic buckling of antisymmetric rectangular laminates. Composite Structures. 12, 1-15.
[9]. Budiansky, B. (1964). Dynamic buckling of elastic structures: criteria and estimates, in dynamic stability of structures, Pergamon, New York.
[10]. Budiansky, B. and Hutchinson, J.W. (1966). Dynamic bucking of imperfection sensitive structures. In: Gortler, H. (eds) Applied Mechanics. Springer, Berlin Heidelberg. 636-651https://doi.org/10.1007?978-3-662-29364-5-85.
[11]. Budiansky, B. and Roth, R.S. (1962).Axisymmetric dynamic buckling of damped shallow spherical shells.Collected papers on instability of shell structures. NASA TND-15ID.
[12]. Capiez-Lernout, E., Soize, C. and Mignolet, M.P. (2013). Computational nonlinear stochastic dynamics with model uncertainties and non-stationary stochastic excitation. 11th conference on structural safety andreliability.1-30.
[13]. Chitra, V. and Priyadarsini, R.S. (2013). Dynamic buckling of composite cylindrical shells subjected to Axial Impulse. Int. J. of Sci and Eng. Research.4(5), 2229-5518.
[14]. Crocco, L (1972). Coordinate perturbation and multiple scales in gas dynamics. Phl. Trans. Roy. Soc. A272, 275-301.http://doi.org/10.1098/rsta.1972.0051.
[15]. Elishakoff, I. (1985). Reliability study on the random imperfection sensitivity of column.ActaMechanica.55,151-170.
[16]. Enrico Ferri, Emilio Antinucci, M.Y. He, John W. Hutchinson, Frank W. Zok and Anthony G. Evans. (2006). Dynamic buckling of Impulsively loaded prismatic cores. Journal of Mechanics. 1(8), 1345-1365. Doi: 10.2140/jomms.2006.1.1345.

[17]. Ette, A.M. (1992). On the buckling of a finite imperfect simply supported column on quadratic-cubic Foundation under step loading-analytical approach. J. of Nigerian Math. Soc. 11(1),99-113.

[18]. Ette, A.M. (1997). Dynamic buckling of an imperfect spherical shell under axial impulse.Int. J.Non-linearMech.32(1), 201-209.

[19]. Ette, A.M. (2006). Asymptotic solution on the dynamic buckling of a column stressed by a dynamically slowly varying load. J. of Nigerian Assoc. Math.Physics.10, 1997-2006.

[20]. Ette, A.M. (2007). Perturbation analysis on the dynamic buckling of a lightly damped spherical cap modulated by a slowly varying sinusoidal load (1). J. Nigerian Assoc. Math. Physics.11, 327-322.

[21]. Ette, A.M. (2008). Perturbation technique on the dynamic stability of a lightly damped cylindrical shell axially stressed by an impulse. J. Nigerian Assoc. Math. Physics.12,103-120.

[22]. Ette, A.M. (2009). On a randomly imperfect spherical cap pressurized by a random dynamic Load. J. Nigerian Assoc. Math. Physics. 14(1)

[23]. Ferri, E., Antinucci, E., He, M.Y., Hutchinson, J.W., Zok, F.W. and Evans, A.G. (2006). Dynamic buckling of impulsively loaded prismatic cores. J. of Mech. of Materials and Structures.1(8), 1345-1365.

[24]. Gladden, J.R., Handzy, N.Z., Belmonte, A. and Villermaux, E., (2005). "Dynamic Buckling and Fragmentation in Brittle Rods", Phys. Rev. Lett. 94. https://doi.org/10.1103/PhysRevLett.9.

[25]. Hanson, J.S. and Roorda, J. (1974). On a probabilistic stability theory for imperfection sensitivestructures. Int.J. Solids and Structures.10, 341 – 359.

[26]. Hutchinson, J.W. and Budiansky, B (1966). Dynamic Buckling estimates, A.I.A.A.J. 4(3), 525-530.https://doi.org/10.2514/3.3468

[27]. Karagiozova, D. (2004). Dynamic plastic and dynamic progressive, buckling of elastic-plastic circular shells–revisited. Latin American J. Of Solids and Structures.1, 423-441.

[28]. Lei, Y., Adhikari, S. and Riswell, M.I., (2013). Vibration of nonlocal Kelvin-voigt visco elastic damped Timoshenko beams. Int. J. of Eng. SC. 66(1), 1-13.

[29]. Luke, J.C. (1996). Perturbation method for Nonlinear dispersive wave problem, Proc. Roy. Soc.Ser. A. 292, 403-412.

[30]. Mcshane, G.J. Pingle, S.M, Deshpande, V.S. and Fleck, N.A. (2013). Dynamic buckling of an inclined struct. Int. J. of Solids and Structures.49, 2830-2838.

[31]. Onuoha, N.O. and Ette A.E. (2017). Asymptotic analysis of the dynamic stability of a viscously damped elastic model structure under step load. Journal of the Nigerian Association of Mathematical Physics. 41, 131-140.

[32]. Onuoha, N.O.(2023). Analytical Analysis of the dynamic buckling load of a geometrical imperfect column lying on a nonlinear elastic foundation trapped by a time dependent load. International Journal of Scientific and research Publications. 13(7), 32-52.

[33]. Osuji A.O, A.M. Ette and J.U. Chukwuchekwa (2021). Static buckling analysis of a quadratic-cubic model structure using the phase plane method and method of asymptotics. Earthline Journal of Mathematical Sciences.7(1), 181-193. https://doi.org/10.34198/ejms.7121.181193.

[34]. Roth, R.S. and Klosner J.M. (1964). Nonlinear response of cylindrical shells subjected to dynamic axial loads. A.I.A.A.J. 2(10), 1788-1794.

[35]. Nima, Aghdam and Kai-Uwe, Scroeder (2021). Dynamic buckling of crash boxes under animpact load. Proceedings of the 8th International Conference on Coupled Instabilities in Metal Structures. (CIMS 2021),Available at SSRN. https://dx.doi.org/10.2139/ssrn.3868183.

[36]. Sahu, S.K. and Datta, P.K. (2007). Research Advances in the dynamic stability behavior of plates and shells for conservative systems, Appl. Mech. Rev. 60, 65-75.

[37]. Sapsis, T.P., DaneQuinn, D., Vakakis, A.F and Bergman, L.A. (2012). Effective stiffening and damping enhancement of structures with strongly nonlinear local attachments, Journal of Vibration and Acoustics. 1(34), 1-12.

[38]. Simitses, G.J. (1983). Effect of static preloading on the dynamic stability of structures. A.I.A.A.J. 21(8), 1174-1180.

[39]. Slim, A.C., Teichman, J. and Mahaderan, L. (2011). Buckling of a thin –layer Coquette flow. J. Fluid Mech.1-24.

[40]. Song-Hak U, Yong-II So, and Wang-Myong So (2022). Dynamic buckling of composite structures subjected to impulse loads using Lyapunov exponent. International Journal of Structural Stability and Dynamics. 22(8) https://doi.org/10.1142/S0219455422500869.

[41]. Svalbonas, V. and kalnins, A. (1977). Dynamic buckling of shells: evaluation of various methods, Nuclear. Eng. Des.44,331-356.

[42]. Tamura, Y.S. and Babcock, C.D. (1975). Dymamic stability of cylindrical shells under step loading. J. Applied Mech. 43, Ses. E., 190-194.

[43]. Thuc Phuong Vo, Jaehong Lee and kihak Lee, (2010). On triply coupled vibrations of axially loaded thin-walled composite beams. Journal of Computers & Structures. 88(3-4), 144-153.https://doi.org/10.1016/j.compstruc.2009.08.015

[44]. Udo-Akpan I.U. and Ette A.M. (2016): On the dynamic buckling of a model structure with nonlinearities struck by a step load superposed on a quasi-static load. Journal of Nigerian Assoc. of Math., Physics. 35, 461-472.