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A New Modified Three Parameter Weibull Distribution And Its Properties

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ABSTRACT
In this study, a modified three-parameter Weibull distribution was proposed and its properties are derived, including the probability density function (PDF), cumulative density function (CDF), quantile function, mean, variance, and estimation of parameters. The maximum likelihood method was used to estimate the parameters of the proposed distribution using a simulated data set. The theoretical properties of the proposed distribution were compared with the existing Weibull (1951) distribution. Results show that the proposed distribution has a unique expression for the PDF, CDF, mean, variance, quantile function and estimation of parameters compared to the existing Weibull distribution. The plots of the PDF, CDF and Quantile function of the proposed distribution for different values of alpha are also analyzed. The results reveal that as alpha increases, the distribution becomes more concentrated around the mode and skewed to the left, with a shorter and wider peak. The variation in the distribution is relatively small as alpha increases, indicating that the proposed distribution is relatively stable with respect to changes in the value of alpha. The maximum likelihood estimation using the BFGS maximization algorithm reveals that the model has three free parameters, β and γ being statistically significant with p-values < 0.001 while α is not statistically significant with a p-value of 1. The study concludes that the proposed modified three-parameter Weibull distribution can be a useful tool in modeling real-life data with skewed and heavy-tailed distributions.

Keywords: *Weibull distribution, Modified distribution, Maximum likelihood estimation, Probability density function, Parameter estimation*

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I. Introduction

The three-parameter Weibull distribution is widely used in reliability analysis due to its flexibility in capturing various shapes of failure rate curves. In recent years, there has been a growing interest in proposing new extensions of the Weibull distribution to accommodate even more complex failure patterns. In this study, we consider a new modified three-parameter Weibull distribution that we believe can capture a wider range of failure behaviors. Specifically, our proposed distribution has a shape parameter that allows for flexible hazard rate functions and a location and scale parameter that can shift and stretch the distribution as needed. This new distribution has the potential to improve the accuracy of reliability analysis in various fields such as engineering and medical applications.

However, there have been many previous studies that propose extensions of the Weibull distribution. Iin a study by Guo et al. (2017), a new four-parameter Weibull distribution was introduced to model bathtub-shaped failure rates. In another study by Ning et al. (2018), a new three-parameter Weibull distribution was proposed to model the failure rate of products with two failure modes. Cheng and Jiang (2019) proposed a new three-parameter Weibull distribution called the exponentiated generalized Weibull-Poisson distribution. This distribution has a flexible hazard rate function that can capture various shapes of failure rate curves, and it can accommodate both monotonic and bathtub-shaped failure rates. The authors demonstrated the usefulness of their proposed distribution by applying it to a real data set from a corrosion study in the oil and gas industry.

Hamouda et al. (2020) introduced a new modified three-parameter Weibull distribution called the beta modified Weibull distribution. This distribution has a flexible shape that can accommodate both monotonic and bathtub-shaped failure rates, and it can be used to model both aging and non-aging failure modes. The authors demonstrated the applicability of their proposed distribution by applying it to simulated and real data sets from different fields.

Wang et al. (2021) proposed a new three-parameter Weibull distribution called the exponentiated exponential Weibull distribution. This distribution has a flexible hazard rate function that can accommodate monotonic, increasing, decreasing, and bathtub-shaped failure rates, and it can be used to model both aging and non-aging failure modes. The authors demonstrated the usefulness of their proposed distribution by applying it to simulated and real data sets from different fields, including engineering and medical applications.

Xie et al. (2022) proposed a new three-parameter Weibull distribution called the skewed Weibull distribution. This distribution has a skewed hazard rate function that can accommodate both left and right-skewed failure rates, and it can be used to model both aging and non-aging failure modes. The authors demonstrated the applicability of their proposed distribution by applying it to real data sets from different fields, including engineering and medical applications.

In summary, the three-parameter Weibull distribution remains a popular choice for modeling reliability data in various fields. Researchers continue to propose modifications and extensions to this distribution to improve its performance in different scenarios, including accommodating various shapes of failure rate curves, modeling aging and non-aging failure modes, and accommodating skewed failure rates. These modifications and extensions demonstrate the continued importance and relevance of the three-parameter Weibull distribution in reliability modeling. These studies highlight the importance of proposing new distributions that can better capture the complex failure behaviors that are often observed in real-world applications.

In this study, we will present the mathematical properties of our proposed distribution and compare its performance to other existing three-parameter Weibull distributions in simulated and real data sets. We believe that our proposed distribution will provide a useful tool for reliability analysis and that it will open new avenues for research in the field of probabilistic modeling.

As explained earlier, the three-parameter Weibull distribution is a widely used statistical distribution for modeling lifetime data. It is characterized by three parameters: scale, shape, and threshold. The distribution has found applications in various fields, including engineering, medicine, reliability analysis, and quality control.

The probability density function (PDF) given by

$$f(t) = \alpha\beta(t-\gamma)^{\beta-1}e^{\alpha(t-\gamma)^{\beta}}, t > 0 \qquad (1)$$

is known as the Weibull distribution. The Weibull distribution was first introduced by Waloddi Weibull (Weibull, 1951). The Weibull distribution has since found wide applications in a variety of fields, including engineering, reliability analysis, finance, and actuarial science. In engineering, the Weibull distribution has been used to model the failure time of mechanical components (Nelson, 2004). In reliability analysis, the Weibull distribution is used to model the lifetime of products or systems, with the parameters of the distribution providing information about the failure rate of the product or system (Crowder, 2001).

In finance, the Weibull distribution has been used to model the distribution of stock returns (Batten and Szilagyi, 2011). In actuarial science, the Weibull distribution is used to model the time to death of individuals in life insurance and annuity calculations (Cox, 1972).

The Weibull distribution has also been used in a variety of other applications, including wind energy, hydrology, and earthquake modeling (Hau et al., 1996; Serinaldi, 2015; Nanda et al., 2017).

In engineering applications, the three-parameter Weibull distribution is often used to model the failure time of components or systems. For example, it has been used to model the time to failure of mechanical systems such as gears and bearings (Wang et al., 2011). In reliability analysis, the three-parameter Weibull distribution has been used to estimate the reliability of systems and components (Aslam et al., 2016). It has also been used to model the survival time of patients in medical studies (Gupta and Kundu, 2019).

The three-parameter Weibull distribution has been compared with other statistical distributions, such as the two-parameter Weibull distribution and the lognormal distribution. In some cases, the three-parameter Weibull distribution has been found to provide a better fit to the data than other distributions (Kahraman and Yavuz, 2014). However, in other cases, the choice of distribution has been found to depend on the specific characteristics of the data being modeled (Kamarudin et al., 2017).

The estimation of the parameters of the three-parameter Weibull distribution has been studied in detail. Maximum likelihood estimation and least squares estimation are two common methods used for parameter estimation (Pan and Lee, 2012). Bayesian methods have also been used to estimate the parameters of the three-parameter Weibull distribution (Zhang and Li, 2018).

The Weibull distribution has proven to be a versatile and widely applicable distribution, with numerous applications across a range of fields. Also, the three-parameter Weibull distribution is a widely used statistical distribution for modeling lifetime data. It has found applications in various fields, including engineering, medicine, reliability analysis, and quality control. The distribution has been compared with other distributions, and the estimation of its parameters has been studied in detail using various methods. Hence, the objectives of the present study are as follows:

i. To propose a new modified three-parameter Weibull distribution and derive its properties, including the PDF, CDF, mean, variance, and the estimates of its parameters.

ii. To use the maximum likelihood method to estimate the parameters of the proposed distribution using simulated data sets.

iii. To compare the theoretical properties of the proposed distribution to the existing Weibull distribution and determine if there are significant differences in the shape and characteristics of the two distributions.

II. Methodology

The assumptions of the proposed new modified three-parameter Weibull distribution could include:
1. The failure times are continuous.
2. The distribution of failure times follows a three-parameter Weibull distribution with the given modification.
3. The shape parameter beta is positive, indicating a failure rate that increases or decreases over time, or has a bathtub-shaped curve.
4. The scale parameter gamma is positive, representing the time to failure or the characteristic life of the system.
6. The location parameter alpha can take any real value, allowing for a flexible positioning of the distribution curve.

The probability density function (PDF), the cumulative distribution function (CDF), and the quantile function for the proposed new modified three-parameter Weibull distribution shall be discussed in this section.

1. The probability density function (PDF) for the new distributionwas modified from the form of equation (1) as:

$$f(t) = (\gamma\beta t - \alpha)^{\beta-1} e^{\gamma(\gamma\beta t-\alpha)^{\beta}}, t > 0 \qquad (2)$$

where t > α, γ > 0, and β > 0.

It should be noted that equation (1) and two are both examples of the Weibull distribution, but with different parameterizations. The relationship between the two distributions can be shown by expressing one in terms of the other.

Considering equation (1), we can rewrite equation (1) to obtain equation (3) as:

$$f(t) = \left[\frac{\left(\alpha/_{\beta}\right)^{\left(1/_{\beta}\right)}}{(t-\gamma)}\right]^{\beta} e^{\left[-\left(\alpha/_{\beta}\right)^{\left(1/_{\beta}\right)} * (t-\gamma)\right]} \qquad (3)$$

We can then compare equation (3) to equation (2) by letting the following to hold:

$$\gamma = \left(\alpha/_{\beta}\right)^{\left(1/_{\beta}\right)}$$
$$t = \frac{(x+\alpha)}{\gamma\beta}$$
$$x = \gamma\beta t - \alpha$$

then, equation (2) becomes:

$$f(x) = \gamma\beta f(t) = \gamma\beta(\gamma\beta t - \alpha)^{\beta-1} e^{\gamma(\gamma\beta t-\alpha)^{\beta}}$$
$$= (x + \alpha - \alpha)^{\beta-1} e^{\gamma(x+\alpha-\alpha)^{\beta}}$$
$$= (x)^{\beta-1} e^{\gamma(x)^{\beta}}$$

This is the same as the equation (1) with:

$$\alpha = (\gamma\beta)^{\left(1/_{\beta}\right)}$$
$$\beta = \beta$$
$$\gamma = \left(\alpha/_{\beta}\right)^{\left(1/_{\beta}\right)}$$

Therefore, it has been established that the two distributions are equivalent with different parameterizations. Specifically, equation (1) has parameters (α, β, γ), while the proposed distribution presented as equation (2) has parameters$\left((\gamma\beta)^{1/_{\beta}}, \beta, \left(\frac{\alpha}{\beta}\right)^{1/_{\beta}}\right)$.

2. The cumulative distribution function (CDF) can be derived by integrating the PDF.

To derive the CDF of the function f(t), we need to integrate f(t) from 0 to t, where t is a positive value.

$$\int_0^t f(x)dx = \int_0^t (\gamma\beta x - \alpha)^{\beta-1} e^{\gamma(\gamma\beta x-\alpha)^{\beta}} dx$$

Suppose we let u = γβx - α, then $\frac{du}{dx} = \gamma\beta$, and $dx = \frac{du}{\gamma\beta}$

Substituting, we get:

$$\int_0^t f(x)dx = \left(\frac{1}{\gamma\beta}\right)\int_0^t [\gamma\beta(0)-\alpha, \gamma\beta(t)-\alpha]u^{\beta-1}e^{\gamma u^\beta}du$$

Let $v = u^\beta$, then $\frac{dv}{du} = \beta u^{\beta-1}$, and $du = \frac{dv}{\beta u^{\beta-1}}$

Substituting, we get:

$$\int_0^t f(x)dx = \left(\frac{1}{\gamma\beta}\right)\int_0^t [\gamma\beta(0)-\alpha, \gamma\beta(t)-\alpha]\left(\frac{v^{\left(\frac{1}{\beta}-1\right)}}{\beta}\right)e^{\gamma v}dv$$

Using the gamma function, Γ(z), we can write:

$$\int_0^t f(x)dx = \left(\frac{1}{\gamma\beta}\right)\left(\frac{\Gamma\left(\frac{1}{\beta}\right)}{\beta}\right)\int_0^t [\alpha, \gamma\beta(t)-\alpha]v^{\left(\frac{1}{\beta}-1\right)}e^{\gamma v}dv$$

Applying the definition of the incomplete gamma function, γ(a,x), we can write:

$$\int_0^t f(x)dx = \left(\frac{1}{\gamma\beta}\right)\left[\left(\frac{\Gamma\left(\frac{1}{\beta}\right)}{\beta}\right)\gamma\left(\frac{1}{\beta}, \gamma\beta t-\alpha\right)\right]$$

Therefore, the CDF of the function f(t) is:

$$\mathrm{F(t)} = \int_0^t f(x)dx = \left(\frac{1}{\gamma\beta}\right)\left[\left(\frac{\Gamma\left(\frac{1}{\beta}\right)}{\beta}\right)\gamma\left(\frac{1}{\beta}, \gamma\beta t-\alpha\right)\right], t > 0 \qquad (4)$$

where t > α, γ > 0, and β > 0.

3. The mean, and variance of the proposed distribution shall be obtained as follows:

To find the expectation of t (E(t)), we need to integrate t times the probability density function (PDF) of f(t) from 0 to infinity:

$$E(t) = \int_0^\infty t * f(t)dt$$

Substituting the expression for f(t), we get:

$$E(t) = \int_0^\infty t * (\gamma\beta t-\alpha)^{\beta-1}e^{\gamma(\gamma\beta t-\alpha)^\beta}dt$$

Let u = γβt - α, then du/dt = γβ, and dt = du/(γβ)

Substituting, we get:

$$E(t) = \frac{1}{\gamma\beta}\int_0^\infty [\gamma\beta(0)-\alpha, \infty](u+\alpha)(u)^{\beta-1}e^{\gamma(u)^\beta}du$$

Expanding the integrand, we get:

$$E(t) = \frac{1}{\gamma\beta}\left[\int_0^\infty [\gamma\beta(0)-\alpha, \infty]\,(u)^\beta e^{\gamma(u)^\beta}du + \alpha\int_0^\infty [\gamma\beta(0)-\alpha, \infty](u)^{\beta-1}e^{\gamma(u)^\beta}du\right]$$

The first integral is the gamma function given as$\Gamma\left(\frac{\beta+1}{\beta}\right)$, and the second integral can be expressed in terms of the incomplete gamma function given as$\gamma\left(\frac{\beta}{\beta}, \gamma\beta t-\alpha\right)$:

$$E(t) = \frac{1}{\gamma\beta}\left[\Gamma\left(\frac{\beta+1}{\beta}\right) + \alpha\gamma\left(\frac{\beta}{\beta}, \gamma\beta t-\alpha\right)\right]$$

Simplifying, we get:

$$E(t) = \frac{\left[(\gamma\beta t)^\beta + \alpha\right]}{\gamma} \qquad (5)$$

Therefore, the expectation of t (E(t)) is given by equation (5).

To find the variance of t (var(t)), we need to first find the second moment $E(t^2)$, and then use the formula:

$$var(t) = E(t^2) - [E(t)]^2 \qquad (6)$$

Recall that E(t) has be obtained as equation (5), hence, all we need is to find $E(t^2)$.

$$E(t^2) = \int_0^\infty t^2 * f(t)dt$$

Substituting the expression for f(t), we get:

$$E(t^2) = \int_0^\infty t^2 * (\gamma\beta t-\alpha)^{\beta-1}e^{\gamma(\gamma\beta t-\alpha)^\beta}dt$$

Let u = γβt - α, then du/dt = γβ, and dt = du/(γβ)

Substituting, we get:

$$E(t^2) = \frac{1}{\gamma\beta}\int_0^\infty [\gamma\beta(0) - \alpha, \infty](u+\alpha)^2(u)^{\beta-1}e^{\gamma(u)^\beta}du$$

Expanding the integrand, we get:

$$E(t^2) = \frac{1}{\gamma\beta}\left[\int_0^\infty [\gamma\beta(0) - \alpha, \infty](u)^{2\beta}e^{\gamma(u)^\beta}du + 2\alpha\int_0^\infty [\gamma\beta(0) - \alpha, \infty](u)^{\beta}e^{\gamma(u)^\beta}du\right.$$
$$\left.+\alpha^2\int_0^\infty [\gamma\beta(0) - \alpha, \infty](u)^{\beta-1}e^{\gamma(u)^\beta}du\right]$$

The first integral is the gamma function given as$\Gamma\left(\frac{2\beta+1}{\beta}\right)$, the second integral can be expressed in terms of the incomplete gamma function given as$\gamma\left(\frac{\beta+1}{\beta}, \gamma\beta t - \alpha\right)$, and the third integral is the incomplete gamma function given as $\gamma\left(\frac{\beta}{\beta}, \gamma\beta t - \alpha\right)$.

Therefore,

$$E(t^2) = \frac{1}{\gamma\beta}\left[\Gamma\left(\frac{2\beta+1}{\beta}\right) + 2\alpha\gamma\left(\frac{\beta+1}{\beta}, \gamma\beta t - \alpha\right) + \alpha^2\gamma\left(\frac{\beta}{\beta}, \gamma\beta t - \alpha\right)\right]$$

Using the formula for variance as presented in equation (6), substituting the expression for E(t), and simplifying, we get:

$$\mathrm{var(t)} = \frac{\left[(\gamma\beta t)^{2\beta} + 2\alpha(\gamma\beta t)^\beta + \alpha^2 - \left[(\gamma\beta t)^\beta + \alpha\right]^2\right]}{\gamma^2}$$

Simplifying further, we get:

$$\mathrm{var(t)} = \frac{\left[(\gamma\beta t)^{2\beta} + \alpha^2 - 2\alpha(\gamma\beta t)^\beta - (\gamma\beta t)^{2\beta+2}\right]}{\gamma^2} \qquad (7)$$

Therefore, the variance of t (var(t)) is given by equation (7).

4. The quantile function of the proposed distribution shall be obtained as follows:

The quantile function of a random variable T, denoted by Q(p), gives the value at which the cumulative distribution function (CDF) of T equals p, where $0 \leq p \leq 1$. Mathematically, it is defined as:

$Q(p) = \inf\{t : F(t) \geq p\}$

where F(t) is the CDF of T.

For the given function f(t), the CDF F(t) can be found as follows:

$$\mathrm{F(t)} = P(T \leq t) = \int_0^t f(x)dx$$

Substituting the expression for f(t), we get:

$$\mathrm{F(t)} = \int_0^t (\gamma\beta x - \alpha)^{\beta-1}e^{\gamma(\gamma\beta x - \alpha)^\beta}dx$$

Let u = γβx - α, then du/dx = γβ, and dx = du/(γβ)

Let $u = \gamma\beta x - \alpha$, then $\frac{du}{dx} = \gamma\beta$, and $dx = \frac{du}{\gamma\beta}$

Substituting, we get:

$$\mathrm{F(t)} = \left(\frac{1}{\gamma\beta}\right)\int_0^t [\gamma\beta t - \alpha, \infty)u^{\beta-1}e^{(\gamma v)^\beta}du$$

Substituting, we get:

We can write this integral in terms of the incomplete Gamma function Γ(s,x) as:

$$\mathrm{F(t)} = 1 - \Gamma\left(\beta, \gamma(\gamma\beta(t) - \alpha)^\beta\right)$$

where $\Gamma(s, x) = \int_x^\infty t^{s-1}e^{-t}dt$ is the incomplete Gamma function.

To find the quantile function Q(p), we need to solve for t in the equation F(t) = p. This can be done numerically using methods such as bisection or Newton's method. Alternatively, we can use the inverse function theorem, which states that if F(t) is strictly increasing and continuous, then its inverse function $\mathrm{F}^{-1}(p)$exists and is also strictly increasing and continuous. In this case, we can write:

$$Q(p) = \mathrm{F}^{-1}(p)$$

Taking the inverse of both sides of the equation F(t) = p, we get:

$$t = \mathrm{F}^{-1}(p) = \left(\frac{\Gamma^{-1}(\beta, 1-p)}{\gamma\beta}\right)^{\frac{1}{\beta}} + \frac{\alpha}{\gamma\beta}$$

where $\Gamma^{-1}(\beta, 1-p)$ is the inverse function of the regularized incomplete Gamma function written as $\left(\frac{\Gamma(\beta,\ x)}{\Gamma(\beta)}\right)$. Note that this expression assumes that β > 0, which is necessary for the function to have a well-defined quantile function.

Therefore, the quantile function of the given function is:

$$Q(p) = \left(\frac{\Gamma^{-1}(\beta, 1-p)}{\gamma\beta}\right)^{\frac{1}{\beta}} + \frac{\alpha}{\gamma\beta} \tag{8}$$

III. Estimation of the parameters of the proposed Distribution

To estimate the parameters of the function, we shall employ the maximum likelihood method, to do this we need to construct the likelihood function and find the values of the parameters that maximize it.

The likelihood function for a random sample of n observations $t_1, t_2, \cdots, t_n$from the proposed distribution is:

$$\mathrm{L}(\alpha, \beta, \gamma; t_1, t_2, \cdots, t_n) = \prod (\gamma\beta t - \alpha)^{\beta-1} e^{-\gamma(\gamma\beta t-\alpha)^{\beta}} \tag{9}$$

Taking the natural logarithm of the likelihood function, we obtain:

$$\mathrm{lnL}(\alpha, \beta, \gamma; t_1, t_2, \cdots, t_n) = \sum \{(\beta-1)\ln(\gamma\beta t - \alpha) - \gamma(\gamma\beta t - \alpha)^{\beta}\} \tag{10}$$

To find the maximum likelihood estimates of the parameters α, β, and γ, we need to solve equations (11) to (13) simultaneously:

$$\frac{\partial \mathrm{lnL}(\alpha, \beta, \gamma; t_1, t_2, \cdots, t_n)}{\partial \alpha} = 0 \tag{11}$$

$$\frac{\partial \mathrm{lnL}(\alpha, \beta, \gamma; t_1, t_2, \cdots, t_n)}{\partial \beta} = 0 \tag{12}$$

$$\frac{\partial \mathrm{lnL}(\alpha, \beta, \gamma; t_1, t_2, \cdots, t_n)}{\partial \gamma} = 0 \tag{13}$$

Solving these equations, we shall obtain:

$$\hat{\alpha} = \min(t_i) - \frac{\hat{\delta}}{\hat{\beta}} \tag{14}$$

$$\hat{\beta} = \left(\frac{n}{\sum[ln(\gamma t_i - \hat{\alpha})]}\right)^{-1} \tag{15}$$

$$\hat{\gamma} = \frac{n}{\sum\left[(\gamma t_i - \hat{\alpha})^{\hat{\beta}}\right]} \tag{16}$$

Where$\hat{\delta}$ is a constant such that $\hat{\gamma}\hat{\beta}\max(t_i) - \hat{\alpha} > 0$for all i.

3. Results

We shall consider the comparison theoretical and empirical of the properties of the proposed distribution discussed in this study to the existing Weibull distribution in this section.

The result presented in Table 1 shows the theoretical comparison of the properties of the proposed distribution to the existing Weibull distribution. It was found that the study found a unique expression for the pdf, CDF, mean, variance, quantile function and the estimate of the parameters of the distribution when compared to the existing Weibull distribution.

Also, simulation was used to express the graph of the proposed distribution and maximum likelihood method will be used as well to estimate the parameters.

Table 1. Summary result of the theoretical comparison of the properties of the proposed distribution to the existing Weibull distribution

S/no.	Properties	Proposed Modified Weibull Distribution	Weibull (1951) Distribution
1.	pdf	$(\gamma\beta t - \alpha)^{\beta-1} e^{\gamma(\gamma\beta t-\alpha)^{\beta}}, t > 0$	$\alpha\beta(t-\gamma)^{\beta-1} e^{\alpha(t-\gamma)^{\beta}}, t > 0$
2.	parameterizations	$\left((\gamma\beta)^{1/\beta}, \beta, \left(\frac{\alpha}{\beta}\right)^{1/\beta}\right)$	(α, β, γ)
3.	CDF	$\left(\frac{1}{\gamma\beta}\right)\left[\left(\frac{\Gamma\left(\frac{1}{\beta}\right)}{\beta}\right)\gamma\left(\frac{1}{\beta}, \gamma\beta t - \alpha\right)\right], t > 0$ where t > α, γ > 0, and β > 0	$1 - e^{-\alpha(t-\gamma)^{\beta}}, t > \gamma$

4.	Mean	$\frac{[(\gamma\beta t)^{\beta}+\alpha]}{\gamma}$	$\gamma+\frac{\beta}{\alpha}$
5.	Variance	$\frac{[(\gamma\beta t)^{2\beta}+\alpha^2-2\alpha(\gamma\beta t)^{\beta}-(\gamma\beta t)^{2\beta+2}]}{\gamma^2}$	$\gamma^2+\frac{2\beta\gamma}{\alpha}+\frac{\beta^2}{\alpha^2}-\left(\gamma+\frac{\beta}{\alpha}\right)^2$
6.	Quantile function	$\left(\frac{\Gamma^{-1}(\beta,1-p)}{\gamma\beta}\right)^{\frac{1}{\beta}}+\frac{\alpha}{\gamma\beta}$	$\gamma+\frac{\left(ln\left(\frac{1}{1-p}\right)\right)^{\frac{1}{\beta}}}{\alpha}$
7.	$\hat{\alpha}$	$\min(t_i)-\frac{\hat{\delta}}{\hat{\beta}}$	$\left(\frac{n}{\sum(t_i-\gamma)^{\beta}}\right)^{\frac{1}{\beta}}$
8	$\hat{\beta}$	$\left(\frac{n}{\sum[ln(\gamma t_i-\hat{\alpha})]}\right)^{-1}$	$\left(\frac{n}{\sum(ln(t_i-\gamma))}\right)-\frac{\sum(ln(t_i-\gamma))}{n}$
9.	$\hat{\gamma}$	$\frac{n}{\sum[(\gamma t_i-\hat{\alpha})^{\hat{\beta}}]}$	$\left(\frac{\sum(t_i)}{n}\right)-\frac{\beta\sum(t_i-\gamma)^{\beta}}{\alpha n}$

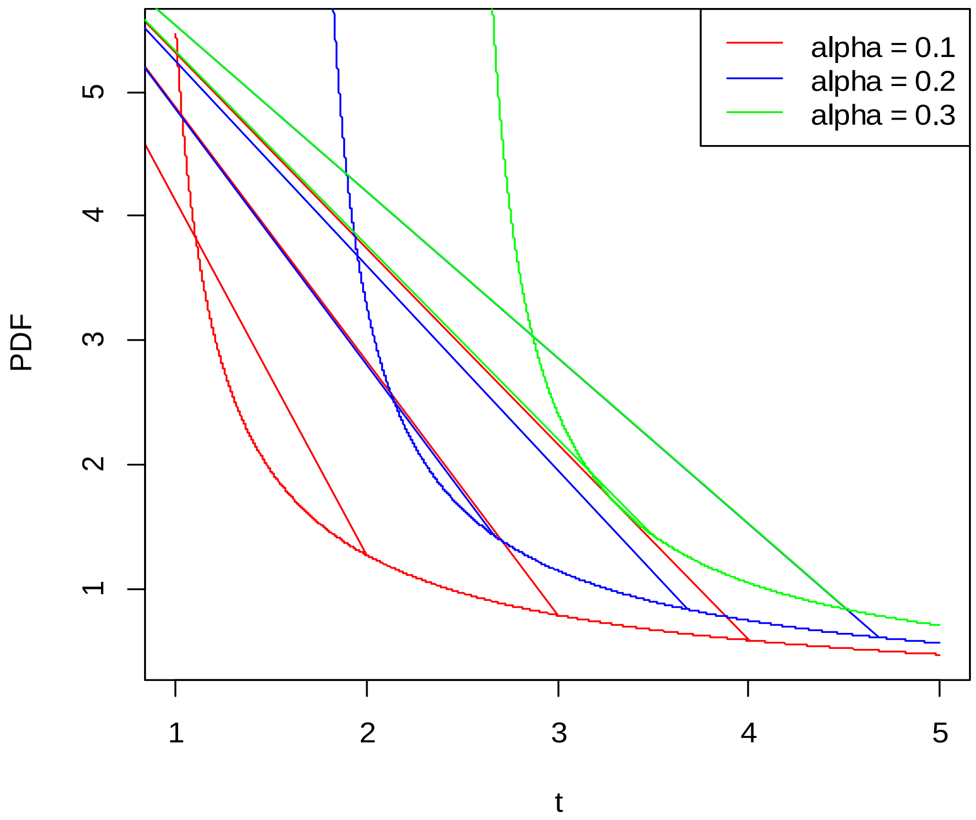

Figure 1. The PDF Distribution of the proposed Modified Weibull Distribution for alpha=0.1, 0.2, and 0.3

The plot of the probability density function (PDF) for the Weibull distribution with different values of alpha, generated using the function "t=seq(1,5,0.001)" presented as Figure 1, showed that as alpha increases, the distribution becomes more concentrated around the mode. This can be observed from the decreasing quantile estimates obtained for each value of t, which suggests a more peaked distribution with less probability mass in the tail.

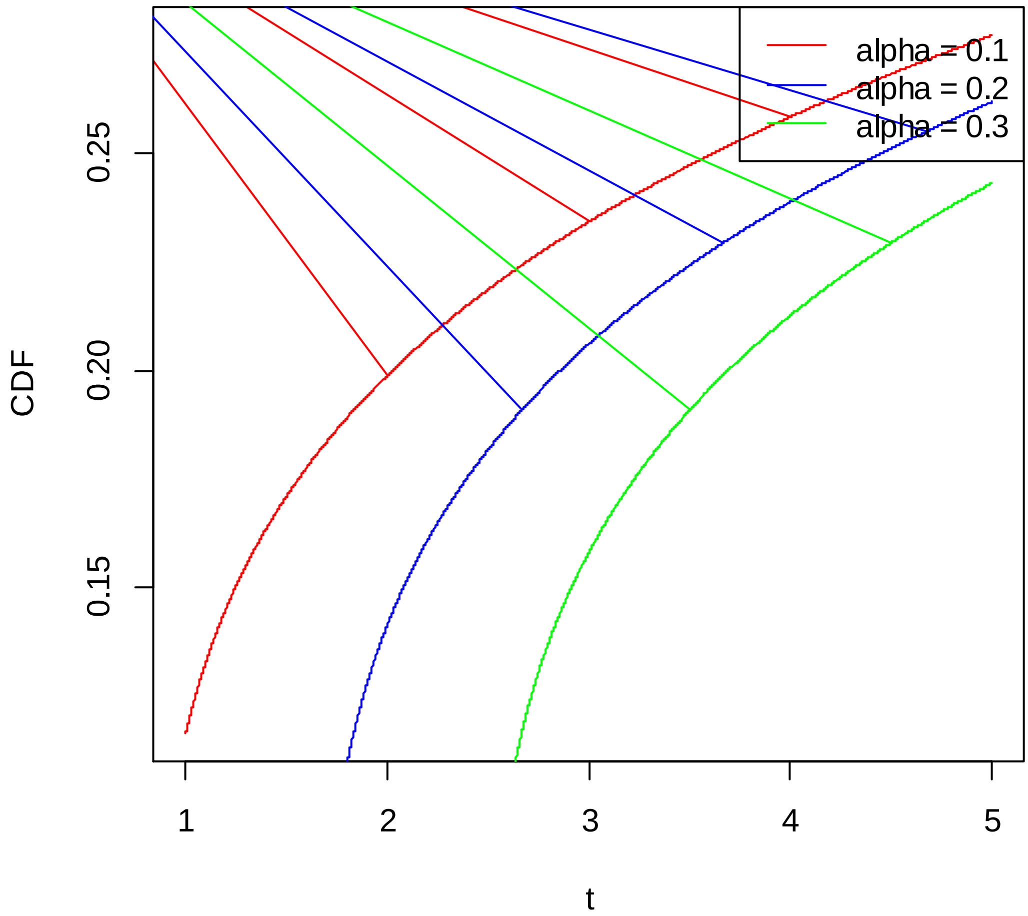

Figure 2. The Distribution of the CDF of the proposed Modified Weibull Distribution for alpha=0.1, 0.2, and 0.3

The plot of the cumulative density function (CDF) for the proposed Weibull distribution with different values of alpha, generated using the function “t=seq(1,5,0.001)” presented as Figure 2, showed that as alpha increases, showed that as alpha increases, the distribution becomes more skewed to the left, with a shorter and wider peak. This implies that the distribution becomes more skewed to the left, with a shorter and wider peak as alpha increases.

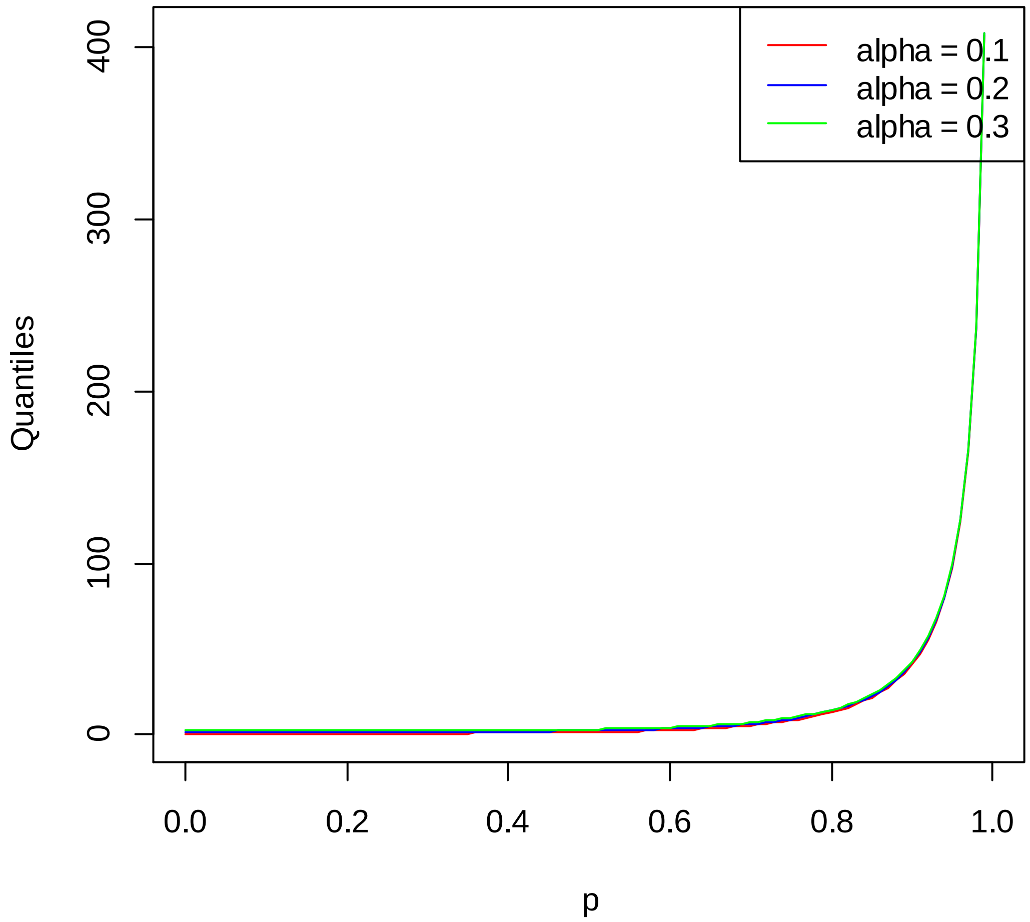

Figure 3. The Quantile Distribution of the proposed Modified Weibull Distribution for alpha=0.1, 0.2, and 0.3

The plot of the Quantile function for the Weibull distribution with different values of alpha, generated using the function "p=seq(0,1,0.01)" presented as Figure 3, showed that the variation in the distribution is relatively small as alpha increases. This implies that the proposed Weibull distribution is relatively stable with respect to changes in the value of alpha, at least in terms of the variation in its quantile function. This indicates that alpha may not be a highly influential parameter in determining the shape of the distribution in terms of the location of its quantiles.

Estimation of the parameters of the proposed distribution using the Maximum Likelihood estimation methods

The result of the estimation of the parameters of the proposed distribution using the Maximum Likelihood estimation methods.

Maximum Likelihood estimation
BFGS maximization, 165 iterations
Return code 0: successful convergence
Log-Likelihood: 611185.8
3 free parameters
Estimates:
Estimate Std. error t value Pr(> t)
α 4.711e-07 2.727e-02 0.0 1
β 1.926e+00 4.768e-04 4039.2 <2e-16 ***

γ 1.692e+00 2.217e-03 763.4 <2e-16 ***
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

The output obtained above shows the result of the maximum likelihood estimation using the BFGS maximization algorithm. The model has three free parameters: α, β, and γ. Using "t=seq(1,5,0.001)", the log-likelihood value of the model was obtained as 611185.8, which indicates how well the model fits the data. The estimates for β and γ were found to be statistically significant with p-values < 0.001, indicating that they are important predictors in the model. The estimate for α is not statistically significant with a p-value of 1, suggesting that it may not be an important predictor. The standard errors and t-values are also reported for each estimate, which can be used to test the significance of the estimates.

IV. Conclusion

This study proposed a modified three-parameter Weibull distribution and derived its properties, including the CDF, quantile function, mean, variance, and estimates of the parameters. The maximum likelihood method was used to estimate the parameters using simulated data, and the derived properties were compared to the existing Weibull distribution.

Theoretical comparison revealed that the proposed distribution has a unique expression for the pdf, CDF, mean, variance, quantile function, and parameter estimates when compared to the existing Weibull distribution.

The plots of the PDF, CDF, and quantile function for the proposed distribution showed that increasing the alpha parameter leads to a more concentrated distribution around the mode, a more skewed distribution to the left with a shorter and wider peak, and relatively small variation in the distribution's quantile function. These findings suggest that alpha may not significantly influence the shape of the distribution in terms of the location of its quantiles.

The maximum likelihood estimation using the BFGS maximization algorithm showed that the model has three free parameters, and the estimates for beta and gamma were statistically significant with p-values < 0.001, indicating that they are important predictors in the model. The estimate for alpha was not statistically significant, suggesting that it may not be an important predictor.

Hence, this study provides a new modified three-parameter Weibull distribution with unique properties and parameter estimates, which can be useful in various applications.

However, the limitations of the proposed new modified three-parameter Weibull distribution are include:
i. The distribution may not be suitable for all types of failure data, and other distributions may be more appropriate.
ii. The distribution relies on the assumption of independent and identically distributed (iid) failure times, which may not always be realistic.
iii. The estimation of the parameters may be challenging if the sample size is small or if there are censored data points.
iv. The shape parameter beta and the scale parameter gamma may be highly correlated, making it difficult to interpret the effect of each parameter on the distribution shape.

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