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Adaptive Local Discontinuity Galerkin Method For Convection-Diffusion Eigenvalue Problems

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Abstract:
The convection-diffusion equation is a very important branch of partial differential equations, with wide applications in many fields such as fluid mechanics and gas dynamics. Since it is difficult to obtain analytical solutions for the convection-diffusion equation, solving the equation and its eigenvalue problems using various numerical methods has great value in numerical analysis and is currently a hot topic in computational mathematics. This paper studies the local discontinuous Galerkin (LDG) method for convection-diffusion eigenvalue problems, provides both a priori and a posteriori error estimates, analyzes the reliability of eigenvalue estimates, and conducts adaptive experiments. Combining theoretical analysis, it is demonstrated that our method achieves optimal convergence rates.
***Key Word**: eigenvalues of convection-diffusion; local discontinuous Galerkin method; a posteriori error; adaptive.*

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I. Introduction

The application of convection-diffusion eigenvalue problems in multiple fields such as fluid mechanics, environmental science, energy development, and electronics has provided it with various physical backgrounds, leading to increasing attention from scholars for solving convection-diffusion eigenvalue problems. Reference [1] discusses the formulation of the convection-diffusion equation and estimates its solutions. Reference [2] discusses the hp-local discontinuous Galerkin finite element method. Reference [3] discusses a posteriori error estimation and adaptive algorithms. The idea of using a posteriori error estimation and adaptive finite element algorithms was first proposed by Babuska Rheinbolt in 1978 in reference [4]. Reference [5] discusses the Crouzeix-Raviart element bimodal mesh discretization method for convection-diffusion eigenvalue problems. Reference [6] discusses the two-level correction method for convection-diffusion eigenvalue problems. Reference [7] discusses the function value recovery algorithm. Reference [8] discusses the adaptive continuation scheme, and so on. Adaptive finite element methods are the mainstream of scientific computing, and in recent years, this method has been extensively studied and applied to many problems.The local discontinuous Galerkin method was proposed by Cockburn and Shu for solving convection-diffusion equations. The idea is to introduce auxiliary variables to transform high-order equations into a system of first-order equations, and then apply the discontinuous finite element method to the first-order equations, which facilitates h-p adaptivity. This method has been applied to the numerical solution of convection-diffusion equations, traditional KdV equations, and other equations with high-order derivative terms, achieving good results. This paper first uses the local discontinuous Galerkin method to compute convection-diffusion eigenvalue problems, establishes a posteriori error estimation, and verifies the reliability and effectiveness of the local discontinuous Galerkin method's eigenfunction a posteriori error estimation through adaptive calculations.

II. Basic Theory Preparation

Let $\Omega \subset \mathrm{R}^2$ be a bounded domain with a Lipshitz boundary $\partial\Omega$, and let **n** be the unit outward normal vector to $\partial\Omega$. Consider the eigenvalue problem with Dirichlet boundary conditions: Find $\lambda \in C$ and $u \in H_0^1(\Omega)$, such that

$$\begin{cases} -\Delta u + \mathbf{r}\cdot\nabla u + cu = \lambda u, & in \quad \Omega, \\ \qquad\qquad u = 0, & on \quad \partial\Omega, \end{cases} \tag{2.1}$$

Denote

$$(u, v) := \int_\Omega uvdx,$$

and define a continuous bilinear form

$$a(u,v):=(\nabla u,\nabla v)+(\mathrm{r}\cdot\nabla u,v)+(cu,v),\qquad \forall u,v\in H_0^1(\Omega).$$

Assume that **r** and c are bounded functions on the Ω, $\nabla\cdot\mathbf{r}$ exists and satisfies

$$-\frac{1}{2}\nabla\cdot\mathrm{r}+c\geqslant 0,\qquad in\quad \Omega.$$

Under these assumptions, there are two positive numbers A and B independent of u,v such that the bilinear form

a(· , ·) satisfies

$$|a(u,v)|\leqslant A\|u\|_{1,\Omega}\|v\|_{1,\Omega},\qquad \forall u,v\in H_0^1(\Omega)$$
$$a(v,v)\geqslant B\|v\|_{1,\Omega}^2,\qquad \forall v\in H_0^1(\Omega) \tag{2.2}$$

The weak form of (2.1) is to find$(\lambda,u)\in C\times H_0^1(\Omega),\ \ u\neq 0$,so that the following equation holds

$$a(u,v)=\lambda(u,v),\quad \forall v\in H_0^1(\Omega). \tag{2.3}$$

Let $\mathcal{T}_h=\{\kappa\}$ be a shape-regular mesh of Ω, where the diameter of an element κ is denoted by h_κ. An interior edge of $\mathcal{T}_h$ is a non-empty interior $\partial\kappa^+\cap\partial\kappa^-$where κ^+ and κ^- are two adjacent elements of the $\mathcal{T}_h$ and do not necessarily match, and one of the outer edges of the $\mathcal{T}_h$ is a non-empty interior $\partial\kappa^+\cap\partial\Omega$. Let $\mathcal{E}:=\mathcal{E}_\mathcal{J}\cup\mathcal{E}_\mathcal{D}$, where $\mathcal{E}_\mathcal{J}$ denotes the set of interior edges, and $\mathcal{E}_\mathcal{D}$ denotes the set of edges on the boundary $\partial\Omega$.

The degree of polynomials in the element $\kappa\in\mathcal{T}_h$ is denoted by $p_\kappa\geqslant$1, and the LDG finite element space is now defined as:

$$S^h(\mathcal{T}_h)=\{u\in L^2(\Omega):u|_\kappa\in S^{p_\kappa}(\kappa),\quad \forall\kappa\in\mathcal{T}_h\},$$

where $S^{p_\kappa}(\kappa)$ denotes the polynomial space $p^{p_K}(K)$ of degree p_κ on κ.

Introduce the space of piecewise functions over the mesh $\mathcal{T}_h$:

$$H^s(\mathcal{T}_h)=\{v\in L^2(\Omega):v|_\kappa\in H^s(\kappa),\quad \forall\kappa\in\mathcal{T}_h\}.$$

Introduce the auxiliary variable $\mathbf{q}=\nabla u$, then the equation (2.1) can be rewritten as:

$$\begin{cases}-\nabla\cdot\mathbf{q}+\mathbf{r}\cdot\nabla u+cu=\lambda u, & in\quad \Omega,\\ \qquad\qquad u=0, & on\quad \partial\Omega,\end{cases} \tag{2.4}$$

Using $V_h=S^h(\mathcal{T}_h)$ and $\boldsymbol{Q_h}=\boldsymbol{S^h}(\boldsymbol{\mathcal{T}_h})^2$ to represent the LDG finite element spaces, the LDG formulation of the approximation problem (2.5) is: Find $(\lambda_h,u_h)\in C\times V_h$ for all $\kappa\in\mathcal{T}_h$

$$\int_\kappa\mathbf{q}_h\cdot\nabla v dx-\int_{\partial\kappa}\widehat{\mathbf{q}}_h\cdot\mathbf{n}_\kappa v ds+\int_\kappa(\mathbf{r}\cdot\nabla u_h+cu_h)v dx=\int_\kappa\lambda_h u_h v dx,\quad \forall v\in V_h, \tag{2.5}$$

$$\int_\kappa\mathbf{q}_h\cdot\mathbf{t}dx-\int_{\partial\kappa}\hat{u}_h\cdot\mathbf{n}_\kappa\mathbf{t}ds+\int_\kappa u_h\nabla\cdot\mathbf{t}dx=0,\quad \forall\mathbf{t}\in\mathbf{Q}_h, \tag{2.6}$$

where $v\in V_h,\ n_\kappa$is the unit outward normal vector of $\partial\kappa$, $\hat{\mathbf{u}}_h$ and $\widehat{\mathbf{q}}_h$ are the numerical fluxes, which are the trace of u and **q**, respectively.

Define the average and the jump of v on e:

$$\{\{v\}\}=\frac{1}{2}(v^++v^-),[[v]]=v^+\boldsymbol{n}^++v^-\boldsymbol{n}^-,$$

where $e=\partial\kappa^+\cap\partial\kappa^-,v^+=v|_{\kappa^+},v^-=v|_{\kappa^-},\boldsymbol{n}$, n is the unit outward normal vector from κ^+ to κ^-. If $e\in\mathcal{E}_\mathcal{D}$, define the average and the jump of v on e:

$$\{\{v\}\}=v,[[v]]=v\mathrm{n}.$$

With the above definition, it can be obtained that

$$\sum_{\kappa\in\mathcal{T}}\int_{\partial\kappa}v\mathbf{q}\cdot\mathbf{n}ds=\int_\mathcal{E}\{\{\mathbf{q}\}\}\cdot[[v]]ds+\int_{\mathcal{E}_\mathcal{J}}[[\mathbf{q}]]\{\{v\}\}ds.$$

Define numerical fluxes:

$$\hat{u}|_e=\begin{cases}\{\{u\}\}+\mathbf{b}\cdot[[u]] & e\subset\mathcal{E}_\mathcal{J}\\ 0 & e\subset\mathcal{E}_\mathcal{D}\end{cases}\qquad \widehat{\mathbf{q}}|_e=\begin{cases}\{\{\mathbf{q}\}\}-\eta[[u]]-\mathbf{b}[[\mathbf{q}]] & e\subset\mathcal{E}_\mathcal{J}\\ \mathbf{q}-\eta u\mathbf{n} & e\subset\mathcal{E}_\mathcal{D}\end{cases},$$

The parameters η and b should be selected appropriately.

In order to define the parameter η, the function h is introduced in the relevant local mesh size and approximation in $L^\infty(\mathcal{E})$, we have:

$$h=h(x)=\begin{cases}min\{h_\kappa,h'_\kappa\}, & x\in e_{\kappa\kappa'},\\ \\ h_\kappa & x\in e_{\kappa\Omega},\end{cases}$$

Define the discontinuous stability parameter $\eta\in L^\infty(\mathcal{E})$ as $\eta=\alpha h^{-1}$, and select the parameter **b** so that $\|\mathbf{b}\|_{\infty,\mathcal{E}_\mathcal{J}}\leqslant\beta$, where $\alpha>0$ and $\beta\geqslant 0$ are constants independent of the grid size.

Define the lifting operator $\Psi(v)\in\mathbf{Q}_h$ for $v\in V(h):=V_h+H_0^1(\Omega)$, such that

$$\int_\Omega\Psi(v)\cdot\mathbf{t}d\mathbf{x}=\int_{\mathcal{E}_\mathcal{J}}(\{\{\mathbf{t}\}\}-\mathbf{b}[[\mathbf{t}]])\cdot[[v]]ds+\int_{\mathcal{E}_\mathcal{D}}v\mathbf{t}\cdot\mathbf{n}ds,\quad \forall\mathbf{t}\in\mathbf{Q}_h. \tag{2.7}$$

Since $\mathbf{q}=\nabla u$, then

$$\int_\Omega\boldsymbol{q}\cdot\boldsymbol{t}dx=\int_\Omega(\nabla_h u-\Psi(u))\cdot\boldsymbol{t}dx,\quad \forall\mathbf{t}\in\mathbf{Q}_h, \tag{2.8}$$

using (2.6) and (2.8), we obtain:

$$\int_{\Omega} (\nabla_h u - \Psi(u)) \cdot \nabla v dx - \int_{\mathcal{E}_I} (\{\{\mathbf{q}\}\} - \eta[[u]] - \mathbf{b}[[\mathbf{q}]]) \cdot [[v]] ds$$
$$- \int_{\mathcal{E}_D} (\mathbf{q} - \eta u \cdot \mathbf{n}) \cdot \mathbf{n} v ds + \int_{\Omega} (\boldsymbol{r} \cdot \nabla u + cu) v dx = \int_{\Omega} \lambda u v dx. \quad (2.9)$$

Because

$$\int_{\Omega} \Psi(v) \cdot (\nabla_h u - \Psi(u)) dx = \int_{\Omega} \Psi(v) \cdot q dx,$$
$$= \int_{\mathcal{E}_{\mathcal{I}}} ([[v]] \cdot \{\{\mathbf{q}\}\} - \mathbf{b} \cdot [[v]][[\mathbf{q}]]) ds + \int_{\mathcal{E}_{\mathcal{D}}} v\mathbf{n} \cdot \mathbf{q} ds, \quad (2.10)$$

using (2.9) and (2.10), we obtain

$$a_h(u, v) := \int_{\Omega} (\nabla_h u - \Psi(u)) \cdot (\nabla v - \Psi(v)) dx + \int_{\mathcal{E}_I} \eta[[u]] \cdot [[v]] ds + \int_{\mathcal{E}_D} \eta u v ds$$
$$+ \int_{\Omega} (\mathbf{r} \cdot \nabla u + cu) v dx = \int_{\Omega} \lambda u v dx. \quad (2.11)$$

The finite element approximation of (2.3) is to find $(\lambda_h, u_h) \in C \times V_h, u_h \neq 0$, such that

$$a_h(u_h, v_h) = \lambda_h(u_h, v_h), \quad \forall v_h \in V^h. \quad (2.12)$$

The source problem for (2.3) is to find $w \in H_0^1(\Omega)$ such as

$$a(w, v) = (f, v), \quad \forall v \in H_0^1(\Omega). \quad (2.13)$$

Its local discontinuous finite element approximation is to find $w_h \in V_h$, such that

$$a_h(w_h, v_h) = (f, v_h), \quad \forall v_h \in V_h. \quad (2.14)$$

Define the linear bounded operator $T: L^2(\Omega) \to H_0^1(\Omega)$ and $T_h: L^2(\Omega) \to V_h$ satisfying $Tf := w$ and $\mathrm{T_h f} := \mathrm{w_h}$.

The equivalent operator form of (2.3) is given by:

$$Tu = \frac{1}{\lambda} u. \quad (2.15)$$

The equivalent operator form of (2.12) is given by:

$$T_h u_h = \frac{1}{\lambda_h} u_h. \quad (2.16)$$

The duality problem of (2.3) is: Find $(\lambda^*, u^*) \in C \times H_0^1(\Omega),\ u^* \neq 0$, such that

$$a(v, u^*) = \lambda^*(v, u^*), \quad \forall v \in H_0^1(\Omega). \quad (2.17)$$

(2.17) is the source problem for $w^* \in H_0^1(\Omega)$, such that

$$a(v, w^*) = (v, g), \quad \forall v \in H_0^1(\Omega). \quad (2.18)$$

Define the linear bounded operator $T^*: L^2(\Omega) \to H_0^1(\Omega)$ satisfies

$$a(v, T^* g) = (v, g), \quad v \in H_0^1(\Omega), \quad (2.19)$$

The equivalent operator form of (2.17) is given by:

$$T^* u^* = \frac{1}{\lambda^*} u^*. \quad (2.20)$$

The LDG approximation of (2.17) is to find $(\lambda_h^*, u_h^*) \in C \times V_h,\ u_h^* \neq 0$,such that

$$a_h(v_h, u_h^*) = \lambda_h^*(v_h, u_h^*), \quad \forall v_h \in V^h. \quad (2.21)$$

The LDG approximation of (2.18) is to find $w_h^* \in V^h$ such that

$$a_h(v_h, w_h^*) = (v_h, g), \quad \forall v_h \in V_h. \quad (2.22)$$

We introduce the direct sum space $V(h) = V_h + H_0^1(\Omega)$ equipped with the DG norm

$$\|v\|_h^2 = \|\nabla_h v\|_{0,\Omega}^2 + \|v\|_{0,\Omega}^2 + \alpha \|h^{-1/2}[[v]]\|_{0,\mathcal{E}_{\mathcal{I}}}^2 + \alpha \|h^{-1/2} v\|_{0,\mathcal{E}_{\mathcal{D}}}^2, \quad (2.23)$$

From lemma 3.2 and (2.5) in Ref. [9] it holds the Galerkin orthogonality

$$a_h(w - w_h, v_h) = 0, \quad \forall v_h \in V_h, \quad (2.24)$$

$$a_h(v_h, w^* - w_h^*) = 0, \quad \forall v_h \in V_h. \quad (2.25)$$

From lemma 6.1 and (2.7) in Ref. [10] it holds the continuity and coercivity properties

$$|a_h(u_h, v_h)| \lesssim \|u_h\|_h \|v_h\|_h, \quad \forall u_h, v_h \in V(h), \quad (2.26)$$
$$\|u_h\|_h^2 \lesssim a_h(u_h, u_h), \quad \forall u_h \in V_h. \quad (2.27)$$

Let $w$and w^* be the solution of the equations (2.13) and (2.20), respectively. Assume the following regularity estimate holds:

$$\|w\|_{1+r} \lesssim \|f\|_{0,\Omega} \quad \left(\frac{1}{2} < r \leqslant 1\right), \quad (2.28)$$

$$\|w^*\|_{1+r} \lesssim \|g\|_{0,\Omega} \quad \left(\tfrac{1}{2} < r \leqslant 1\right). \tag{2.29}$$

Lemma 2.1[11] Let $\kappa \in \mathcal{T}_h$ and $v \in H^{s_\kappa}(\kappa)$, $s_\kappa > \frac{3}{2}$, then there is a polynomial $\Pi^{h_\kappa} v \in S^{h_\kappa}$, satisfying

$$\| v - \Pi_{p_\kappa}^{h_\kappa} v \|_{m,\kappa} \lesssim h_\kappa^{s_\kappa - m} \|v\|_{s_\kappa,\kappa}, \qquad (0 \leqslant m \leqslant s_\kappa) \tag{2.30}$$

$$\| v - \Pi_{p_\kappa}^{h_\kappa} v \|_{0,\partial\kappa} \lesssim h_\kappa^{s_\kappa - \frac{1}{2}} \|v\|_{s_\kappa,\kappa}. \tag{2.31}$$

Now we introduce the global interpolation operator $\Pi^h: H_0^1(\Omega) \to V_h$, such that $\Pi^h(u)|_\kappa = \Pi^{h_\kappa}(u|_\kappa)$, for the vector-value function $\mathbf{r} = (\mathbf{r_1}, \mathbf{r_2})$, define $\Pi^h(\mathbf{r})|_\kappa = (\Pi^h \mathbf{r_1}|_\kappa, \Pi^h \mathbf{r_2}|_\kappa)$.

Theorem 2.1 Let w and w^* be solutions to the (2.13) and (2.14) respectively, w satisfies $w|_\kappa \in H^{s_\kappa}(\kappa)$ and for all $\kappa \in \mathcal{T}_h$ and $s_\kappa > \frac{3}{2}$, the following inequality holds

$$\|w - w_h\|_h \lesssim \inf_{v_h \in V_h} \|w - v_h\|_h, \tag{2.32}$$

$$\|w - w_h\|_h \lesssim \left(\textstyle\sum_{\kappa \in \mathcal{T}_h} (h_\kappa^{s_\kappa - 1} \|w\|_{s_\kappa,\kappa})^2\right)^{\frac{1}{2}}. \tag{2.33}$$

Proof. First, we prove (2.32) by utilizing equations (2.24), (2.26), and (2.27):

$$\begin{aligned} \|v_h - w_h\|_h^2 &\lesssim |a_h(v_h - w_h, v_h - w_h)| \\ &\lesssim a_h(v_h - w, v_h - w_h) + a_h(w - w_h, v_h - w_h) \\ &\lesssim \|v_h - w\|_h \|v_h - w_h\|_h. \end{aligned} \tag{2.34}$$

Using the triangle inequality we reach at

$$\|w - w_h\|_h \lesssim \|w - v_h\|_h + \|v_h - w_h\|_h. \tag{2.35}$$

Next, we prove the estimate (2.33) via (2.23). Let $E_h(w) = w - \Pi^h w$ we have

$$\begin{aligned} \|E_h(w)\|_h^2 &\lesssim \textstyle\sum_{\kappa \in \mathcal{T}_h} (\|\nabla_h E_h(w)\|_{0,\kappa}^2 + \|E_h(w)\|_{0,\kappa}^2) + \alpha \sum_{\kappa \in \mathcal{T}_h} \left(\sum_{e \subset \partial\kappa} h^{-\frac{1}{2}} \|E_h(w)\|_{0,e}^2\right) \\ &:= I_1 + I_2 \end{aligned} \tag{2.36}$$

I_1 is estimated from (2.26):

$$(\|\nabla_h E_h(w)\|_{0,\kappa}^2 + \|E_h(w)\|_{0,\kappa}^2) \quad \lesssim (h_\kappa^{s_\kappa - 1} \|w\|_{s_\kappa,\kappa})^2. \tag{2.37}$$

I_2 is estimated from (2.27):

$$h^{-\frac{1}{2}} \|E_h(w)\|_{0,e}^2 \quad \lesssim (h_\kappa^{s_\kappa - 1} \|w\|_{s_\kappa,\kappa})^2. \tag{2.38}$$

By using (2.37) and (2.38), we obtain

$$\| w - \Pi^h w \|_h \lesssim \left(\textstyle\sum_{\kappa \in \mathcal{T}_h} (h_\kappa^{s_\kappa - 1} \|w\|_{s_\kappa,\kappa})^2\right)^{\frac{1}{2}} \tag{2.39}$$

and the interpolation error estimation

$$\inf_{v_h \in V_h} \| w - v_h \|_h \lesssim \| w - \Pi^h w \|_h \tag{2.40}$$

From (2.32), (2.39), (2.40) and (2.33), the proof is complete.

Theorem 2.2 Let w and w^* be the solutions to the equations (2.13) and (2.14), respectively. Assume that w satisfies $w|_\kappa \in H^{s_\kappa}(\kappa)$ and for all $\kappa \in \mathcal{T}_h$ and $s_\kappa > \frac{3}{2}$, then the following inequalities hold

$$\|w - w_h\|_{0,\Omega} \lesssim h^r \|w - w_h\|_h, \tag{2.41}$$

$$\|w - w_h\|_{0,\Omega} \lesssim \left(\textstyle\sum_{\kappa \in \mathcal{T}_h} (h_\kappa^{s_\kappa - 1 + r} \|w\|_{s_\kappa,\kappa})^2\right)^{\frac{1}{2}}. \tag{2.42}$$

Proof. First, we prove (2.41) by considering the source problem of the dual problem of (2.3), denoted as $a(v, w^*) = (v, g)$, $\forall v \in H_0^1(\Omega)$. We derive using the Galerkin orthogonality (2.25) and equation (2.26):

$$\begin{aligned} (w - w_h, g) &= a_h(w - w_h, w^*) = a_h(w - w_h, w^* - w_h^*) \\ &\lesssim \|w - w_h\|_h \|w^* - w_h^*\|_h) \end{aligned} \tag{2.43}$$

By utilizing (2.33) and the regularity (2.29),

$$\|w^* - w_h^*\|_h \lesssim h^r \|w^*\|_{1+r,\Omega} \lesssim h^r \|g\|_{0,\Omega} \tag{2.44}$$

By using (2.43) and (2.44), we obtain:

$$\| w - w_h \|_{0,\Omega} = \sup_{g\in L^2(\Omega)} \frac{|(w - w_h, g)|}{\| g \|_{0,\Omega}} \lesssim h^r \| w - w_h \|_h$$

Thus, we can obtain (2.41).

Next, we prove (2.42) by using (2.33) and (2.41):

$$\| w - w_h \|_{0,\Omega} \lesssim h^r \|w - w_h\|_h \lesssim (\sum_{\kappa\in\mathcal{T}_h} (h_\kappa^{s_\kappa -1+r} \|w\|_{s_\kappa,\kappa})^2)^{\frac{1}{2}}.$$

Thus, we can obtain (2.42).

Taking $s_\kappa = 1 + r\left(\frac{1}{2} < r \leqslant 1\right)$ from (2.33) and (2.28) we have the stability estimation:

$$\|T_h f\|_h \lesssim \|T_h f - Tf\|_h + \|Tf\|_h \lesssim \|f\|_{0,\Omega} \tag{2.45}$$

Let λ be the j-th eigenvalue of (2.3), with the algebraic multiple q and the ascent α, where $\lambda_j = \lambda_{j+1} = \cdots = \lambda_{j+q-1}$. When $\|T_h - T\|_{0,\Omega} \to 0$, the q eigenvalues of (2.12) will converge to λ. Let $M(\lambda)$ be the generalized eigenvector space of (2.3) related to λ, and $M_h(\lambda)$ is the direct sum of the generalized eigenvector space of (2.12) related to λ_h that converges to λ.

The following theorem can be proved using a similar argument as that of Theorem 3.1 in [3].

Theorem 2.3 Let $M(\lambda) \subset H^{1+r}(\Omega)(1 \geqslant r > \frac{1}{2})$, then the following inequality holds

$$|\lambda_h - \lambda| \lesssim h^{\frac{2r}{\alpha}}. \tag{2.46}$$

Let $u_h \in M_h(\lambda)$ then there is the eigenfunction u of (2.3) such that

$$\|u - u_h\|_{0,\Omega} \lesssim h^{\frac{2r}{\alpha}}, \tag{2.47}$$

$$\|u - u_h\|_h \lesssim h^{\frac{2r}{\alpha}} + h^r, \tag{2.48}$$

If $\alpha = 1$, then

$$\|u - u_h\|_h \lesssim h^r. \tag{2.49}$$

$$\|u - u_h\|_{0,\Omega} \lesssim h^r \|u - u_h\|_h. \tag{2.50}$$

III. A-posteriori error estimation

i. Estimators of eigenfunctions and their reliability

Let (λ_h, u_h) be the eigenpair of (2.12). We define the element residuals and the edge residuals on each element $\kappa \in \mathcal{T}_h$ and $e \in \mathcal{E}$, respectively, as follows,

$$R_\kappa = \Delta u_h + (\lambda_h - c)u_h - \mathbf{r}\cdot\nabla u_h$$

$$J_{F,1} = [[\nabla u_h]], \quad \forall e \in \mathcal{E}_I; \quad J_{F,2} = [[u_h]], \quad \forall e \in \mathcal{E}_I; \quad J_{F,3} = u_h, \quad \forall e \in \mathcal{E}_D$$

Define the local error indicators on $\kappa \in \mathcal{T}_h$

$$\begin{aligned}\eta_\kappa^2 = h_\kappa^2 \|\Delta u_h + (\lambda_h - c)u_h - \mathbf{r}\cdot\nabla u_h\|_{0,\kappa}^2 + \sum_{e\in\mathcal{E}_I} h_e \|J_{F,1}\|_{0,e}^2 \\ + \sum_{e\in\mathcal{E}_I} \alpha h_e^{-1} \|J_{F,2}\|_{0,e}^2 + \sum_{e\in\mathcal{E}_D} \alpha h_e^{-1} \|J_{F,3}\|_{0,e}^2.\end{aligned} \tag{3.1}$$

The global error indicator is

$$\eta(u_h) = (\sum_{\kappa\in\mathcal{T}_h} \eta_\kappa^2)^{1/2}. \tag{3.2}$$

Next, we will prove that this error estimator is reliable.

Theorem 3.1 Let (λ, u) and (λ_h, u_h) be the eigenpairs of (2.3) and (2.12), respectively, then for any $v \in H_0^1(\Omega)$, the following formula holds

$$\| u - u_h \|_h \lesssim \sup_{v\in H_0^1(\Omega)} \frac{|(\lambda_h u_h, v) - a_h(u_h, v)|}{\|v\|_h} + \inf_{v\in H_0^1(\Omega)} \| u_h - v \|_h. \tag{3.3}$$

Proof. Note that on $H_0^1(\Omega) \times H_0^1(\Omega), a = a_h$. Let $w \in H_0^1(\Omega)$, which can be deduced from the coercivity and continuity of the bilinear form

$$\begin{aligned}\|u-w\|_h^2 &\lesssim |a_h(u-w,u-w)| \lesssim |a_h(u,u-w)-a_h(w,u-w)| \\ &\lesssim |\lambda(u,u-w)-a_h(w,u-w)| \\ &\lesssim |(\lambda u,u-w)-a_h(w+u_h-u_h,u-w)| \\ &\lesssim |(\lambda u,u-w)-a_h(u_h,u-w)|+|a_h(u_h-w,u-w)| \\ &\lesssim |(\lambda_h u_h,u-w)-a_h(u_h,u-w)|+\|u_h-w\|_h\|u-w\|_h\end{aligned} \tag{3.4}$$

Take $v=u-w$, we have:

$$\begin{aligned}\|u-w\|_h &\lesssim \sup_{v\in H_0^1(\Omega)} \frac{|(\lambda u,u-w)-a_h(u_h,u-w)|}{\|v\|_h}+\|u_h-w\|_h \\ &\lesssim \sup_{v\in H_0^1(\Omega)} \frac{|(\lambda u,v)-a_h(u_h,v)|}{\|v\|_h}+\|u_h-w\|_h\end{aligned} \tag{3.5}$$

By the triangle inequality, we have:

$$\begin{aligned}\|u-u_h\|_h &\lesssim \|u-w\|_h+\|u_h-w\|_h \\ &\lesssim \sup_{v\in H_0^1(\Omega)} \frac{|(\lambda u,v)-a_h(u_h,v)|}{\|v\|_h}+\|u_h-w\|_h\end{aligned} \tag{3.6}$$

From the arbitrariness of w, the proof concludes.

Lemma 3.1[12][13] For arbitrary $\varphi\in H_0^1(\Omega)$, there exists a piecewise linear interpolation $I^h\varphi\in V_h$ such that:

$$\|\varphi-I^h\varphi\|_{0,\kappa}+h_\kappa\|\nabla(\varphi-I^h\varphi)\|_{0,\kappa}\lesssim h_\kappa\|\nabla\varphi\|_{0,U_\kappa},\quad \forall\kappa\in\mathcal{T}_h \tag{3.7}$$

$$\|\varphi-I^h\varphi\|_{0,e}\lesssim h_e^{\frac{1}{2}}\|\nabla\varphi\|_{0,U_e},\quad \forall e\in\mathcal{E}, \tag{3.8}$$

where U_κ is the union of all elements that share at least one node with κ, and U_e is the union of all edges that share at least one node with edge e.

Theorem 3.2 Let (λ,u) and (λ_h,u_h) be the eigenpairs of (2.3) and (2.12), respectively. It holds for any $v\in H_0^1(\Omega)$.

$$\|u-u_h\|_h\lesssim\eta(u_h)+\|\lambda u-\lambda_h u_h\|_{0,\Omega}. \tag{3.9}$$

Proof . From the interpolation property, we get $[[v-I^hv]]=0$. Using Green's formula, we have

$$\begin{aligned}B &\equiv \lambda(u,v-I^hv)-a_h(u_h,v-I^hv) \\ &= \sum_{\kappa\in\mathcal{T}_h}\int_\kappa(\Delta u_h-\mathbf{r}\cdot\nabla u_h-cu_h)(v-I^hv)dx+\int_\Omega\lambda u(v-I^hv)dx \\ &\quad -\sum_{\kappa\in\mathcal{T}_h}\int_\kappa\Psi(u)\cdot\nabla(v-I^hv)dx-\sum_{e\in\mathcal{E}}\int_e[[\nabla u_h]](v-I^hv)ds \\ &\equiv B_1+B_2+B_3+B_4\end{aligned} \tag{3.10}$$

By using the Cauchy-Schwarz inequality, (3.7) and (3.8), we have:

$$|B_1|+|B_2|\lesssim\left(\sum_{\kappa\in\mathcal{T}_h}h_\kappa^2\|\Delta u_h+(\lambda_h-c)u_h-\mathbf{r}\cdot\nabla u_h+\lambda u-\lambda_h u_h\|_{0,\kappa}^2\right)^{1/2}\|v\|_h \tag{3.11}$$

$$|B_3|\lesssim\left(\left(\sum_{e\in\mathcal{E}_I}h^{-1}\|[[u_h]]\|_{0,e}^2\right)^{\frac{1}{2}}+\left(\sum_{e\in\mathcal{E}_D}h^{-1}\|u_h\|_{0,e}^2\right)^{\frac{1}{2}}\right)\|v\|_h \tag{3.12}$$

$$|B_4|\lesssim\left(\sum_{e\in\mathcal{E}}h_e\|[[\nabla u_h]]\|_{0,e}^2\right)^{\frac{1}{2}}\|v\|_h \tag{3.13}$$

By (2.3) we have

$$(\lambda u,v)-a_h(u_h,v)=(\lambda u,v-I^hv)-a_h(u_h,v-I^hv) \tag{3.14}$$

From Ref. [14,15], it can be seen that for arbitrary $v\in V_h$, there is a rich operator $E_h\colon V_h\to V_h\cap H_0^1(\Omega)$ such that

$$\sum_{\kappa\in\mathcal{T}_h}\left(h_\kappa^{-2}\|v-E_hv\|_{0,\kappa}^2+\|\nabla(v-E_hv)\|_{0,\kappa}^2\right)\lesssim\sum_{e\in\mathcal{E}_I}h_e^{-1}\|[[v]]\|_{0,e}^2. \tag{3.15}$$

For the second term to the right-hand side of (3.3), using the formulas (2.23) and (3.15), and noting that $[[E_hu_h]]=0$, we have:

$$\inf_{v\in H^1(\Omega)}\|u_h-v\|_h^2\lesssim\|E_hu_h-u_h\|_h^2$$

$$= \sum_{\kappa\in\mathcal{T}_h} \left(\|\nabla(E_h u_h - u_h)\|_{0,\kappa}^2 + \|(E_h u_h - u_h)\|_{0,\kappa}^2\right)$$

$$+ \sum_{e\in\mathcal{E}_I} \alpha h^{-1}\|[[E_h u_h - u_h]]\|_{0,e}^2 + \sum_{e\in\mathcal{E}_D} \alpha h^{-1}\|E_h u_h - u_h\|_{0,e}^2$$

$$\lesssim \sum_{e\in\mathcal{E}_I} \alpha h_e^{-1}\|[[u_h]]\|_{0,e}^2 + \sum_{e\in\mathcal{E}_D} \alpha h_e^{-1}\|u_h\|_{0,e}^2. \quad (3.16)$$

Substituting (3.10) and (3.16) into (3.3), we obtain (3.9). The proof is completed.

By Theorem 2.3, we know that when the ascent $\alpha = 1$, both $\|\lambda u - \lambda_h u_h\|_{0,\Omega}$ and $\|u - u_h\|_{0,\Omega}$ are higher-order terms of $\|u - u_h\|_h$. Therefore, (3.9) tells us that the error estimator $\eta(u_h)$ is one of the upper bounds of $\|u - u_h\|_h$, indicating that the error estimator is reliable.

ii. The effectiveness of the eigenfunction estimator

In order to ensure the effectiveness of the estimator for practical adaptive refinement, our next goal is to prove that the local error estimator η_κ provides a local lower bound for the error on κ. Let $b_\kappa \in H_0^1(\kappa)$ be the standard unit bubble function and $b_e \in H_0^1(U_e)$ be the bubble function on the face, where U_e is the union of the two elements κ^+ and κ^- sharing e. By utilizing the bubble function technique developed by Verfürth[16], we introduce the following.

Lemma 3.2 For all polynomial functions $v \in P_k(\kappa)$,

$$\|v\|_{0,\kappa} \lesssim \left\|b_\kappa^{\frac{1}{2}} v\right\|_{0,\kappa}. \quad (3.17)$$

For all polynomial functions $w \in P_k(e)$, we have

$$\|w\|_{0,e} \lesssim \left\|b_e^{\frac{1}{2}} w\right\|_{0,e}. \quad (3.18)$$

For each $b_e w$, there is an extended $W_b$satisfying $W_b|_e = b_e w, W_b \in H_0^1(U_e)$

$$\|W_b\|_{0,w_e} \lesssim h_e^{\frac{1}{2}}\|w\|_{0,e}, \quad (3.19)$$

$$\|\nabla W_b\|_{0,w_e} \lesssim h_e^{-1/2}\|w\|_{0,e}. \quad (3.20)$$

Based on the above lemma and standard arguments (see Lemma 3.13 in Ref. [17]), we can prove the efficiency of the local error estimator as follows.

Lemma 3.3 Let (λ, u) and (λ_h, u_h) be the eigenpairs of (2.3) and (2.12), respectively. We have the following:

(a) For arbitrary $\kappa \in \mathcal{T}_h$,

$$h_\kappa\|\Delta u_h + (\lambda_h - c)u_h - \mathbf{r}\cdot\nabla u_h\|_{0,\kappa} \lesssim \|\nabla(u - u_h)\|_{0,\kappa} + h_\kappa\|u - u_h\|_{0,\kappa}$$

$$+ h_\kappa\|\lambda u - \lambda_h u_h\|_{0,\kappa}$$

(b) Let $e \in \mathcal{E}_I$,

$$h_e^{1/2}\|J_{F,1}\|_{0,e} \lesssim \sum_{\kappa\in U_e} \left(\|\nabla(u - u_h)\|_{0,\kappa} + h_\kappa\|u - u_h\|_{0,\kappa} + h_\kappa\|\lambda u - \lambda_h u_h\|_{0,\kappa}\right),$$

With $U_e = \{\kappa^+, \kappa^-\}$

(c) For each edge e$\in \mathcal{E}_I$,

$$h_e^{-1}\|J_{F,2}\|_{0,e}^2 = h_e^{-1}\|[[u_h]]\|_{0,e}^2 = h_e^{-1}\|[[u - u_h]]\|_{0,e}^2$$

(d) For each edge e$\in \mathcal{E}_D$,

$$h_e^{-1}\|J_{F,3}\|_{0,e}^2 = h_e^{-1}\|u_h\|_{0,e}^2 = h_e^{-1}\|u - u_h\|_{0,e}^2$$

Proof . (a) Let $v_h = \Delta u_h + (\lambda_h - c)u_h - \mathbf{r}\cdot\nabla u_h$ and $v_b = b_\kappa v_h$. Noting that $\Delta u + (\lambda - c)u - \mathbf{r}\cdot\nabla u = 0$ in $L^2(\kappa)$, $v_b = 0$ on $\partial\kappa$, by utilizing integration by parts, we have:

$$\left\|b_\kappa^{1/2}v_h\right\|_{0,\kappa}^2=\int_\kappa \nabla(u-u_h)\nabla v_b dx+\int_\kappa \mathrm{r}\cdot\nabla(u-u_h)v_b dx+\int_\kappa (\lambda_h u_h-\lambda u)v_b dx$$

$$+\int_\kappa c(u-u_h)v_b dx \tag{3.21}$$

By utilizing (3.17) and the Cauchy-Schwarz inequality, we obtain:

$$h_\kappa\|v_h\|_{0,\kappa}\lesssim\|\nabla(u-u_h)\|_{0,\kappa}+h_\kappa\|u-u_h\|_{0,\kappa}+h_\kappa\|\lambda_h u_h-\lambda u\|_{0,\kappa}$$

then (a) is proved.

(b) For any $e\in\mathcal{E}_I$, let $w_h=[[\nabla u_h]]$， $w_b=b_e w_h$. Let $W_b\in H_0^1(U_e)$ be an extension of W_b satisfying (3.22) and (3.23). Note that $[[\nabla \mathrm{u}]]=0$, by using Green's formula, we have:

$$\begin{aligned}\left\|b_e^{1/2}w_h\right\|_{0,e}^2&=\int_e [[\nabla u_h]]w_b ds=\int_e [[\nabla u_h-u]]w_b ds\\&=\sum_{\kappa\in U_e}\left(\int_\kappa(\triangle u_h-\triangle u)W_b dx+\int_\kappa\nabla(u_h-u)\cdot\nabla W_b dx\right)\\&\lesssim\sum_{\kappa\in U_e}(\|\Delta u_h+(\lambda_h-c)u_h-\mathbf{r}\cdot\nabla u_h\|_{0,\kappa}\|W_b\|_{0,\kappa}+\|\nabla(u_h-u)\|_{0,\kappa}\|\nabla W_b\|_{0,\kappa}\\&\quad+\|\mathbf{r}\cdot\nabla(u_h-u)\|_{0,\kappa}\|W_b\|_{0,\kappa}+\|u_h-u\|_{0,\kappa}\|W_b\|_{0,\kappa}+\|\lambda u-\lambda_h u_h\|_{0,\kappa}\|W_b\|_{0,\kappa}\end{aligned}\tag{3.22}$$

From (3.18), (3.19), (3.20), it can be deduced that

$$\begin{aligned}h_e^{1/2}\|w_h\|_{0,e}\lesssim h_e^{\frac{1}{2}}\left\|b_e^{\frac{1}{2}}w_h\right\|_{0,e}&\lesssim\sum_{\kappa\in U_e}(h_e\|\Delta u_h+(\lambda_h-c)u_h-\mathbf{r}\cdot\nabla u_h\|_{0,\kappa}+\|\nabla(u_h-u)\|_{0,\kappa}\\&\quad+h_e\|\mathbf{r}\cdot\nabla(u_h-u)\|_{0,\kappa}+h_e\|u_h-u\|_{0,\kappa}+h_e\|\lambda u-\lambda_h u_h\|_{0,\kappa}).\end{aligned}\tag{3.23}$$

Combining the bound of $\|\Delta u_h+(\lambda_h-c)u_h-\mathbf{r}\cdot\nabla u_h\|_{0,\kappa}$, in (a) and the shape-regularity of the mesh yields

$$h_e^{1/2}\|[[\nabla u_h]]\|_{0,e}\lesssim\sum_{\kappa\in U_e}\left(\|\nabla(u-u_h)\|_{0,\kappa}+h_\kappa\|u-u_h\|_{0,\kappa}+h_\kappa\|\lambda u-\lambda_h u_h\|_{0,\kappa}\right),$$

Which gives (b).

(c) For any $e\in\mathcal{E}$, we have $[[\mathrm{u}]]=0$. Then (c) is proved.

(d) For any $e\in\mathcal{E}_D$, we have $u=0$. Then (d) is proved.

Theorem 3.3 Under theorem 3.1, the following estimate holds:

$$\begin{aligned}\eta_\kappa\lesssim&\sum_{\kappa\in w_\kappa}\left(\|\nabla(u-u_h)\|_{0,\kappa}+h_\kappa\|u-u_h\|_{0,\kappa}+h_\kappa\|\lambda u-\lambda_h u_h\|_{0,\kappa}\right)\\&+\sum_{e\in\mathcal{E}_\mathcal{I}}h_e{}^{-1}\|[[u-u_h]]\|_{0,e}+\sum_{e\in\mathcal{E}_\mathcal{D}}h_e{}^{-1}\|u-u_h\|_{0,e},\end{aligned}\tag{3.24}$$

$$\eta(u_h)\lesssim\|u-u_h\|_h+h\|\lambda u-\lambda_h u_h\|_{0,\Omega}\tag{3.25}$$

Proof . Through the definition of η_κ and lemma 3.3, (3.24) can be obtained, and by using the definition of the energy norm $\|\cdot\|_h$, (3.25) can be obtained.

Theorem 3.3 shows that the error estimator $\eta(u_h)$ is efficient.

iii. The reliability of the indicator with respect to eigenvalue errors

Lemma 3.4 (Lemma 4.6 in Ref. [3]) Let (λ,u) and (λ_h,u_h) be the feature pairs of (2.3) and (2.12), respectively, let (λ^*,u^*) and (λ_h^*,u_h^*) be the feature pairs of (2.17) and (2.21), respectively, $(u_h,u_h^*)\neq0$, then

$$\lambda-\lambda_h=\lambda\frac{(u-u_h,u^*-u_h^*)}{(u_h,u_h^*)}-\frac{a_h(u-u_h,u^*-u_h^*)}{(u_h,u_h^*)}\tag{3.26}$$

Proof . By (2.28) and (2.29), we can obtain:

$$a(u,v)=\lambda(u,v),\quad\forall v\in V_h\tag{3.27}$$

$$a(v,u^*)=\lambda(v,u^*),\quad\forall v\in V_h\tag{3.28}$$

From (2.3), (2.12), (3.27), and (3.28), we can obtain:

$$\begin{aligned}&\lambda(u-u_h,u^*-u_h^*)-a_h(u-u_h,u^*-u_h^*)\\&=\lambda(u,u^*)-\lambda(u,u_h^*)-\lambda(u_h,u^*)+\lambda(u_h,u_h^*)\\&\quad-a_h(u,u^*)+a_h(u,u_h^*)+a_h(u_h,u^*)-a_h(u_h,u_h^*)\\&=\lambda(u_h,u_h^*)-a_h(u_h,u_h^*)=(\lambda-\lambda_h)(u_h,u_h^*)\end{aligned}\tag{3.29}$$

Dividing (u_h, u_h^*) on both sides of the above equation leads to (3.26).

Theorem 3.4 Under the condition of Lemma 3.4 , we have:

$$|\lambda - \lambda_h| \quad \lesssim \eta(u_h)^2 + \eta(u_h^*)^2 \tag{3.30}$$

Proof . Theorem 3.1 shows that $\|u - u_h\|_{0,\Omega}$ is of higher order than $\|u - u_h\|_h$, and $\|u^* - u_h^*\|_{0,\Omega}$ is also of higher order than $\|u^* - u_h^*\|_h$. Therefore, from (3.26),(3.9) and the estimator of u_h^*, we can obtain

$$|\lambda - \lambda_h| \lesssim \|u - u_h\|_h \|u^* - u_h^*\|_h \lesssim \eta(u_h)^2 + \eta(u_h^*)^2$$

This completes the proof.

From Theorem 3.2 and Theorem 3.3, it can be concluded that the error indicator $\eta(u_h)^2 + \eta(u_h^*)^2$ for the characteristic function error $\|u - u_h\|_h^2 + \|u^* - u_h^*\|_h^2$ is reliable and efficient. Therefore, the adaptive algorithm based on this error indicator can generate a well-refined grid. The approximate characteristic function achieves optimal convergence rate of $O(dof^{-m})$ in the norm $\|\cdot\|_h^2$. From equation (3.30), we have $|\lambda - \lambda_h| \leq dof^{-m}$. Thus, $\eta(u_h)^2 + \eta(u_h^*)^2$ can be considered as an error indicator for λ_h. The numerical experiments in Section 4 demonstrate that $\eta(u_h)^2 + \eta(u_h^*)^2$ serves as a reliable and efficient error indicator for λ_h.

IV. Numerical experiments

In this section, we will report some numerical experiments to demonstrate the effectiveness of our method. We consider problem (2.1), where $\mathrm{r} = (0,0)^T, (1,1)^T, (2,0)^T$ and $c = 0$. Our program is compiled under the iFEM software package, and we use the LDG method with parameters $\mathrm{b} = (0,0)^T$ and $\alpha = 400$. We consider the following two test domains: the L-shaped domain $\Omega_L = (-1,1)^2 \setminus ([0,1) \times (-1,0])$ and the crack domain $\Omega_{SL} = (-1,1)^2 \setminus \{0 \leqslant x \leqslant 1, y = 0\}$. We take the reference eigenvalue $\lambda_1 = |\mathrm{r}|^2/4 + 9.63972384472$ in the L-shaped domain, and the reference eigenvalue $\lambda_1 = |\mathrm{r}|^2/4 + 8.3713297112$ in the crack domain Ω_{SL}.

Table 1: Numerical results using quadratic LDG method for Ω_L,Ω_{SL} with an initial grid of h=1/8 when $\mathbf{r} = (0,0)^T$

Domain	l	dof	λ_1	Error
Ω_L	1	2304	9.663013953	0.02329011
	5	2640	9.643980310	0.00425647
	15	21408	9.639770483	4.6638E-05
	20	71124	9.639728536	4.6917E-06
Domain	l	dof	λ_1	Error
Ω_{SL}	1	3072	8.469827928	0.09849822
	5	3252	8.397502155	0.02617244
	15	9396	8.372217175	0.00088746
	20	24336	8.371487145	0.00015743

Table 2: Numerical results using quadratic LDG method for Ω_L,Ω_{SL} with an initial grid of h=1/8 when $\mathbf{r} = (1,1)^T$

Domain	l	dof	λ_1	Error
Ω_L	1	2304	10.16275287	0.02302903
	5	2652	10.14372296	0.00399912
	15	22566	10.13977018	4.6335E-05
	20	75030	10.13972894	5.0953E-06
Domain	l	dof	λ_1	Error
Ω_{SL}	1	3072	8.969524856	0.09819514
	5	3252	8.897240431	0.02591072
	15	9432	8.872201806	0.00087209
	20	24354	8.871486251	0.00015654

Table 3: Numerical results using quadratic LDG method for Ω_L,Ω_{SL} with an initial grid of h=1/8 when $\mathbf{r} = (2,0)^T$

Domain	l	dof	λ_1	Error
Ω_L	1	2304	10.66272197	0.02299813
	4	2412	10.64604925	0.00632541
	15	21198	10.63977213	4.8285E-05
	18	43170	10.63973735	1.3505E-05
Domain	l	dof	λ_1	Error
Ω_{SL}	1	3072	9.469498641	0.09816893
	5	3252	9.397205004	0.02587529
	15	9492	9.372176457	0.00084675

	20	23988	9.371484809	0.00015511

Figure.1. Adaptive meshes and error curves using the quadratic LDG method for $\mathbf{r} = (0,0)^T$

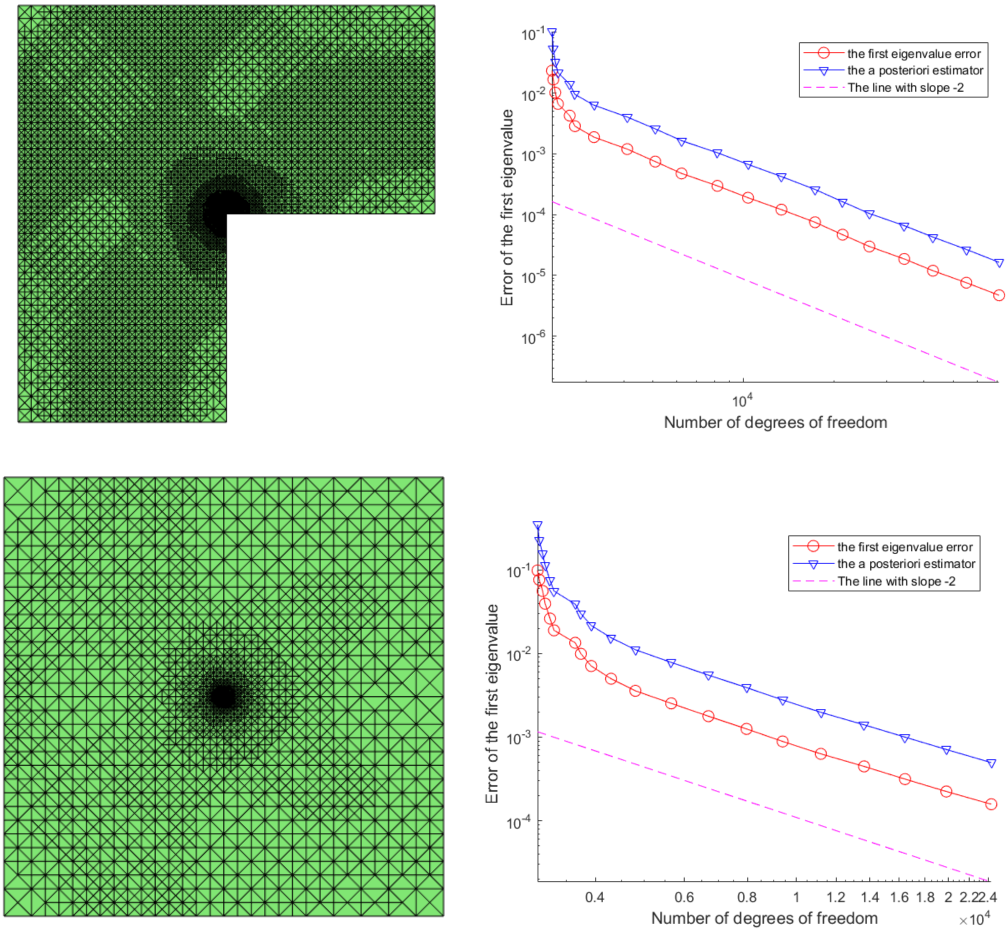
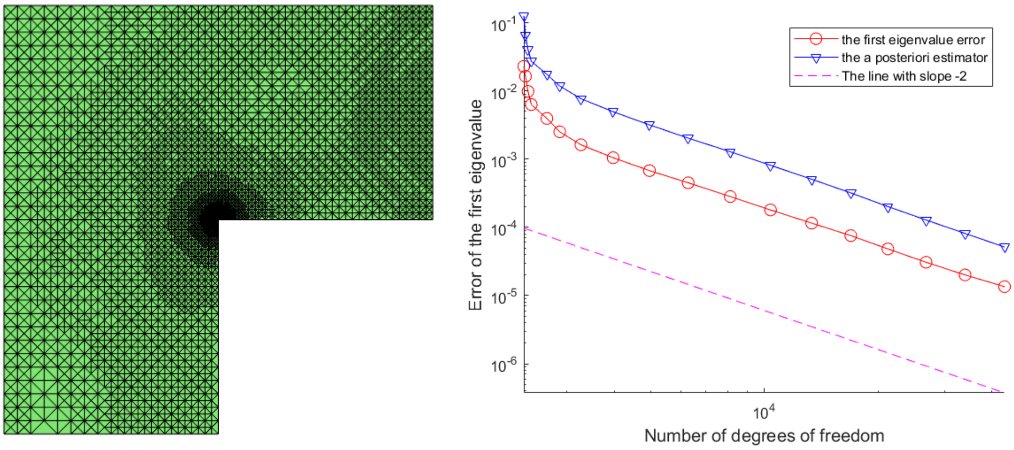


Figure.2. Adaptive meshes and error curves using the quadratic LDG method for $\mathbf{r} = (1,1)^T$



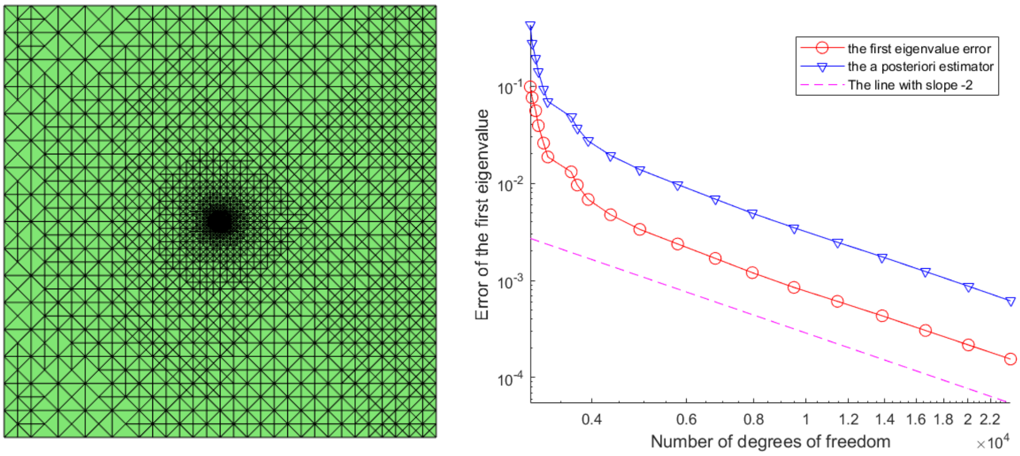
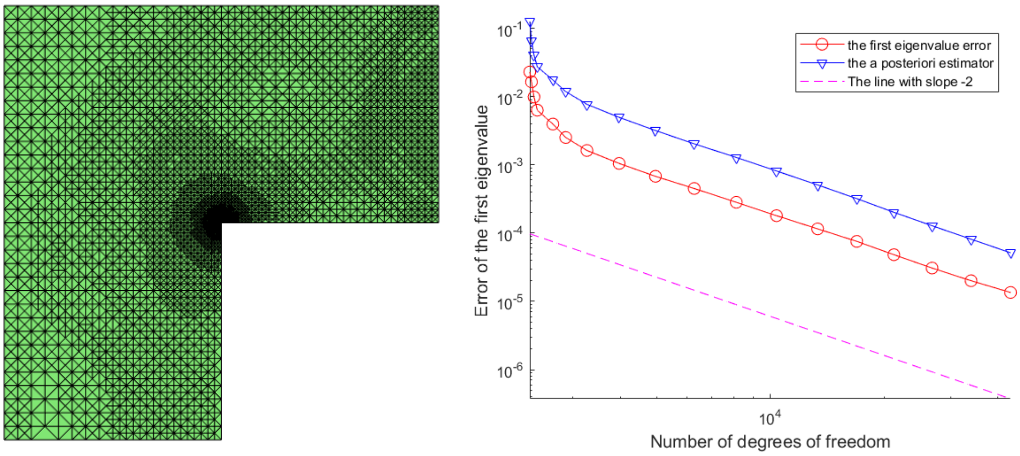
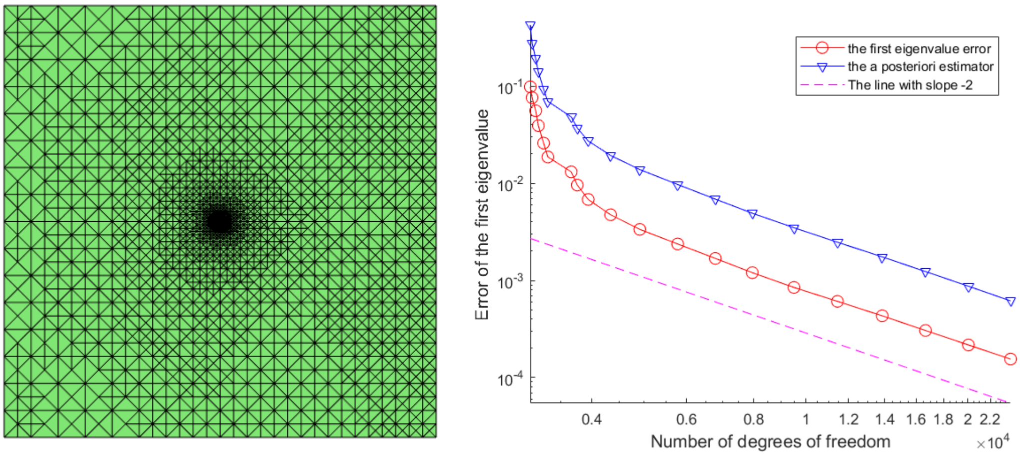


Figure.3. Adaptive meshes and error curves using the quadratic LDG method for $\mathbf{r} = (2,0)^T$





We present the numerical results of the computed eigenvalues obtained through adaptive computations in Tables 1 to 3, and illustrate the adaptive grids and error curves in Figures 1 to 3. From Figures 1 to 3, we observe that the error curves of the quadratic LDG method are approximately parallel to a line with a slope of -2 when $\mathbf{r} = (0,0)^T, (1,1)^T, (2,0)^T$. The results indicate that the adaptive algorithm achieves the optimal convergence rate. Additionally, from the error curves, it can be observed that, using the same degree of freedom (dof), the approximation obtained by the adaptive algorithm is more accurate compared to the approximation computed on uniform grids.

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