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One Dimensional Torus

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Abstract
Consider a transformation T which is measurable and measure pre-serving from a measure space (X, β, ν) to itself on the circle group of one-dimensional torus, where ν is a non-negative countably set additive function and T is Ergodic. We consider the multiplication theory of uni-tary operation. Our work concerns the abelian group of the unit circle. We first prove that the set of all eigenvalues (spectrums) of T forms a subgroup of the unit circle. This result implies that the absolute value of every eigenvalue is a constant, every spectrum is simple and 1 is a simple spectrum. We next prove that T induces a linear operator on the complex measurable functions f for each measurable function f and the transformation T , defined by $T_z = cz$ is not weak mixing when c is not a root of unity. Finally we prove that T is ergodic if and only if 1 is a simple spectrum.

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I. Introduction

Ergodic Theory started in the beginning of the nineteenth century when Poincare deliberated on the solving of differential equations from a new viewpoint [1]. From this viewpoint, one focused on the set of all possible solutions in-lieu of the specific solution[1],[2]. This in due course lead to the idea of the phase space and what came to be called the qualitative theory of differential equations[1]. A further inspiration for ergodic theory arises from statistical mechanics where one of the central questions was the equitability of phase (space) means and time means for certain physical systems or ergodichypothesis[1]. The mathe-matical origination of ergodic theory is generally regarded to have occurred in 1931 when G.D. Birkhoff proved the pointwiseergodic theorem. It was at this point that ergodic theory turn a legitimate mathematical discipline[1].

II. Preliminaries

In this section , we give some basic definitions that will be needed in the subse-quent sections.

Definition 2.1
A measure space is a non- empty set X ; together with a specified sigma algebra
β of subsets of X[4] and a measure ν, defined on that algebra, that is the triple (X, β, ν)[12].

Remark 2.2
(i) A sigma algebra, β is a class of sets closed under the forma-tion of complements and countable unions[12].
(ii) A measure ν non-negative (possibly infinite) countably additive set func-tion.
(iii) The sets in the domain of a measure ν are called measurable subsets ofX[4].

Definition 2.3
A single-valued function T from a measure space (X_1, β_1, ν_1)
into a measure space (X_2, β_2, ν_2)[4] is said to be;

(i) A measurable transformation if $T^{-1}\beta_2 \in \beta_1$, that is, the inverse image $T^{-1}A$ of each element A of β_2 is an element of β_1[11].
i.e. $T^{-1}A \subset \beta_1 \forall A \in \beta_2$
Note: $T^{-1}E = w : T(w) \in E$ for every subset E of X_2[11].

(ii) A measure preserving transformation if T is a measurable transformation such that the inverse image of every set has the same measure as the original set[10], that is, $\nu_1(T^{-1}A) = \nu_2(A)$ for each $A \in \beta_2$[5].
(iii) An invertible transformation, if T is a measurable transformation, which is also bijective, such that the inverse T^{-1} of T is a measurable transforma-tion[11].
(iv) An endomorphism if T is a measure preserving transformation for which the two measure spaces coincide.

(v) A homeomorphism if T is a measure-preserving transformation[5].
(vi) An automorphism if T is an invertible measure-preserving transforma-tion[2].
Note that; (a) A measure preserving transformation (MPT) of a measure space to itself is a quartet (X, β, ν, T), where (X, β, ν) is a measure space[9] and
(i) T is measurable means if $E \in \beta \Rightarrow T^{-1}E \in \beta$.
(ii) ν is T-invariantmeans $\nu(T^{-1}E) = \nu(E) \forall E \in \beta$[5].
(b) A measurable transformation T from a measure space (X_1, β_1, ν_1) into a measure space (X_2, β_2, ν_2) is invertible if there exists a measure transformation S from (X_2, β_2, ν_2) into (X_1, β_1, ν_1) such that both ST and T S are equal to the identity transformation (in their respective domains)[12].

The transformation S is uniquely determined by T , it is called the inverse of T and it is denoted by T^{-1}.

Note: If a measure-preserving transformation is invertible its inverse is also a measure preserving transformation[2],[3].

Definition 2.4

A transformation T is said to have a discrete spectrum if there is a basis f_1 of L_2 (i.e complete orthonormal set) each term of which is a proper vector of the induced unitary operator U_T defined by $U_T f(x) = f(T(x)$[3].

Definition 2.5

A unitary operator U on a Hilbert space H is an isomorphism of H if $U U^* = U^*U = I$[5]. Where U^* is the adjoint operator of U and I is the identity operator of H i.e, U is a linear bijective map preserving the linear product $((U_x, U_y) = (x, y), \forall x, y \in H)$. It follows that U is continuous[5].

Definition 2.6

A transformation T from a measure space (X, β, ν) into itself is said to be de-composable if T is a measurable transformation and there are elements A, B of

β,such that
(i) $X = A \cup B$, (ii)$B = X \setminus A$,
(iii) T is a measurable transformation. (iv)$T^{-1}A = A$ and $T^{-1}B = B$.
(v) $\nu(A) > 0$ and $\nu(B) > 0$, where ν is a positive measure. So, T is said to be a decomposable transformation if T is a measurable for which there∃ two disjoint members A, B of β with $T^{-1}A = A$, $T^{-1}B = B$, such that $A \cup B = X$ satisfying $\nu(A) > 0$, $\nu(B) > 0$,where ν is a positive measure.

Definition 2.7

A transformation T from a measure space (X, β, ν) into itself is said to be er-godic, If T is measurable and non-decomposable, that is if,
(i) T is a measurable transformation and ,
(ii) $A \in \beta$ and $T^{-1}A = A$ imply either $\nu(A) = 0$ or $\nu(X \setminus A) = 0$,where ν is a positive measure.

Remark 2.8

i. If $\nu(X) = 1$, then, the condition (ii) above says that $A \in \beta$ and $T^{-1}A = A$ imply either $\nu(A) = 0$ or $\nu(A) = 1$.
(ii) $A \in \beta$ and $T^{-1}A = A$ imply $B = X \setminus A$ lies in β and $T^{-1}B = (T^{-1}X) \setminus (T^{-1}(A)) = (T^{-1}X) \setminus A = X \setminus A = B$ (as $T^{-1}X = X$ and $T^{-1}A = A$) , where T is measurable, so (ii) says that T is non-decomposable(as $B = X \setminus A$).

Remark2.9

A measurable transformation T of a measure space to itself which is a bijective is not always an invertible measurable transformation because $T^{-1}1$ may not be a measurable transformation. For instance, Let X = Z , (The set of all integers and let β be the σ -algebra generated by the sets $A_1 = (k \in z : k \geq 1)$, $A_n = n, \forall n \in Z$, $(n \neq o)$.
Define $T : Z \to Z$, by $T(n) = n - 1, \forall n \in Z$[4].
Here T is bijective and measurable but its inverse T^{-1} is not a measurable transformation[2].

III. ERGODIC TRANSFORMATIONS

In this section , we bring in the necessary and sufficient condition for a trans-formation T to be ergodic and the multiplication theorem.

Theorem 3.1 Let T be a measurable transformation of a measure space (X, β, ν) into itself. Then, T is ergodic if and only if every invariant func-tionf : $X \to C'$ under T , is a constant, ν -almost everywhere on X, (ν is apositive measure)[1][7].

Definition 3.2

i. A measure space (X, β, ν) is said to be sigma- finite if X is the union of countably many sets of finite measure.

ii. A set w∈ β is called wandering, if ($T^{-n}w$: n ≥ 0) are pair wise disjoint[9].

Definition 3.3

Let (X, β, ν, T) be a measure preserving transformation on an infinite σ –finite measure space. A measure preserving transformation T is called conser-vative, if every wandering set has a measure of zero[9].

Definition 3.4

Let X be a measure space with a normalized measure m, and let β be the set of all equivalence classes of measurable sets, then two measurable sets E and F are called equivalent if and only if their difference E - F has measure zero, that is m(E - F) = 0 implies m(E) = m(F) [6].

THEOREM 3.5 (MULTIPLICATION THEOREM)

A unitary operator U on L_2 is induced by an automorphism T of β if and only if both U and U^{-1} send every bounded function onto a bounded function and U(f g) = (U f)(U g) whenever f and g are bounded functions[2].

Remark 3.7

A probability space (X, β, ν) is a measure space for which ν(x) = 1, (ν being a positive measure on β)[8].

IV. The main results

Definition 4.1

A linear operator U : H → H (H a complex Hilbert space[7]) is said to be aunitary operator if:

i. U is bijective and

ii.∀f , g∈ H U f, U g = f, g (where < ., . > is the inner product of H)[5],[7],

Theorem 4.2

If T is an automorphism of a probability space (X, β, ν) , then

i. T induces a linear operator, denoted U_T on the complex measurable function on (X, β) , given by $U_T f = f \circ T$ for each complex measurable function on (X, β) ,

ii.If f∈ L_2(X, β, ν), we have that $U_T f$ belongs to L_2(X, β, ν), and

iii. U_T is a unitary operator from the Hilbert space L_2(X, β, ν) onto itself[5].

Proof:

i. Let T : X → X be an automorphism of a given probability space (X, β, ν).

If f, g are complex measurable functions on (X, β) , and if C is a complex con-stant, then f +g is a complex measurable function on (X, β), and cf is a complexmeasurable function on (X, β)by definition of U_T ,

$U_T (f + g)(w) \equiv U_T (f(w) + g(w)) \equiv (f + g)(T (w)) = f(T (w) + g(T (w)) =$

$f \circ T (w) + g \circ T (w) = U_T f(w) + U_T g(w)$, and $U_T (cf)(w) = (cf \circ T)(w) = (cf)(T (w)) = cf(T (w)) = cU_T f(w)$.

So, U_T is a linear operator on the vector space of all complex measurable func-tion on (X, β) .

ii. Let f∈ L_2(X, β, ν) . Then by definition, $_x |U_T f|^2 = _X |f \circ T|^2 d\nu = _x f(T(x)f(T(x))d\nu$ (as T is an invertible measure-preserving transformation)

[redacted]

f (X, β,

^{2}dν which is finite. o, U belongs to L^2 ν) with

$= _x f _2\circ f d\nu = _x |f|$ 2

$||U_T f||_{L2(X,\beta,\nu)} = ||f||_{L2(X,\beta,\nu)}$.

iii. From (2) above, we have $||U_T f||_{L2(\nu)} = ||f||_{L2(\nu)}$. so U_T is 1-1 because f, g are in $L^2(\nu)$, then $U_T f = U_T g \Rightarrow U_T (f - g) = 0_{L2} (\nu)$, (as U_T is linear), hence $||f - g||_{L2(\nu)} = ||U_T (f - g)||_{L2(\nu)} = ||0_{L2(\nu)}|| = 0$,giving $f = g \in L^2(\nu)$.

U_T is also onto, as given $f \in L^2(\nu)$, we have that $f \circ T^{-1} \in L^2(X, \beta, \nu)$,

2 T ∘ ∘ ∘

(with T as T^{-1} above) and if U f $_T$ −1 = (f T^{-1}) T = f . Finally, if f, g are in

[redacted] [redacted]

f∈ L (X, β, ν), then $(U_T f, U_T g) =$ $_x U_T f(U_T g)d\nu =$ $_x f(T (w))(g(T (w)))d\nu$
(as T is an invertible measure preserving transformation.[2],[5]) = $_x f \circ gd\nu = (f, g)$
therefore U_T is a unitary operator on $f \in L^2(X, \beta, \nu)$.

Definition 4.3

Two automorphisms T_1 of a measure space (X_1, β_1, ν_1) and T_2 of a measure space (X_2, β_2, ν_2) are said to be spectrally isomorphic if there exists a unitaryoperator $U : L^2(X_1, \beta_1, \nu_1) \to (X_2, \beta_2, \nu_2)$ such that $U \circ U_{T1} = U_{T2} \circ U$, where$U_{T k} : L^2(X_k, \beta_k, \nu_k) \to L^2(X_k, \beta_k, \nu_k)$ is giving by $U_{T k} \circ f = f \circ T_k$ for every
f $\in L^2(X_k, \beta_k, \nu_k)$, (k = 1and2). The property p above is said to be spectrally invariant if it is preserved under spectral isomorphism, that is if T_1 has the property p and if T_1 and T_2 are spectrally isomorphic, then T_2 must have the property p. e,gErgodicity is spectrally invariant[2].

Definition 4.4

Two endomorphisms $T_1 : (X_1, \beta_1, \nu_1) \to (X_1, \beta_1, \nu_1)$and $T_2 : (X_2, \beta_2, \nu_2) \to (X_2, \beta_2, \nu_2)$ are said to be isomorphic transformations if there exists an invertible measure preserving transformation ϕ from a measurespace (X_1, β_1, ν_1) onto the measure space (X_2, β_2, ν_2) such that $\phi \circ T_1 \circ \phi^{-1} = T_2$. A property p is said to be isomorphism invariant if when T_1 has the property p and T_1, T_2 are isomorphic transformations, then T_2 must have the property p[2],[5].

Remark 4.5

Isomorphism of automorphisms implies spectral isomorphism, as $\phi \circ T \circ \phi^{-1} = T_2 \Rightarrow \phi \circ T_1 = T_2 \circ \phi \Rightarrow U \circ U_{T1} = U_{T2} \circ U$ where $U = U_\phi$ and T_1, T_2 are automorphisms.

Definition 4 .6

A complex number α is called an eigenvalue of an automorphismT : (X, β, ν) → (X, β, ν) , if there is f∈ $L^2(X, \beta, \nu)$ with $f \neq 0_{L^2(\nu)}$ such that $U_T f = \alpha f, \Rightarrow f \circ T = \alpha f$.
When that is the case, we say that f is the eigenvector corresponding to the eigenvalue α of T [2] [7].

Definition 4.7

An automorphismT : (X, β, ν) → (X, β, ν) , is said to have a discrete spectrum (or pure point spectrum) if the eigenvectors span $L^2(X, \beta, \nu)$[1].

Definition 4.8

An eigenvalue of an automorphismT : (X, β, ν) → (X, β, ν)is said to be simple eigenvalue of T if∀f, g∈ $L^2(X, \beta, \nu)$, ($U_T f = \alpha f$ and $U_T g = \alpha g$)⇒ f = λg for some complex constant λ[2].

Definition 4.9

An automorphismT : (X, β, ν) → (X, β, ν) , (ν is a positive measure[2] on β) is said to have a continuous spectrum if 1 is the only eigenvalue of T and it is a simple eigenvalue of T .
Note: Spectral Isomorphism does not imply isomorphism of two automorphisms T_1 and T_2.

Definition 4.10

i. A set E is said to be invariant under a transformation T if and only if $T^{-1}E = E$[12] ;this means that x belongs to E if and only if T_x belongs to E. Clearly E is invariant if and only if its characteristic function is invariant[7].
ii. A function f is said to be invariant under a transformation T if and only if $f(T_x) = f(x)$ for all x[12] .

THEOREM 4.11

The transformation $T_z = cz$, z∈ G where (G, ·) is the circle group z∈C : |z| = 1, · multiplication,if then T is not weak mixing if c is not a root of unity. [2]

Proof

Let f(z) = z∀z∈ G. Then $U_T f = f \circ T = cf$, as f(T (z)) = f(cz) = cz =

cf(z)∀z∈ G. So∀n ≥ 1, $U_T^n f = c^n f$. Thus $\frac{1}{n}$ $_{i=0}^{n-1}$ | U_T^i f, f − f, 1 1, f | =

1 n−1 j j
n j=0 |c | = 1 (as |c | = |c| = 1)∀n ≥ 1) as G f(z) · T dm = G 1 · f(z)dm = 0
(with z = e^{it}, t∈ [0, 2π)) because z · z = |z| = 1

⇒ z = ←−^{1} and U_T f, f = c^nf, f [9] z

n n n 2 n n
= cn f, f = c G f(z)f(z)dm = c G |z| dm = c G 1dm = c m(G)
= c , (m being the normalized Haar measure for G). T is ergodic transformation.

THEOREM 4.12

1) An automorphism T of a probability space (X, β, ν) is ergodic if and only if

1 is a simple Eigenvalue of the unitary operator U : L^2(X, β, ν) → L2(X, β, ν) [2],[5]. Proof: "If part" Suppose that 1 is a simple eigenvalue, we are required to prove that T must be ergodic. If f∈ L^2(X, β, ν) is an invariant function,then U_T f ≡ f ◦ T = f , (as f is invariant by definition), which implies that f(w)λX_x(w) holds for almost all points w of X , where λ is some complex con-stant, as both f and X_x are eigenvectors corresponding to the eigenvalues 1 and 1 is simple by our hypothesis so, f(w) = λ for almost all points w of X , as X_x(x) = 1∀x∈ X.

∴ T is ergodic.

"Only if part"

Suppose that T is ergodic. We need to prove that 1 must be a simple eigen-value. If f and g are eigenvectors corresponding to the eigenvalue 1, then, U_T f = f and U_T g = g holds in f∈ L^2(X, β, ν) and f = 0_L2 (ν), g = 0_L2 (ν). Now,h = $_g{}^1$≠ 0_L2 (ν) ,(as g is finite, ν almost everywhere on X and

ν (w∈ X : g(w) = 0) = 0 and h∈ L^2(X, β, ν). Hence U_T f · h (By definition of

of complex functions) =

U_T) =(f · h) ◦ T (By definition of multiplication ·, f

(f ◦ T) · (h ◦ T) = f · h (As f ◦ T = f andh ◦ T = h) = g(ν almost everywhere by the ergodicity of T .)∴ f = λg holds (ν almost everywhere on X), where λ is some complex constant. So, 1 is simple.

THEOREM 4 .13:

Let an automorphism T of probability space (X, β, ν) be ergodic, then; i. The absolute value |f| of every eigenvalue f is constant ν− a.e. ii. Every eigenvalue

is simple[2]. Proof: Since T is ergodic and U_T is unitary, each eigenvalue
2 ∈ T 2 ′ L ∈
has an absolute value 1, as α C with U f = αf for some f = 0 2(ν)

L (X, β, ν) ⇒ U_T f, U_T f $_2$= αf, αf = αα¯ f, $_2$f = |α| f, f [12] . But
U_T f, U_T f = f, f 2 | || || / L2 ∴ | |
.So, α = 1, as f, f = f = 0 (ν)[12]. α = 1.

Next, if f≠ 0_L2 (ν)∈ L (X, β, ν), x is an eigenvector of U_T corresponding to an eigenvalue α∈ C.

' , then U_T f = f ◦ T = αf , hence |f|(T (x)) = |f|(x) holds for almost all points x of X as |f ◦ T | = |f|∈ L^2(ν) because |α| = 1 . So by ergodicity of T , |f| is a constant, (νalmosteverywhereonXbeinganinvariantfunctionon(X ,β, ν)

2 To prove that every eigenvalue is simple. Let α∈ C be any eigenvalue of U_T . Suppose that f and g are eigenvectors corresponding to the eigenvalue α.

Then f = 0_L2 (ν) and g = 0_L2 (ν) with U_T f = αf andU_T g = αg So, U_T (f ·h)f ·h (by theorem 4.12), where h = $_g{}^1$≠ 0_L2 (ν), and so h∈ L^2(X, β, ν). Hence $^f{}_g$ = f · h is a constant, (ν almost everywhere on X , by the ergodicity of T ,being an invariant function on (X, β, ν) .

So f = λg for some complex constant λ.

∴α is simple.

THEOREM 4.14:

Let an automorphism T of the probability space (X, β, ν) be ergodic,then the set of all eigenvalues forms a subgroup of the unit circle group G = (z∈ C′ : |z| = 1)[1], [2].

Proof: Let E be the set of all eigenvalues α of U_T .

Then , $1 \in E$ as 1 is always an eigenvalue of U_T for any automorphism T of the probability space (X, β, ν) .

So is a non-empty, subset of the circle group $G = (z \in C^{'} : |z| = 1)$ under mul-tiplication of complex numbers. Let δ and σ be any points of E . Then we have that U_T f = δf for some $f \neq 0_L 2$ (ν)$\in L^2$(X, β, ν), and U_T g = σgforsomeg$\neq 0_L 2$ ()$\in L^2$(X, β, ν). So U_T (f ·h)◦T = (f ·h)◦T [9] = (h·f)◦T = (h◦T)·(f ◦T), where h = $\frac{1}{g} \neq 0_L 2$ (ν), h$\in L^2$(X, β, ν) .

$U_T \frac{f}{g} = \sigma^{-1}(h \cdot (\delta f)) = \delta\sigma^{-1}(h \cdot f) = \delta\sigma^{-1}\frac{f}{g}$. Hence $\delta\sigma^{-1} \in E$ as $\frac{f}{g} \neq 0_L 2$ (ν) and $\frac{f}{g} \in L^2$(X, β, ν).

∴ E is a subgroup of the circle (G, ·).

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