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Characterization Of The First Coefficient λ1 In The Partition Function As An Equivalent To The Riemann Hypothesis

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Abstract
We construct a new equivalent of the Riemann Hypothesis by means of the first coefficient λ_1 alone. Some comments are also specified for λ_n, at any n>1.
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The new equivalent

We consider the formula for the Li-Keiper coefficients λ_n given by [1, 2, 3]:

$$\lambda_n = \sum_{\varrho} \left(1 - \left(1 - \frac{1}{s}\right)^n\right)$$

(1)

where the sum is on all nontrivial zeros of the Zeta function, and $s = \rho_k + i \cdot t_k$.
For n = 1, we have a sum of positive terms like:(one zero→four zeros).

$$\frac{2 \cdot \varrho_k}{(\varrho_k^2 + t_k^2)} + \frac{2 \cdot (1 - \varrho_k)}{((1 - \varrho_k)^2 + t_k^2)}$$

(2)

We notice now that an absolute maximum in Eq.(2) is obtained (independent of the t_k's) with $\rho_k = ½$ for all k's; the above sum is given by:

$$\lambda_1 = \sum_{k=1}^{\infty} \frac{2}{\frac{2}{\left(\frac{1}{4} + t_k^2\right)}}$$

(3)

We divided by 2 assuming that the zeros are simple. We also notice that if the expression for λ_1 given by:

$$1 + \frac{\gamma}{2} - \frac{1}{2} \cdot \log(4 \cdot \pi) = 0.0230957 \ldots$$

(4)

exhausts the above sum (3), this is equivalent to the truth of the Riemann Hypothesis (RH). See [4] for general

Equivalents to the Riemann Hypothesis, and for the expression of λ_1 in the binary system [5].
Below, as an illustration we present the plot of the term at the first zero i.e. formula (2) where t_1=14.134 (the first zero) and ρ_k= x as a function of x in the range [0,1].

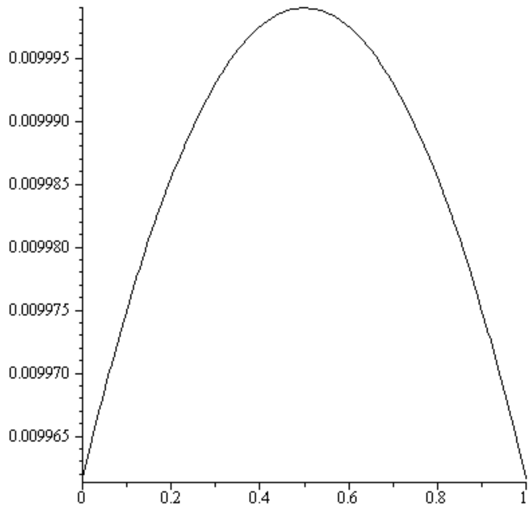

Fig.1 The first term in Eq.(2) as a function of ρ_k = x in [0,1] with the maximum at x=1/2 .

As noticed, in the sum (Eq.(3)), we divided by 2 in order to have simplicity of the zeros in the case that $\rho_k \to 1/2$ for all k.
At this point, we remark the following:

if we assume that all the zeros are simple, then the contribution of 1/2+i.t and 1/2-i.t is given by $1/(1/4 + t_k^2)$ while for a zero outside the critical line, still by the assumption that the zeros are simple, (i.e. we have now 4 distinct zeros) the amount is given by Eq.(2) above i.e.

$$\frac{2 \cdot \varrho_k}{(\varrho_k^2 + t_k^2)} + \frac{2 \cdot (1 - \varrho_k)}{((1 - \varrho_k)^2 + t_k^2)}$$

(5)

a value which is greater than $1/(1/4 + t_k^2)$ as it may be verified for all ρ_k in the range (0,1) and for all

$$t > \frac{\sqrt{2}}{2} = 0.707 \ldots$$

(at the value $t = \frac{\sqrt{2}}{2}$ the amount for ρ_k=1, i.e. $2/(1+t^2)$ is equal to $1/(1/4+t^2)$.

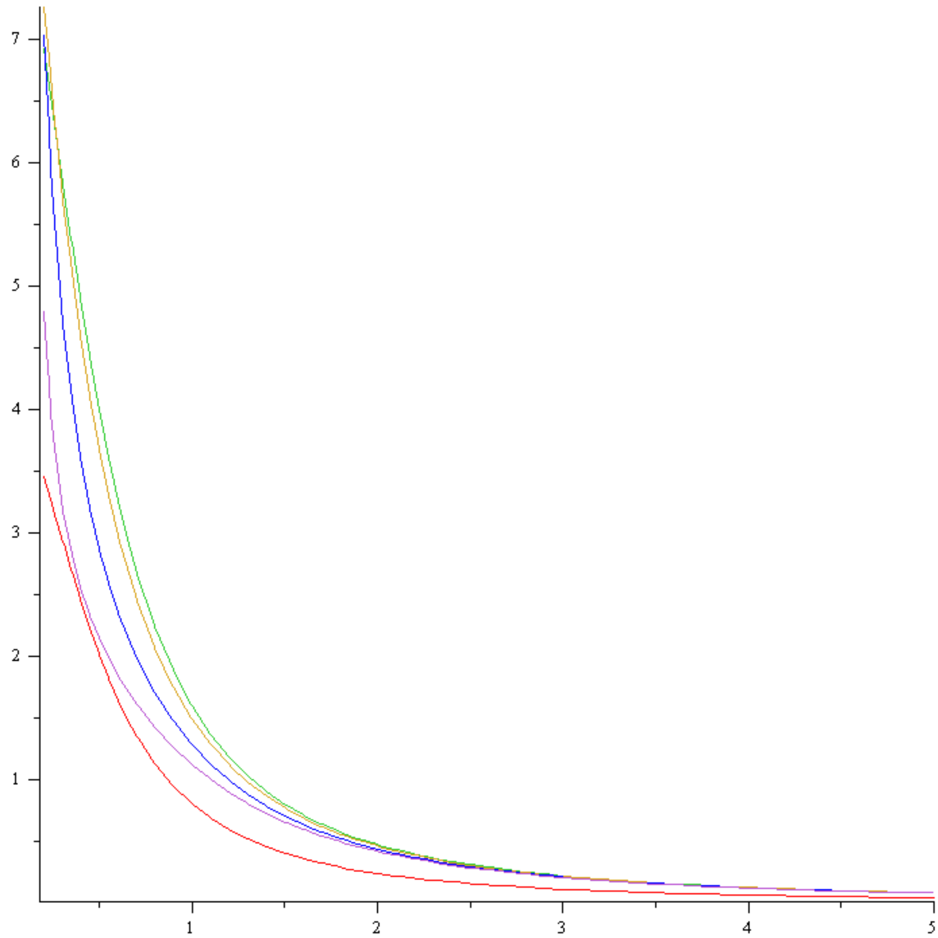

Fig. 2 Plot of Eq.(2) for various ρ_k (0.5, 0.55, 0.7, 0.85, 0.94) in the range t=0.2.. 5, (notice that for $t > \frac{\sqrt{2}}{2}$ the red curve($1/(1/4+t^2)$) is below of any plot for any ρ_k in the range (0..1)).

Moreover, since it has been rigorously proven that the height of the first non-trivial zero t_o in the critical strip verify the inequality $t > \frac{\sqrt{2}}{2}$, for rigorous results on zero free regions (for an example, see [7]), we have the following equivalent:

The absolute minimum (with respect to the (ρ_k)) for the first Li-Keiper coefficient λ_1 (for any distribution of heights ($t_k > \frac{\sqrt{2}}{2}$) is equivalent to the truth of the Riemann hypothesis.

Comments and plots for λ_n, n>1

The situation for λ_n, n > 1 is different and is now discussed with some examples.

We consider one term in λ_n, i.e. one with a zero off the critical line given, following the definition Eq.(1) by:

$$\Delta(\rho_k, n, t) = 1 - \left(1 - \frac{1}{(\rho_k + i \cdot t)}\right)^n + 1 - \left(1 - \frac{1}{(\rho_k - i \cdot t)}\right)^n +$$

$$+1 - \left(1 - \frac{1}{(1 - \rho_k + i \cdot t)}\right)^n + 1 - \left(1 - \frac{1}{(1 - \rho_k - i \cdot t)}\right)^n$$

(6)

and the function (½)· Δ(ρ_k=1/2, n, t) (for a zero on the critical line, assuming that all the zeros are simple or not, in this case we divide by the multiplicity of the four zeros) and take as a first example t=4·π. The two plots are given below at the value ρ_k=1 (which as it is seen from the example on Fig.1 is the minimum value), as a function of n. We then check that for each n,

$$\frac{1}{2}\cdot\Delta\left(\rho_k=\frac{1}{2},n,t=4\pi\right)>\Delta(\rho_k,n,t=4\pi)$$

(7)

in the intervals around each minimum of (½)·Δ(ρ_k=1/2, n, t=4π), see Fig. 3, and thus the function (½)·Δ(ρ_k=1/2, n, t=4π) is not a minimum in such (even if short) intervals of n.

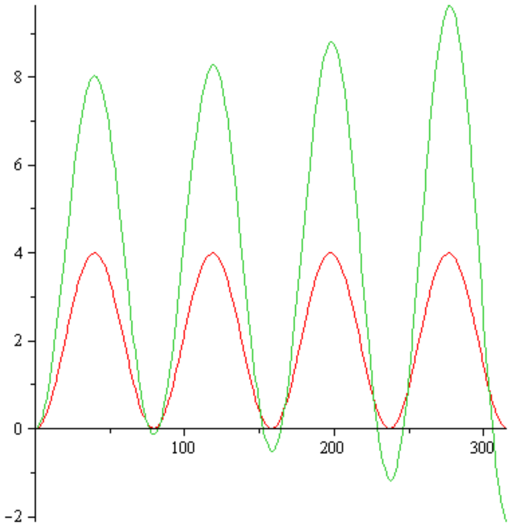

Fig. 3 The functions $\frac{1}{2}\cdot\Delta\left(\rho_k=\frac{1}{2},n,t=4\pi\right)$ (in red) and Δ(ρ_k=1, , n, t=4π)(in green).

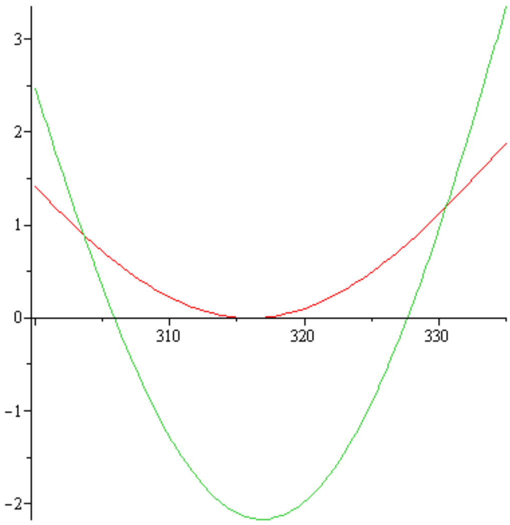

Fig. 4 The two functions around n= 315 still for t=4π.

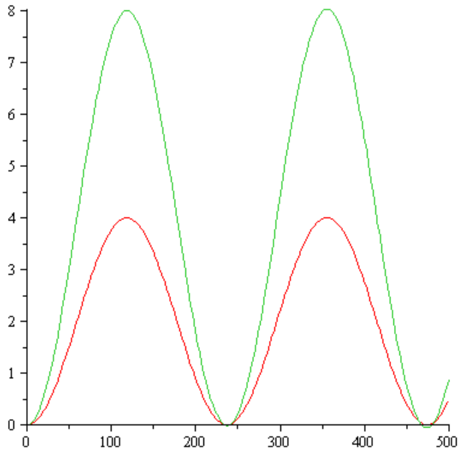

Fig. 5 The two functions (½). Δ(ρ$_k$=1/2, n, t=12π)(in red) and Δ(ρ$_k$=1, n, t=12π) (in green) now at t=12π.

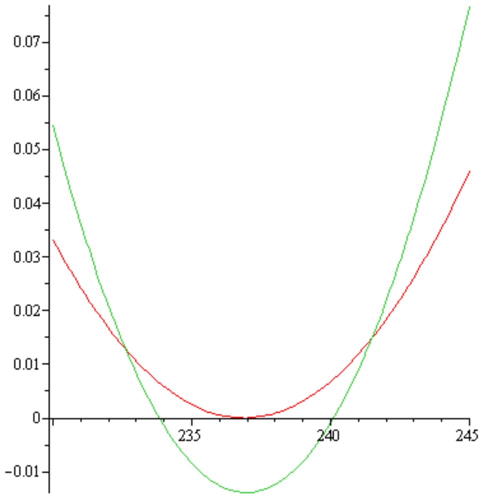

Fig. 6 The two functions in the interval n=232..241 still for t=12π.

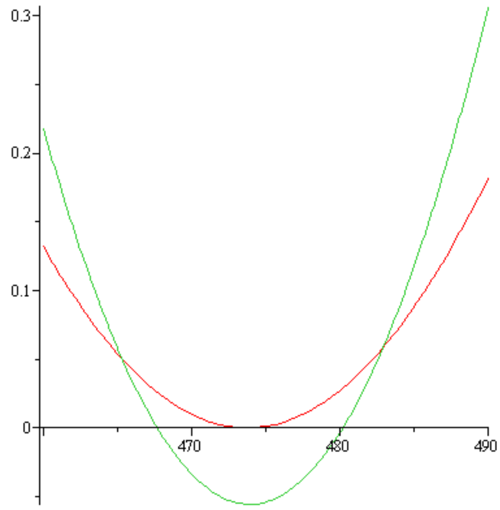

Fig. 7 The two functions in the interval n=465..483 still for t= 12π.

Notice that increasing the value of t, as an example from t=4π to t=12π, the range of the above inequality still persists and negativity increases with increasing n as illustrated above.

In conclusion, for n >1 at a fix value of t, in some intervals of n the inequality:

$$\frac{1}{2}\cdot\Delta\left(\rho_k=\frac{1}{2},n,t=4\pi\right)<\Delta(\rho_k,n,t=4\pi) \tag{8}$$

is violated and a conclusion as for n=1 cannot be established. This is in agreement with the Li criterion of positivity of λ_n, $\forall n$, for the truth of the Riemann Hypothesis [6].
We now decrease the value of the real part of the zero off the line ρ.

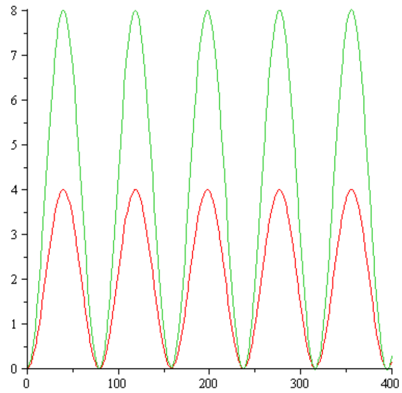

Fig. 8 The two functions for t=4π the green one at ρ = 0.55.

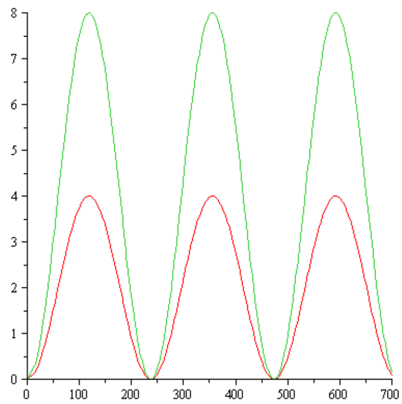

Fig. 9 The two functions for t=12π the green one at ρ=0.55.

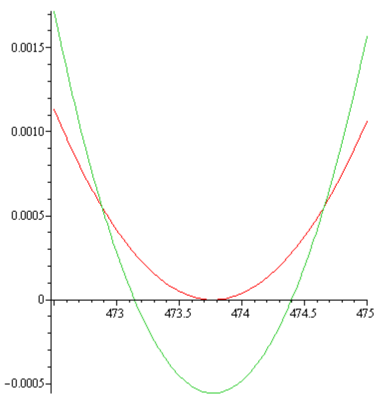

Fig.10 The two functions for t=12π the green one at ρ=0.55 in the range n=473..474.5.

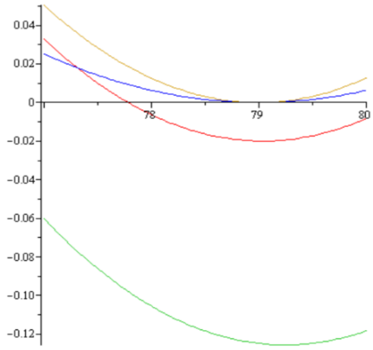

Fig. 11 Δ(ρ, n, t=4π) for ρ=1(green), ρ=0.7(red) ρ=0.501 (yellow) and (½)·Δ (1/2, n, t=4π)(blue), around n=79.

From the above plots the message is that for all n>1and any height t, the absolute minimum for λ_n is reached (for all n>1) and for any $t > \frac{\sqrt{2}}{2}$ at ρ_k=1/2. (In this case, the function (plot in red) is exactly ½ of the function (plot in green), i.e. the disappearance of the exploding factor for $\rho_k \neq$ ½, ∀k (see Appendix)).
Finally, in the Fig. 15 we give the plot of the two function at ρ=0.501, at n=90, this time as a function of t in the range (0..23).

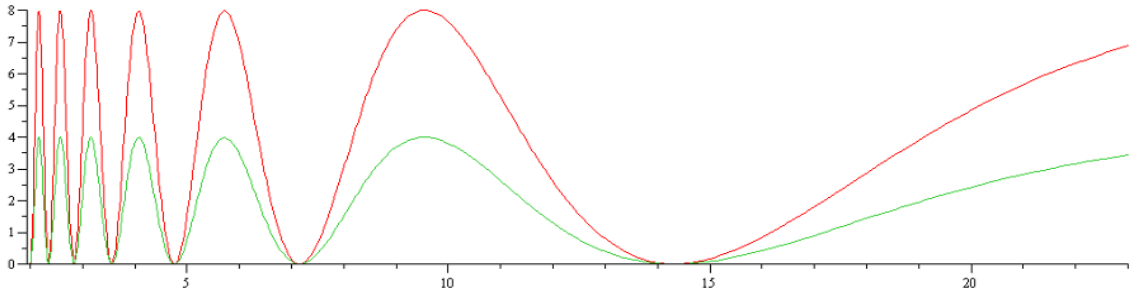

Fig. 12 The two function as for t in the range t= (1..23); (½)·Δ(ρ_k=1/2, n= 90, t) (in green), Δ(ρ_k=0.501, n= 90, t) (in red).

I. Conclusion

The Riemann Hypothesis has fascinated and involved many scholars with interesting works, using different points of view [7, 8, 9, 10, 11].

In this work, the equivalent by the extreme values of the first coefficient is simple and we add that it bears some analogy with the partition function -as an example- of the 2-d Ising model in zero field solved by Onsager where, after a transformation, the zeros in s= sinh(2·β·J) sitting on the unit circle (with some boundary conditions) becomes zeros in z sitting on the critical line z=1/2 +i·t, as discussed in [12, 13].

To the best of our knowledge, the above equivalent in terms of the first Li-Keiper coefficients is new or has not been derived along the above lines and this reinforces the conjecture on the truth of the Riemann Hypothesis by means of a minimum principle.

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Appendix

An explicit Formula for the Li-Keiper coefficients is obtained from the Definition: if a zero is off the critical line, then, there are at least four zero and

$$\Delta(\rho,n,t) = \left(1-\frac{1}{(\rho+i\cdot t)}\right)^n + \left(1-\frac{1}{(\rho-i\cdot t)}\right)^n + \left(1-\frac{1}{(1-\rho+i\cdot t)}\right)^n + \left(1-\frac{1}{(1-\rho-i\cdot t)}\right)^n =$$

$$= 4 - 2\cdot\left[\left(\frac{\rho^2+t^2}{(1-\rho)^2+t^2}\right)^{\frac{n}{2}} + \left(\frac{(1-\rho)^2+t^2}{\rho^2+t^2}\right)^{\frac{n}{2}}\right]\cdot\cos(a)$$

Where: $a = \left[n\cdot\left(arctan\left(\frac{t}{\rho}\right) + arctan\left(\frac{t}{1-\rho}\right) + \alpha\right)\right]$ with α =p·π and some p.

The exploding factor, (the factor of 2·cos(a)) reduces to the nonexploding factor equal to 2 only for ρ=1/2 and 4-4·cos(a) ≥ 0 in agreement with the Li criterion of positivity of λ_n (for every n). But for n=1 alone, the above equivalent is given by an absolute minimum of λ_1 alone.

(Returning now at the case n=1, if we apply the measure (dt/2·π) / (1/4+t^2) to the first term in z of the function -log(ξ(z)) ~ - λ_1·|z | i.e. with z=1-1/(1+i·t) and if we define (here z~0 (i.e. S~1) means

|z|=|1-1/s |= |1-1(1+i·t)| = t/|1+i·t| instead of z=0, then

$$\beta\cdot f_1 = -\lambda_1\cdot\left(\frac{1}{2\pi}\right)\cdot\int_R dt\cdot\frac{1}{\left(\frac{1}{4}\right)+t^2}\cdot\left|1-\frac{1}{1+i\cdot t}\right| =$$

$$= -\lambda_1\cdot\left(\frac{1}{2\pi}\right)\cdot\int_R dt\cdot\frac{1}{\left(\frac{1}{4}\right)+t^2}\cdot\frac{|t|}{[(1+t^2]^{1/2}} \sim -\lambda_1\cdot 0.4840513..$$

This first contribution $\beta\cdot f_1$ to β·f (as given above) is also a maximum amount since λ_1 is a minimum. The same computations for the quadratic terms and the cubic terms i.e.:

$\left(-\frac{1}{2}\right)\cdot\lambda_2\cdot z^2$ and for the cubic: $\left(-\frac{1}{3}\right)\cdot\lambda_3\cdot z^3$give:

$$\beta\cdot f_2 = -\lambda_2\cdot\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2\pi}\right)\cdot\int_R dt\cdot\frac{1}{\left(\frac{1}{4}\right)+t^2}\cdot\left|1-\frac{1}{1+i\cdot t}\right|^2 = -\lambda_2\cdot\left(\frac{1}{2}\right)\cdot(0.33333333)$$

$$\beta\cdot f_3 = -\lambda_3\cdot\left(\frac{1}{3}\right)\cdot(0.26306275)\ ...$$