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A Study on Operators Satisfying Property (Sab) and Browder Type Properties

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Abstract
In this paper, examples of an operator satisfying generalized Browder's theorem and generalized a-Browder's theorem but not satisfy the properties (Bb), (Bab) and (Sb) are given. The necessary and sufficient condition for their equivalence with Browder's theorem and generalized a-Browder's theo- rem are found. The equivalence of property (Sab) with operators satisfying generalized a-Browder's theorem is studied. And also, the equivalence rela- tion between properties (Sab) & (Bab), (Sab) & (Bb) and (Sab) & (Sb) are found.
Keywords: *Browder theorem, Generalized a-Browder's theorem and properties (Sab).*

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I. Introduction and Preliminaries

In this paper, we establish the equivalence relationship between the properties (Bb), (Bab), (Sb) and Browder & generalized a-Browder's theorem are found.

Throughout this paper, $L(X)$ denotes the algebra of all bounded linear operators acting on an infinite dimentional complex Banach Space X. We refer[8] for details about notation and terminologies. We give the following notations that will be useful

- Browder spectrum: $\sigma_b(T)$
- Weyl spectrum: $\sigma_w(T)$
- Upper Semi-Weyl spectrum: $\sigma_{ea}(T)$
- Upper Semi-Browder spectrum: $\sigma_{ub}(T)$
- B-Weyl spectrum: $\sigma_{bw}(T)$
- Upper Semi-B-Weyl spectrum: $\sigma_{uBw}(T)$
- Drazin Invertible spectrum: $\sigma_d(T)$

Definition 1.1. *An operator $T \in L(X)$ is said to have the single valued extension property (SVEP) at $\lambda_0 \in C$ if for every open neighbourhood U of λ_0, the only analytic function $f : U \to H$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$.*

An operator T is said to have SVEP, if T has SVEP at every point $\lambda \in C$. Clearly, an operator T has SVEP at every point of the resolvent $\rho(T) = C - \sigma(T)$
Both the operator T and T^* have SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$, in particular at every isolated point of the spectrum.

Here \ denotes the set difference.The set of all isolated points of $K \subseteq C$ is denoted by iso K.

Definition 1.2. *For $T \in L(X)$, define*

$$\begin{aligned} E_0(T) &= \{\lambda \in \text{iso } \sigma(T): 0 < \alpha(\lambda I - T) < \infty\} \\ E_0^a(T) &= \{\lambda \in \text{iso } \sigma_a(T): 0 < \alpha(\lambda I - T) < \infty\} \\ E(T) &= \{\lambda \in \text{iso } \sigma(T): 0 < \alpha(\lambda I - T)\} \\ E^a(T) &= \{\lambda \in \text{iso } \sigma_a(T): 0 < \alpha(\lambda I - T)\} \end{aligned}$$

Definition 1.3. *For* $T \in L(X)$, *define*

$$\begin{aligned} \Pi_0(T) &= \sigma(T)\setminus\sigma_b(T) \\ \Pi_0^a(T) &= \sigma_a(T)\setminus\sigma_{ub}(T) \\ \Pi(T) &= \sigma(T)\setminus\sigma_d(T) \\ \Pi^a(T) &= \sigma_a(T)\setminus\sigma_{ld}(T) \\ \Pi^+(T) &= \sigma(T)\setminus\sigma_{ld}(T) \end{aligned}$$

Clearly
$\Pi_0(T) \subseteq \Pi^a(T) \subseteq \Pi^a(T)$
and $\Pi_0(T) \subseteq \Pi(T) \subseteq \Pi^a(T)$ *holds.*
Now we describe several spetral properties introduced recently in [6], [7], [8] and [9].

Definition 1.4. *An operator* $T \in L(X)$ *is said to have*
(i) *property(Bb) [7] say* $T \in (p-Bb)$ *if*
$$\sigma(T)\setminus\sigma_{bw}(T) = \Pi_0(T).$$
(ii) *property (Bab) [9] say* $T \in (p-Bab)$ *if*
$$\sigma(T)\setminus\sigma_{bw}(T) = \Pi^a(T).$$
(iii) *property (Sb) [6] say* $T \in (p-Sb)$ *if*
$$\sigma(T)\setminus\sigma_{uBw}(T) = \Pi_0(T).$$
(iv) *property (Sab) [8] say* $T \in (p-Sab)$ *if*
$$\sigma(T)\setminus\sigma_{uBw}(T) = \Pi^a(T).$$

Definition 1.5. *An operator* $T \in L(X)$ *is said to have*
(i) *Browder's theorem,* $T \in (Bt)$ *if*
$$\sigma(T)\setminus\sigma_w(T) = \Pi_0(T).$$
(ii) *generalized Browder's theorem,* $T \in (gBt)$ *if*
$$\sigma(T)\setminus\sigma_{bw}(T) = \Pi(T).$$
(iii) *a-Browder's theorem,* $T \in (a-Bt)$ *if*
$$\sigma_a(T)\setminus\sigma_{ea}(T) = \Pi^a(T).$$
(iv) *generalized a-Browder's theorem,* $T \in (a-gBt)$ *if*
$$\sigma_a(T)\setminus\sigma_{uBw}(T) = \Pi^a(T).$$

II. Preliminaries

Theorem 2.1. *([1],Theorem 3.4) If T is a linear operator on a vector space X. Then the following properties hold .*
1. *If* $p(T) < \infty$, *then* $\alpha(T) \le \beta(T)$.
2. *If* $q(T) < \infty$, *then* $\beta(T) \le \alpha(T)$.
3. *If* $p(T) = q(T) < \infty$, *then* $\alpha(T) = \beta(T)$ *(possibilty infinite).*
4. *If* $\alpha(T) = \beta(T)$, *and if either* $p(T)$ *or* $q(T)$ *is finite, then* $p(T) = q(T)$.

Corollary 2.2. *([1], Corollary 3.21) Let* $\lambda_0 \in \sigma(T)$ *and assume that* $\lambda_0 I - T$ *is a Semi-Fredholm operator. Then the following statements are equivalent.*
1. T *and* T^* *have the SVEP at* λ_0.
2. λ_0 *is an isolated point of* $\sigma(T)$.

Remark 2.3. *For* $T \in L(X)$. *If* $\lambda I - T$ *is a quasi-Fredholm, then the following are equivalent.*
1. $p(\lambda I - T) < \infty \Leftrightarrow T$ *has SVEP at* λ
2. $q(\lambda I - T) < \infty \Leftrightarrow T^*$ *has SVEP at* λ

Theorem 2.4 ([2], Theorem 2.4). *Suppose that* $\lambda I - T \in L(X)$ *is B-Weyl. Then the following statements are equivalent:*
1. T *has SVEP at* λ;
2. $\lambda I - T$ *is Drazin invertible;*
3. T^* *has SVEP at* λ.

Theorem 2.5 ([2], Theorem 3.2)**.** *Let $T \in L(X)$. Then the following statements are equivalent:*

1. *T satisfies generalized Browder's theorem;*
2. *T has SVEP at every $\lambda \notin \sigma_{bw}(T)$;*
3. *T^* has SVEP at every $\lambda \notin \sigma_{bw}(T)$;*
4. *T^* satisfies generalized Browder's theorem.*

Theorem 2.6 ([3], Theorem 2.4)**.** *An operator $T \in L(X)$ satisfies generalized a- Browder's theorem if and only if T has SVEP at every $\lambda \notin \sigma_{uBw}(T)$*

Theorem 2.7 ([4], Theorem 2.1)**.** *For every $T \in L(X)$, we have $\Pi^0(T) \subseteq \Pi^a(T) \subseteq E^a(T)$ and $E^0(T) \subseteq E^a(T)$.*

Lemma 2.8 ([5], Theorem 3.6)**.** *For $T \in L(X)$, the following are equivalent.*

1. *$\lambda_0 I - T$ is left Drazin invertible $\Leftrightarrow \lambda_0 I - T$ is quasi-Fredholm with finite ascent.*
2. *$\lambda_0 I - T$ is right Drazin invertible $\Leftrightarrow \lambda_0 I - T$ is quasi-Fredholm with finite descent.*
3. *$\lambda_0 I - T$ is Drazin invertible $\Leftrightarrow \lambda_0 I - T$ is quasi-Fredholm with finite ascent and descent.*

Corollary 2.9 ([9], Corollary 3.8)**.** *Let $T \in L(X)$. Then the following assertions are equivalent.*

1. *$T \in (p-Bab)$*
2. *$T \in (p-Bb)$ and $\Pi_0(T) = \Pi^a(T)$*
3. *$T \in (p-Bb)$ and $\Pi(T) = \Pi^a(T)$*

0

Remark 2.10. *The necessary and sufficient condition for an operator $T \in (gBt)$ to satisfy $T \in (a-gBt)$ is that T has SVEP at every $\lambda \in \sigma_{uBw}(T)^c \cap \sigma_{bw}(T)$.*

III. Property (Bb), Property (Bab), Property (Sb) and Property (Sab)

In this section, it is shown that an operator satisfying property (Bb), (Bab), (Sb) and (Sab) satisfies generalized Browder's theorem and generalized a-Browder's theorem respectively. The necessary and sufficient condition for the equivalence are found.

3.1 Property (Bb)

In this section, it is shown that an operator satisfying property (Bb) satisfies generalized Browder's theorem but the converse is not true. The necessary and sufficient condition for the equivalence is found.

Theorem 3.1. $T \in (p - Bb) \Rightarrow T \in (gBt)$.

Proof. If $\lambda \in \sigma(T) \backslash \sigma_{bw}(T)$

$then\ \lambda \in \Pi_0(T) \subseteq E_0(T)$ [Since $T \in (p - Bb)$]

$\lambda \in$ iso $\sigma(T)$ and $\lambda \notin \sigma_{bw}(T)$ [by definition of $E_0(T)$]

T and T^* has SVEP at λ and $\lambda I - T$ is B-weyl

$\lambda\ I - T$ is Drazin Invertible

$\lambda \in \sigma(T)$ but $\lambda \notin \sigma_d(T)$

$\lambda \in \Pi(T)$

Therefore

$$\sigma(T) \backslash \sigma_{bw}(T) \subseteq \Pi(T)$$

Conversely, if $\lambda \in \Pi(T)$,

$then\ \lambda \in \sigma(T) \backslash \sigma_d(T)$

$\lambda \in \sigma(T)$ but $\lambda \notin \sigma_d(T)$

$\lambda \in \sigma(T)$ but $\lambda \notin \sigma_{bw}(T)$ [Since $\sigma_{bw}(T) \subseteq \sigma_d(T)$]

$\lambda \in \sigma(T) \backslash \sigma_{bw}(T)$

Therefore

$$\Pi(T) \subseteq \sigma(T) \backslash \sigma_{bw}(T)$$

$$\sigma(T)\backslash\sigma_{bw}(T) = \Pi(T)$$

Therefore, $T \in (gBt)$.

The following example shows that an operator satisfying generalized Browder's theorem need not satisfy property (Bb).

Example 3.2. *Let P be the projection operator defined on l^2 as,*

$$P(a_1, a_2, a_3, \ldots) = (0, a_2, a_3, \ldots)\forall(a) = (a_i) \in l^2.$$

Then $\sigma(P) = \sigma_p(P) = \{0,1\}$. $\sigma_{bw}(P) = \emptyset$, $\sigma_b(P) = \{1\}$, $\sigma_d(P) = \emptyset$, $\Pi_0(P) = \{0\}$, $\Pi(P) = \{0,1\}$.

Therefore

$$\sigma(P)\backslash\sigma_{bw}(P) = \Pi(P)$$

and

$$\sigma(P)\backslash\sigma_{bw}(P) \neq \Pi_0(P)$$

Therefore $P \in (gBt)$ *but* $P \notin (p - Bb)$.

Theorem 3.3. $T \in (p - Bb) \Leftrightarrow T \in (gBt)$ *and* $\Pi(T) = \Pi_0(T)$.

Proof. Let $T \in (p - Bb)$.
$T \in (p - Bb)$ implies $T \in (gBt)$.

$$\Pi_0(T) = \sigma(T)\backslash\sigma_{bw}(T) = \Pi(T)$$
$$\Pi_0(T) = \Pi(T)$$

Hence $T \in (gBt)$ and $\Pi(T) = \Pi_0(T)$.
Conversely,
If $T \in (gBt)$ and $\Pi(T) = \Pi_0(T)$

$$\sigma(T)\backslash\sigma_{bw}(T) = \Pi(T) = \Pi_0(T)$$

Hence $T \in (p - Bb)$.

3.2 Property (Bab)

In this section, it is shown that an operator satisfying property (Bab) satisfies generalized Browder's theorem but the converse is not true. The necessary and sufficient condition for the converse to be true is found out.

Theorem 3.4. $T \in (p - Bab) \Rightarrow T \in (gBt)$.
Proof. If $\lambda \in \sigma(T)\backslash\sigma_{bw}(T)$

$$\begin{aligned}
then\ \lambda &\in \Pi_0^a(T) \subseteq E_0^a(T) && [Since\ T \in (p - Bab)]\\
\lambda &\in iso\ \sigma(T)\ but\ \lambda \notin \sigma_{bw}(T) && [by\ definition\ of\ E_0^a(T)]\\
T &\ \ and\ T^*\ has\ SVEP\ at\ \lambda\ and\ \lambda I - T\ is\ a\ B - weyl\ operator\\
\lambda &\in \sigma(T)\ and\ \lambda I - T\ is\ Drazin\ Invertible.\\
\lambda &\in \sigma(T)\ but\ \lambda \notin \sigma_d(T)\\
\lambda &\in \Pi(T)
\end{aligned}$$

Therefore

$$\sigma(T)\backslash\sigma_{bw}(T) \subseteq \Pi(T)$$

Conversely, if $\lambda \in \Pi(T)$

$$\begin{aligned}
then\ \ \lambda &\in \sigma(T)\backslash\sigma_d(T)\\
\lambda &\in \sigma(T) \text{ but } \lambda \notin \sigma_d(T)\\
\lambda &\in \sigma(T) \text{ but } \lambda \notin \sigma_{bw}(T) && [\text{ Since } \sigma_{bw}(T) \subseteq \sigma_d(T)]\\
\lambda &\in \sigma(T)\backslash\sigma_{bw}(T)
\end{aligned}$$

Therefore

$$\Pi(T) \subseteq \sigma(T)\backslash\sigma_{bw}(T)$$

$$\sigma(T)\setminus\sigma_{bw}(T) = \Pi(T)$$

Therefore, $T \in (gBt)$.

The following example shows that an operator satisfying generalized Browder's theorem need not satisfy property (Bab).

Example 3.5. *Let S be an operator defined on the Hilbert Space l^2 as*

$$S(a_1, a_2, a_3, \dots) = \left(0, \frac{a_1}{2}, 0, \dots\right).$$

Then $\sigma(S) = \sigma_p(S) = \{0\}$. $\sigma_a(S) = \{0\}$, $\sigma_{bw}(S) = \emptyset$, $\sigma_{ub}(S) = \{0\}$, $\sigma_d(S) = \emptyset$, $\Pi_0^a(S) = \emptyset$ and $\Pi(S) = \{0\}$.
Therefore $\sigma(S)\setminus\sigma_{bw}(S) = \Pi(S)$ and $\sigma(S)\setminus\sigma_{bw}(S) \neq \Pi_0^a(S)$
Hence $S \in (gBt)$ but $S \notin (p - Bab)$.

Theorem 3.6. $T \in (p - Bab) \Leftrightarrow T \in (gBt)$ *and* $\Pi(T) = \Pi_0^a(T)$.

Proof. Assume that $T \in (p - Bab)$ implies $T \in (gBt)$.

$$\Pi_0^a(T) = \sigma(T)\setminus\sigma_{bw}(T) = \Pi(T)$$
$$\Pi_0^a(T) = \Pi(T)$$

Hence $T \in (gBt)$ and $\Pi(T) = \Pi_0^a(T)$.
Conversely,
If $T \in (gBt)$ and $\Pi(T) = \Pi_0^a(T)$

$$\sigma(T)\setminus\sigma_{bw}(T) = \Pi(T) = \Pi_0^a(T)$$

Hence $T \in (p - Bab)$.

3.3 Property (Sb)

In this section, it is shown that an operator satisfying property (Sb) satisfies generalized a-Browder's theorem but the converse is not true. The necessary and sufficient condition for the converse to be true is found out.

Theorem 3.7. $T \in (p - Sb) \Rightarrow T \in (a - gBt)$.
Proof. Assume that $T \in (p - Sb)$
To prove $T \in (a - gBt)$. It is enough to show that T has SVEP at each $\lambda \notin \sigma_{uBw}(T)$.
Let $\lambda \notin \sigma_{uBw}(T)$.
If $\lambda \in \sigma(T)$ then $\lambda \in \sigma(T)\setminus\sigma_{uBw}(T)$

$\lambda \in \Pi_0(T) \subseteq E_0^a(T)$ [Since $T \in (p - Sb)$]

$\lambda \in iso\sigma_a(T)$

Therefore, T has SVEP at $\lambda \notin \sigma_{uBw}(T)$.
If $\lambda \notin \sigma(T)$ then $\lambda \in C - \rho(T)$, this implies that $\lambda \in \rho(T)$
Therefore T has SVEP at $\lambda \notin \sigma_{uBw}(T)$.
Hence $T \in (a - gBt)$.

The following example shows that, an operator satisfying generalized a-Browder's theorem need not satisfy property (Sb).

Example 3.8. *Let F_s be the forward shift operator on l^2. Given an operator $F_s: l^2 \to l^2$ defined by*

$$F_s(a_1, a_2, a_3, \dots) = (0, a_1, a_2, a_3, \dots) \quad \forall a = (a_i) \in l^2$$

Then $\sigma_p(F_s) = \emptyset$. $\sigma_c(F_s) = \{\lambda||\lambda| = 1\}$, $\sigma_r(F_s) = \{\lambda||\lambda| < 1\}$, $\sigma(F_s) = \{\lambda||\lambda| \leq 1\} = D(0,1)$, $\sigma_a(F_s) = C(0,1)$, $\sigma_{uBw}(F_s) = C(0,1)$, $\Pi^a(F_s) = \emptyset$, $\Pi_0(F_s) = \emptyset$.
Therefore

$$\sigma_a(F_s)\setminus\sigma_{uBw}(F_s) = C(0,1)\setminus C(0,1) = \emptyset$$
$$= \Pi^a(F_s)$$

and

$$\sigma(F_s)\setminus\sigma_{uBw}(F_s) = D(0,1)\setminus C(0,1)$$
$$\neq \Pi_0(F_s)$$

Therefore $F_s \in (a - gBt)$ but $F_s \notin (p - Sb)$.

Theorem 3.9. $T \in (p - Sb) \Leftrightarrow T \in (a - gBt)$ *and* $\Pi^{+}(T) = \Pi_0(T)$

Proof. Assume that $T \in (p - Sb)$ implies $T \in (a - gBt)$.

$$\text{(i.e.,)} \quad \begin{aligned} \sigma_a(T)\backslash\sigma_{uBw}(T) &= \Pi^{a}(T) \\ \sigma_a(T)\backslash\sigma_{uBw}(T) &= \sigma_a(T)\backslash\sigma_{ld}(T) \\ \sigma_{uBw}(T) &= \sigma_{ld}(T) \end{aligned}$$

Therefore

$$\begin{aligned} \Pi_0(T) &= \sigma(T)\backslash\sigma_{uBw}(T) \\ &= \sigma(T)\backslash\sigma_{ld}(T) \\ &= \Pi^{+}(T) \end{aligned}$$

Hence $T \in (a - gBt)$ and $\Pi^{+}(T) = \Pi_0(T)$.
Conversely,
Assume that $T \in (a - gBt)$ and $\Pi^{+}(T) = \Pi_0(T)$.

$$\begin{aligned} \Rightarrow \sigma(T)\backslash\sigma_{uBw}(T) &= \sigma(T)\backslash\sigma_{ld}(T) \\ &= \Pi^{+}(T) \\ &= \Pi_0(T) \end{aligned}$$

Hence $T \in (p - Sb)$.
Corollary 3.10. *If* $T \in L(X)$ *has SVEP at every* $\lambda \notin \sigma_{uBw}(T)$ *then* $T \in (p - Sb) \Leftrightarrow \Pi_0(T) = \Pi^{+}(T)$.
Proof. Assume that T has SVEP at every $\lambda \notin \sigma_{uBw}(T)$ which implies $T \in (a - gBt)$.

$$\text{(i.e.,)} \quad \sigma_{uBw}(T) = \sigma_{ld}(T)$$

then $T \in (p - Sb)$

$$\begin{aligned} \Pi_0(T) &= \sigma(T)\backslash\sigma_{uBw}(T) \\ \Pi_0(T) &= \sigma(T)\backslash\sigma_{ld}(T) \\ \Pi_0(T) &= \Pi^{+}(T) \end{aligned}$$

3.4 Property (Sab)

In this section an example for an operator satisfying property (Sab) is given. It is shown that an operator satisfying property (Sab) satisfies generalized a-Browder's theorem but the converse is not true. The necessary and sufficient condition for the equivalence is found. In this example shows that an operator satisfying property (Sab) is given.

Example 3.11. *Consider the weighted shift operator W on* l^2 *defined as,*

$$W(a_1, a_2, a_3, \dots) = \left(\frac{a_2}{2}, \frac{a_3}{3}, \frac{a_4}{4}, \dots\right).$$

Then $\sigma(W) = \{0\}$. $\sigma_a(W) = \{0\}$, $\sigma_{uBw}(W) = \{0\}$, $\sigma_{ub}(W) = \{0\}$, $\Pi_0^a(W) = \emptyset$. *Therefore*
$\sigma(W)\backslash\sigma_{uBw}(W) \quad = \quad \Pi_0^a(W)$ *Therefore* $W \in (p - Sab)$.

Theorem 3.12. $T \in (p - Sab) \Rightarrow T \in (a - gBt)$.

Proof. Assume that $T \in (p - Sab)$.
To prove $T \in (a - gBt)$. It is enough to show that T has SVEP at each $\lambda \notin \sigma_{uBw}(T)$.
Let $\lambda \notin \sigma_{uBw}(T)$.
If $\lambda \in \sigma(T)$ then $\lambda \in \sigma(T)\backslash\sigma_{uBw}(T)$

$\lambda \in \Pi_0^a(T) \subseteq E_0^a(T)$ [Since $T \in (p - Sab)$]

$\lambda \in iso\sigma_a(T)$

Therefore, T has SVEP at $\lambda \notin \sigma_{uBw}(T)$.
If $\lambda \notin \sigma(T)$ then $\lambda \in C - \rho(T)$, this implies that $\lambda \in \rho(T)$
Therefore T has SVEP at $\lambda \notin \sigma_{uBw}(T)$.
Hence $T \in (a - gBt)$.

The following example shows that, an operator satisfying generalized a-Browder's theorem need not satisfy property (Sab).

Example 3.13. *Consider the Null operator Z defined on the Hilbert space* l^2. *Then* $\sigma(Z) = \sigma_p(Z) = \{0\}$.
$\sigma_a(Z) = \{0\}$, $\sigma_{ub}(Z) = \{0\}$, $\sigma_{uBw}(Z) = \emptyset$, $\sigma_{ld}(Z) = \emptyset$, $\Pi_a^0(Z) = \emptyset$, $\Pi^a(Z) = \Pi^+(Z) = \{0\}$. *Therefore* $\sigma_a(Z)\backslash\sigma_{uBw}(Z) = \Pi^a(Z)$ *and* $\sigma(Z)\backslash\sigma_{uBw}(Z) \neq \Pi_a^0(Z)$ *Therefore* $Z \in (a - gBt)$ *but* $Z \notin (p - Sab)$.

Theorem 3.14. $T \in (p - Sab) \Leftrightarrow T \in (a - gBt)$ *and* $\Pi^+(T) = \Pi_0^a(T)$

Proof. Assume that $T \in (p - Sab)$ implies $T \in (a - gBt)$.

$$\text{(i.e.,)} \quad \sigma_a(T)\backslash\sigma_{uBw}(T) = \Pi^a(T)$$
$$\sigma_a(T)\backslash\sigma_{uBw}(T) = \sigma_a(T)\backslash\sigma_{ld}(T)$$
$$\sigma_{uBw}(T) = \sigma_{ld}(T).$$

Therefore

$$\begin{aligned}\Pi_0^a(T) &= \sigma(T)\backslash\sigma_{uBw}(T)\\ &= \sigma(T)\backslash\sigma_{ld}(T)\\ &= \Pi^+(T)\end{aligned}$$

Hence $T \in (a - gBt)$ and $\Pi^+(T) = \Pi_0^a(T)$.
Conversely,
Assume that $T \in (a - gBt)$ and $\Pi^+(T) = \Pi_0^a(T)$.

$$\begin{aligned}\Rightarrow \sigma(T)\backslash\sigma_{uBw}(T) &= \sigma(T)\backslash\sigma_{ld}(T)\\ &= \Pi^+(T)\\ &= \Pi_0^a(T)\end{aligned}$$

Hence $T \in (p - Sab)$.

Corollary 3.15. *If* $T \in L(X)$ *has SVEP at every* $\lambda \notin \sigma_{uBw}(T)$ *then* $T \in (p - Sab) \Leftrightarrow \Pi_0^a(T) = \Pi^+(T)$.
Proof. Assume that T has SVEP at every $\lambda \notin \sigma_{uBw}(T)$ which implies $T \in (a - gBt)$.

$$\text{(i.e.,)} \quad \sigma_{uBw}(T) = \sigma_{ld}(T)$$

then $T \in (p - Sab)$

$$\begin{aligned}\Pi_0^a(T) &= \sigma(T)\backslash\sigma_{uBw}(T)\\ \Pi_0^a(T) &= \sigma(T)\backslash\sigma_{ld}(T)\\ \Pi_0^a(T) &= \Pi^+(T)\end{aligned}$$

IV. Variations on Browder theorems

In this chapter, the necessary and sufficient conditions for the equivalences of properties (Sab) & (Bab), (Sab) & (Bb) and (Sab) & (Sb) are found. We have the following equivalences.

$$(p - Bb): \sigma(T) \cap \sigma_{bw}(T)^c = \Pi_0(T) \Leftrightarrow T \in (gBt), \Pi(T) = \Pi_0(T).$$
$$(p - Bab): \sigma(T) \cap \sigma_{bw}(T)^c = \Pi_0^a(T) \Leftrightarrow T \in (gBt), \Pi(T) = \Pi_0^a(T).$$
$$(p - Sb): \sigma(T) \cap \sigma_{uBw}(T)^c = \Pi_0(T) \Leftrightarrow T \in (a - gBt), \Pi_0(T) = \Pi^+(T).$$
$$(p - Sab): \sigma(T) \cap \sigma_{uBw}(T)^c = \Pi_0^a(T) \Leftrightarrow T \in (a - gBt), \Pi_0^a(T) = \Pi^+(T).$$

4. 1 Equivalence of Properties (Sab) and (Bab), Properties (Sab) and (Bb) and Properties (Sab) and (Sb)

Theorem 4.1. $T \in (p - Sab) \Rightarrow T \in (p - Bab)$.

Proof. Let $\lambda \in \sigma(T)\backslash\sigma_{bw}(T)$

$$\begin{aligned} \textit{then} \quad \lambda &\in \sigma(T) \text{ but } \lambda \notin \sigma_{bw}(T)\\ \lambda &\in \sigma(T) \text{ but } \lambda \notin \sigma_{uBw}(T)\\ \lambda &\in \sigma(T)\backslash\sigma_{uBw}(T) && [\text{Since } \sigma_{uBw}(T) \subseteq \sigma_{bw}(T)]\\ \lambda &\in \Pi_a^0(T) && [\text{ Since } T \in (p - Sab))]\end{aligned}$$

Therefore

$$\sigma(T)\backslash\sigma_{bw}(T) \subseteq \Pi_0^a(T)$$

Conversely, let $\lambda \in \Pi_0^a(T) \subseteq E_0^a(T)$

$\lambda \in$ iso $\sigma_a(T)$ but $\lambda \notin \sigma_{uBw}(T)$ [Since $T \in (p - Sab)$]
$\lambda \in$ iso $\sigma(T)$ and $\lambda I - T$ is an upper semi B-Fredholm operator
T and T^* has SVEP at λ and $\lambda I - T$ is a quasi-Fredholm.
$0 < p(T - \lambda I) = q(T - \lambda I) < \infty$ and $\lambda I - T$ is a quasi-Fredholm.
$\lambda \in \sigma(T)$ and $\lambda \notin \sigma_d(T)$
$\lambda \in \sigma(T)$ and $\lambda \notin \sigma_{bw}(T)$ [Since $\sigma_{bw}(T) \subseteq \sigma_d(T)$]
$\lambda \in \sigma(T)\backslash\sigma_{bw}(T)$

Therefore,

$$\Pi_0^a(T) \subseteq \sigma(T)\backslash\sigma_{bw}(T)$$

We get

$$\sigma(T)\backslash\sigma_{bw}(T) = \Pi_0^a(T)$$

Therefore $T \in (p - Bab)$.

Theorem 4.2. *{$T \in (p - Sab) \Leftrightarrow T \in (p - Bab)$} $\Leftrightarrow$ {T has SVEP on $\sigma_{uBw}(T)^c \cap \sigma_{bw}(T)$}*

Proof. To prove {$T \in (p - Sab) \Leftrightarrow T \in (p - Bab)$} $\Leftrightarrow$ {T has SVEP on $\sigma_{uBw}(T)^c \cap \sigma_{bw}(T)$}
$T \in (p - Sab) \Rightarrow T \in (p - Bab)$.
It is sufficient to prove that,
$T \in (p - Bab) \Rightarrow T \in (p - Sab) \Leftrightarrow$ {T has SVEP on $\sigma_{uBw}(T)^c \cap \sigma_{bw}(T)$}
{$T \in (p - Bab) \Rightarrow T \in (p - Sab)$} $\Leftrightarrow$ $T \in (gBt)$ and $\Pi(T) = \Pi_0^a(T)$
& $T \in (a - gBt)$ and $\Pi(T) = \Pi^+(T)$
$\Leftrightarrow$ T has SVEP on $\sigma_{bw}(T)^c$
& T has SVEP on $\sigma_{uBw}(T)^c$
$\Leftrightarrow$ T has SVEP on $\sigma_{uBw}(T)^c\backslash\sigma_{bw}(T)^c$
$\Leftrightarrow$ T has SVEP on $\sigma_{uBw}(T)^c \cap \sigma_{bw}(T)$

Theorem 4.3. $T \in (p - Sab) \Rightarrow T \in (p - Bb)$.

Proof. Let $\lambda \in \sigma(T)\backslash\sigma_{bw}(T)$
then $\lambda \in \sigma(T)\backslash\sigma_{bw}(T) \subseteq \sigma(T)\backslash\sigma_{ubw}(T)$
$\lambda \in \Pi_a^0(T) \subseteq E_0^a(T)$ [Since $T \in (P - Sab)$]
$\lambda \in iso\ \sigma_a(T)$ but $\lambda \notin \sigma_{bw}(T)$
$\lambda \in iso\ \sigma(T)$ and $\lambda I - T$ is a B-Fredholm operator
T and T^* has SVEP at λ and $\lambda I - T$ is a B-Fredholm operator
$0 < p(T - \lambda I) = q(T - \lambda I) < \infty$ and $\lambda I - T$ is a Fredholm operator
$\lambda \in \sigma(T)$ and $T - \lambda I$ is a Browder operator
$\lambda \in \sigma(T)\backslash\sigma_b(T) = \Pi_0(T)$

Therefore,

$$\sigma(T)\backslash\sigma_{bw}(T) \subseteq \Pi_0(T)$$

Conversely, let $\lambda \in \Pi_0(T) \subseteq \Pi(T)$

$\lambda \in \sigma(T)$ but $\lambda \notin \sigma_d(T)$
$\lambda \in \sigma(T)\backslash\sigma_{bw}(T)$ [Since $\sigma_{bw}(T) \subseteq \sigma_d(T)$]

Therefore

$$\Pi_0(T) \subseteq \sigma(T)\backslash\sigma_{bw}(T)$$

We get

$$\sigma(T)\backslash\sigma_{bw}(T) = \Pi_0(T)$$

Therefore $T \in (p - Bb)$.

Theorem 4.4. *$\{T \in (p - Sab) \Leftrightarrow T \in (p - Bb)\} \Leftrightarrow \{T$ has SVEP on $\sigma_{uBw}(T)^c \cap \sigma_{bw}(T)\}$*
Proof. To prove $\{T \in (p - Sab) \Leftrightarrow T \in (p - Bb)\} \Leftrightarrow \{$T has SVEP on $\sigma_{uBw}(T)^c \cap \sigma_{bw}(T)\}$
$T \in (p - Sab) \Rightarrow T \in (p - Bb)$.
It is sufficient to prove that,
$T \in (p - Bb) \Rightarrow T \in (p - Sab) \Leftrightarrow \{$T has SVEP on $\sigma_{uBw}(T)^c \cap \sigma_{bw}(T)\}$
$\{T \in (p - Bb) \Rightarrow T \in (p - Sab)\} \Leftrightarrow T \in (gBt)$ and $\Pi(T) = \Pi_0(T)$
& $T \in (a - gBt)$ and $\Pi_0^a(T) = \Pi^+(T)$
$\Leftrightarrow$ T has SVEP on $\sigma_{bw}(T)^c$
& T has SVEP on $\sigma_{uBw}(T)^c$
$\Leftrightarrow$ T has SVEP on $\sigma_{uBw}(T)^c \backslash \sigma_{bw}(T)^c$
$\Leftrightarrow$ T has SVEP on $\sigma_{uBw}(T)^c \cap \sigma_{bw}(T)$

Lemma 4.5. $T \in (p - Bb) \Leftrightarrow \Pi_0^a(T) \backslash \Pi_0(T) = \Pi_0^a(T) \cap \sigma_{bw}(T)$.

Proof. $T \in (p - Bb)$

$$\Leftrightarrow \sigma(T) \cap \sigma_{bw}(T)^c = \Pi_0(T)$$

$$\begin{aligned} T \in (p - Bb) \Leftrightarrow \Pi_0^a(T) \backslash \Pi_0(T) &= \Pi_0^a(T) \cap \Pi_0(T)^c \\ &= \Pi_0^a(T) \cap [\sigma(T) \cap \sigma_{bw}(T)^c]^c \\ &= \Pi_0^a(T) \cap [\sigma(T)^c \cup \sigma_{bw}(T)] \\ &= [\Pi_0^a(T) \cap \sigma(T)^c] \cup [\Pi_0^a(T) \cap \sigma_{bw}(T)] \\ &= \emptyset \cup [\Pi_0^a(T) \cap \sigma_{bw}(T)] \\ &= \Pi_0^a(T) \cap \sigma_{bw}(T) \end{aligned}$$

Therefore,

$$T \in (p - Bb) \Leftrightarrow \Pi_0^a(T) \backslash \Pi(T) = \Pi_0^a(T) \cap \sigma_{bw}(T).$$

Theorem 4.6. $\{T \in (p - Bb) \Rightarrow T \in (p - Bab)\} \Leftrightarrow \Pi_0^a(T) \cap \sigma_{bw}(T) = \emptyset$

Proof. Since

$$\begin{aligned} T \in (p - Bb) &\Leftrightarrow T \in (gBt) \text{ and } \Pi(T) = \Pi_0(T) \\ \& \ T \in (p - Bab) &\Leftrightarrow T \in (gBt) \text{ and } \Pi(T) = \Pi_0^a(T) \end{aligned}$$

$\{T \in (p - Bb) \Rightarrow T \in (p - Bab)\}$ implies $T \in (gBt)$ & $\Pi_0(T) = \Pi_0^a(T)$.
Hence By lemma
$\{T \in (p - Bb) \Rightarrow T \in (p - Bab)\}$ implies $T \in (gBt)$ & $\Pi_0^a(T) \cap \sigma_{bw}(T) = \emptyset$.
Conversely,

$$\begin{aligned} T \in (p - Bb) &\Rightarrow T \in (gBt) \ \& \ \Pi(T) = \Pi_0(T) \\ &\Rightarrow T \in (gBt) \ \& \ \Pi(T) = \Pi_0^a(T) \\ &\Rightarrow T \in (p - Bab) \end{aligned}$$

Hence $T \in (p - Bb) \Rightarrow T \in (p - Bab)$.
Therefore
$\Pi_0^a(T) \cap \sigma_{bw}(T) = \emptyset$ implies $\{T \in (p - Bb) \Rightarrow T \in (p - Bab)\}$.
Hence

$$\{T \in (p - Bb) \Rightarrow T \in (p - Bab)\} \Leftrightarrow \Pi_0^a(T) \cap \sigma_{bw}(T) = \emptyset$$

. **Corollary 4.7**. $\{T \in (p - Bb) \Leftrightarrow T \in (p - Bab)\} \Leftrightarrow \Pi_0^a(T) \cap \sigma_{bw}(T) = \emptyset$

Lemma 4.8. $T \in (p - Sb) \Leftrightarrow \Pi_0^a(T) \backslash \Pi_0(T) = \Pi_0^a(T) \cap \sigma_{uBw}(T)$.
Proof. $T \in (p - Sb)$

$$\Leftrightarrow \sigma(T) \cap \sigma_{uBw}(T)^c = \Pi_0(T)$$

$$
\begin{aligned}
T \in (p-Sb) \Leftrightarrow \Pi_0^a(T)\backslash\Pi_0(T) &= \Pi_0^a(T) \cap \Pi_0(T)^c \\
&= \Pi_0^a(T) \cap [\sigma(T) \cap \sigma_{uBw}(T)^c]^c \\
&= \Pi_0^a(T) \cap [\sigma(T)^c \cup \sigma_{uBw}(T)] \\
&= [\Pi_0^a(T) \cap \sigma(T)^c] \cup [\Pi_0^a(T) \cap \sigma_{uBw}(T)] \\
&= \emptyset \cup [\Pi_0^a(T) \cap \sigma_{uBw}(T)] \\
&= \Pi_0^a(T) \cap \sigma_{uBw}(T)
\end{aligned}
$$

Therefore,

$$T \in (p-Sb) \Leftrightarrow \Pi_0^a(T)\backslash\Pi_0(T) = \Pi_0^a(T) \cap \sigma_{uBw}(T).$$

Theorem 4.9. $\{T \in (p-Sb) \Rightarrow T \in (p-Sab)\} \Leftrightarrow \Pi_0^a(T) \cap \sigma_{uBw}(T) = \emptyset$

Proof. Since

$$
\begin{aligned}
T \in (p-Sb) &\Leftrightarrow T \in (a-gBt) \text{ and } \Pi_0(T) = \Pi^+(T) \\
\& \; T \in (p-Sab) &\Leftrightarrow T \in (a-gBt) \text{ and } \Pi_0^a(T) = \Pi^+(T)
\end{aligned}
$$

$\{T \in (p-Sb) \Rightarrow T \in (p-Sab)\}$ implies $T \in (a-gBt)$ & $\Pi_0(T) = \Pi_0^a(T)$.
Hence By lemma
$\{T \in (p-Sb) \Rightarrow T \in (p-Sab)\}$ implies $T \in (a-gBt)$ & $\Pi_0^a(T) \cap \sigma_{uBw}(T) = \emptyset$.
Conversely,

$$
\begin{aligned}
T \in (p-Sb) &\Rightarrow T \in (a-gBt) \;\&\; \Pi_0(T) = \Pi^+(T) \\
&\Rightarrow T \in (a-gBt) \;\&\; \Pi_0^a(T) = \Pi^+(T) \\
&\Rightarrow T \in (p-Sab)
\end{aligned}
$$

Hence $T \in (p-Sb) \Rightarrow T \in (p-Sab)$.
Therefore
$\Pi_0^a(T) \cap \sigma_{uBw}(T) = \emptyset$ implies $\{T \in (p-Sb) \Rightarrow T \in (p-Sab)\}$.
Hence

$$T \in (p-Sb) \Rightarrow T \in (Sab)\} \Leftrightarrow \Pi_0^a(T) \cap \sigma_{uBw}(T) = \emptyset.$$

Corollary 4.10. $\{T \in (p-Sb) \Leftrightarrow T \in (p-Sab)\} \Leftrightarrow \Pi_0^a(T) \cap \sigma_{uBw}(T) = \emptyset$

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