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Some new characterizations on Abel rings

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Abstract. *In this paper, some characterizations of Abel rings are introduced such as a ring R is an Abel ring if and only if for any e, g $E(R)$, $eR \in Rg = gR$ Re. Also, using the related decompositions of idempotent, we show that R is an Abel ring if and only if every idempotent of R can be written uniquely the difference of an idempotent and an involution. And, in term of the solutions of certain equation, we prove that R is an Abel ring if and only for any $e, g \in E(R)$ and $c \in R$, when $exg = c$ has a solution, there is $c = gce$. Finally, with the inner inverse of regular, we show that R is an Abel ring if and only if for $e \in E(R)$, $e(1) = \{c - ec + e | c \in R\}$.*

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I. Introduction

Let R be an associative ring with identity. The symbols $E(R)$, $N(R)$ and $C(R)$ stand respectively for the set of all idempotent elements, the set of all nilpotent elements and the center of R. If $E(R)$ $C(R)$, then R is called an Abel ring. Lee proved that reduced rings and semicommutative rings are both Abel rings[1]. Liu et al. showed that α *rigid* rings are reduced rings, so α *rigid* rings are also Abel rings[2]. Wei and Li showed that a ring R is an Abel ring if and only if R is a quasi-normal left idempotent reflexive ring[3]. Literature [4–6] have showed some other rings associated with Abel rings. So Abel rings are very important in ring theory. In recent years, there already have been many characterizations of Abel rings. Han et al. showed that a ring R is an Abel ring if and only if every idempotent of R is left semicentral[7]. Zhou et al. proved that a ring R is an Abel ring if and only if $ae = 0$ implies $ea = 0$ for each $e \in E(R)$, $a \in N(R)$[8]. Zhou et al. proved that a ring R is an Abel ring if and only if $1 - xy \in GPE(R)$ implies $1 - yx \in GPE(R)$ for each $x, y \in R$[9]. In this paper, some new characterizations of Abel rings are given.

II. Properties of Abel rings

Let R be a ring and $V_n(R) = \left\{ \begin{pmatrix} a_0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_0 & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} \middle| a_0, a_{ij} \in R, 1 \le i < j \le n \right\}$. Then, with the usual addition and multiplication of matrices, $V_n(R)$ forms a ring.

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Theorem 2.1. *R is an Abel ring if and only if* $E(V_n(R)) = \left\{ \begin{pmatrix} e & 0 & 0 & \cdots & 0 \\ 0 & e & 0 & \cdots & 0 \\ 0 & 0 & e & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e \end{pmatrix} \;\middle|\; e \in E(R) \right\}$.

Proof. $\Longrightarrow$ Assume that R is an Abel ring. We use induction on n.

If $n = 2$, then for any $E = \begin{pmatrix} e & e_{12} \\ 0 & e \end{pmatrix} \in E(V_2(R))$, one has $\begin{pmatrix} e & e_{12} \\ 0 & e \end{pmatrix} = E = E^2 = \begin{pmatrix} e^2 & ee_{12} + e_{12}e \\ 0 & e^2 \end{pmatrix}$, this gives $e^2 = e$ and $e_{12} = ee_{12} + e_{12}e$. Since R is an Abel ring and $e \in E(R), e \in C(R)$, it follows $e_{12} = 2ee_{12}$ and so $ee_{12} = 2ee_{12}$, this leads to $ee_{12} = 0$. Hence $e_{12} = 0$ and $E = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$ with $e \in E(R)$.

Now we assume that $n > 2$ and $E(V_{n-1}(R)) = \left\{ \begin{pmatrix} e & 0 & 0 & \cdots & 0 \\ 0 & e & 0 & \cdots & 0 \\ 0 & 0 & e & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e \end{pmatrix} \;\middle|\; e \in E(R) \right\}$. Set $E = \begin{pmatrix} e & e_{12} & e_{13} & \cdots & e_{1(n-1)} & e_{1n} \\ 0 & e & e_{23} & \cdots & e_{2(n-1)} & e_{2n} \\ 0 & 0 & e & \cdots & e_{3(n-1)} & e_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & e_{(n-1)n} \\ 0 & 0 & 0 & \cdots & 0 & e \end{pmatrix} \in E(V_n(R))$.

We can choose $\alpha = (e_{12} \;\; e_{13} \;\; \cdots \;\; e_{1(n-1)} \;\; e_{1n})$.

$E_1 = \begin{pmatrix} e & e_{23} & \cdots & e_{2(n-1)} & e_{2n} \\ 0 & e & \cdots & e_{3(n-1)} & e_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & e_{(n-1)n} \\ 0 & 0 & \cdots & 0 & e \end{pmatrix} \in V_{n-1}(R)$.

Then $E = \begin{pmatrix} e & \alpha \\ 0 & E_1 \end{pmatrix}$. Noting that $E^2 = E$. Then $\begin{pmatrix} e & \alpha \\ 0 & E_1 \end{pmatrix} = \begin{pmatrix} e^2 & e\alpha + \alpha E_1 \\ 0 & E_1^2 \end{pmatrix}$, this gives

$$e^2 = e, \quad \alpha = e\alpha + \alpha E_1, \quad E_1^2 = E_1.$$

Hence $e \in E(R) \subseteq C(R)$ and $E_1 \in E(V_{n-1}(R))$.

By induction hypothesis, $E_1 = \begin{pmatrix} e & 0 & 0 & \cdots & 0 \\ 0 & e & 0 & \cdots & 0 \\ 0 & 0 & e & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e \end{pmatrix}$, $\alpha = e\alpha + \alpha E_1 = (ee_{12} \;\; ee_{13} \;\; \cdots \;\; ee_{1n}) + (e_{12}e \;\; e_{13}e \;\; \cdots \;\; e_{1n}e)$, $e_{1j} = ee_{1j} + e_{1j}e$, $j = 2, 3, \ldots, n$. Since $e \in E(R) \subseteq C(R)$, $e_{1j} = 0$, $j = 2, 3, \ldots, n$, this implies $\alpha = 0$.

Hence $E = \begin{pmatrix} e & 0 \\ 0 & E_1 \end{pmatrix} = \begin{pmatrix} e & 0 & 0 & \cdots & 0 \\ 0 & e & 0 & \cdots & 0 \\ 0 & 0 & e & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e \end{pmatrix}$. We are done.

$\Leftarrow=$ Assume that $E(V_n(R)) = \left\{ \begin{pmatrix} e & 0 & 0 & \cdots & 0 \\ 0 & e & 0 & \cdots & 0 \\ 0 & 0 & e & \cdots & 0 \\ \cdots & \cdots & \cdot & & \cdot\cdot \\ 0 & 0 & 0 & \cdots & e \end{pmatrix} : e \in E(R) \right\}$. For any $e \in E(R)$ and $a \in R$, let $g = e+(1-e)ae$. Then $eg = e; ge = g$ and $g^2 = g$. Choose $E = \begin{pmatrix} e & 0 & 0 & \cdots & g-e \\ 0 & e & 0 & \cdots & 0 \\ 0 & 0 & e & \cdots & 0 \\ \cdots & \cdots & \cdot & & \cdot\cdot \\ 0 & 0 & 0 & \cdots & e \end{pmatrix}$. Then $E \in E(V_n(R))$, this implies $g - e = 0$ by hypothesis, it follows that $(1 - e)ae = 0$ for any $a \in R$. Therefore R is Abel. □

Corollary 2.2. *R is an Abel ring if and only if $V_n(R)$ is an Abel ring.*

Proof. It is an immediate corollary of Theorem 2.1. □

Theorem 2.3. *R is an Abel ring if and only if $eR \cap gR = eRg$ for any $e, g \in E(R)$.*

Proof. $=\Rightarrow$ Assume that R is an Abei ring and $e, g \in E(R)$. Then $e, g \in C(R)$, one obtains $eRg = geR \subseteq eR \cap gR$. Now for any $x \in eR \cap gR$, we have $x = ex = gx = xg = exg \in eRg$, which implies $eR \cap gR \subseteq eRg$. Hence $eR \cap gR = eRg$.

$\Leftarrow=$ For any $e \in E(R)$, we have $eR \cap (1-e)R = eR(1-e)$, this gives $eR(1-e) = 0$. Hence R is an Abel ring. □

It is well known that R is an Abel ring if and only if $eR = Re$, for each $e \in E(R)$. Hence we have the following proposition.

Proposition 2.4. *R is an Abel ring if and only if for any $e, g \in E(R), eR \cap Rg = gR \cap Re$.*

Proof. The necessity is clear.

THe sufficiency: Let $e \in E(R)$ and $a \in R$. Then $g = e+(1-e)ae \in E(R)$. By hypothesis, one has $eR \cap Rg = gR \cap Re$. Noting that $g = ge$ and $e = eg$. Then $g \in gR \cap Re = eR \cap Rg$, it follows that $g = eg = e$. Hence $(1 - e)ae = 0$ for any $a \in R$, which implies R is an Abel ring. □

Theorem 2.3 and Proposition 2.4 give the following corollary.

Corollary 2.5. *R is an Abel ring if and only if for any $e, g \in E(R), eR \cap gR = gRe$.*

For any $e, g \in E(R)$, we have $eg - ege \in N(R)$. As for Abel rings, we can say more.

Proposition 2.6. *R is an Abel ring if and only if for any $e, g \in E(R)$, $eg - ege \in E(R)$.*

Proof. $=\Rightarrow$ It is obvious because $eg - ege = 0$ for any $e, g \in E(R)$.

$\Leftarrow=$ Suppose that $e \in E(R)$ and $a \in R$. Set $g = e + ea(1 - e)$. Then $eg = g; ge = e$ and $g \in E(R)$. By hypothesis, we have $eg - ege \in E(R)$, that is, $ea(1 - e) = g - e \in E(R)$. However $ea(1 - e) \in N(R)$. Hence $ea(1 - e) = 0$ forany $a \in R$. Thus R is Abel□

Proposition 2.7. *R is an Abel ring if and only if $e + xe - exe \in C(R)$ for each $e \in E(R)$ and $x \in N(R)$.*

Proof. $=\Rightarrow$ It is obvious because $e + xe - exe = e \in C(R)$ for any $x \in R$ and $e \in E(R)$.

$\Leftarrow=$ Let $e \in E(R)$ and $a \in R$. Then, by hypothesis, we have $e + ((1-e)ae)e - e(1-e)ae)e \in C(R)$, that is, $+ (1 - e)ae \in C(R)$. It follows that $e + (1-e)ae = e(e + (1-e)ae) = e$. Hence $(1-e)ae = 0$ for each $a \in R$, this shows that R is Abel. □

Let R be a ring and write $CE(R) = \{x \in R | xe = ex$ for each $e \in E(R)\}$. Clearly, $CE(R)$ is a subring of R and $C(R) \subseteq CE(R)$. Evidently, R is an Abel ring if and only if $CE(R) = R$. Observing the proof of Proposition 2.7, we have the following corollary.

Corollary 2.8. *R is an Abel ring if and only if $e + xe - exe \in CE(R)$ for each $e \in E(R)$ and $x \in N(R)$.*

Observing $e + xe - exe = (e + xe - ex)e$, this implies us to give the following proposition.

Proposition 2.9. *R is an Abel ring if and only if $e + xe - ex \in CE(R)$ for each $e \in E(R)$ and $x \in N(R)$.*

Proof. $=\Rightarrow$ It is evident by Corollary 2.8.

$\Leftarrow=$ Assume that $e+xe-ex \in CE(R)$ for each $e \in E(R)$ and $x \in N(R)$, Then $e+xe-exe = (e+xe-ex)e = e(e+xe-ex)$, it follows that $xe - exe = exe - ex$. Multiplying the equality by e on the left. One has $exe - ex = exe - exe = 0$Hence $ex = exe$, this gives $e + xe - exe = e + xe - ex \in CE(R)$. By Corollary 2.8, we have R is Abel□

Let $g \in E(R)$ and choose $x = (1 - e)ge$. Then $x \in N(R)$ and $ex = 0$, it follows that $e + xe - ex = e + (1 - e)ge = (e + g - eg)e$. Hence Proposition 2.9 leads to the following corollary.

Corollary 2.10. *R is an Abel ring if and only if $e + g - eg \in CE(R)$ for any $e, g \in E(R)$.*

Corollary 2.10 implies us to give the following proposition.

Proposition 2.11. *R is an Abel ring if and only if $e + g - eg \in E(R)$ for any $e, g \in E(R)$.*

Proof. $\Longrightarrow$ It is routine.
$\Longleftarrow$ Assume that $e \in E(R)$ and $a \in R$. Set $g = 1 - e + ea(1 - e)$. Then $eg = ea(1 - e)$, $ge = 0$ and $g^2 = g$. By hypothesis, $g + e - ge \in E(R)$, which implies $eg = 0$. Hence $ea(1 - e) = 0$ for each $a \in R$, this shows R is Abel. □

Let R be a ring and $e, g \in E(R)$. Define $e * g = e + g - eg$. If R is an Abel ring, then $e * g \in E(R)$ by Proposition 2.11. Hence we have the following corollary.

Corollary 2.12. *R is an Abel ring if and only if $(E(R), *)$ is a semigroup.*

3. decompositions of idempotents

Theorem 3.1. *R is an Abel ring if and only if every idempotent of R can be written uniquely the difference of an idempotent and an involution.*

Proof. $\Longrightarrow$ Assume that R is an Abel ring and $e \in E(R)$. Then $e = (1 - e) - (1 - 2e)$ is the difference of an idempotent and an involution. Now let $e = g - u$, where g is idempotent and u is involution. Then $g - u = e = e^2 = (g - u)^2 = g^2 - gu - ug + u^2 = g - gu - ug + 1$. Since R is Abel, $gu = ug$, it follows that $u = 2gu - 1$, this gives $(2g - 1)u = 1$, $u = (2g - 1)^{-1} = 2g - 1$, so $e = g - u = g - (2g - 1) = 1 - g$. Hence $g = 1 - e$ and $u = 2g - 1 = 2(1 - e) - 1 = 1 - 2e$.
$\Longleftarrow$ Suppose that $e \in E(R)$. For $a \in R$. Set $g = e - ea(1 - e)$. Then $eg = g, ge = e, g^2 = g$. Since $(ea(1 - e) + 1 - 2e)^2 = 1$, $ea(1 - e) + 1 - 2e$ is involution. Since $(1 - g) - (1 - 2g) = g = (1 - e) - (ea(1 - e) + 1 - 2e)$, by hypothesis, we have $1 - g = 1 - e$ and $1 - 2g = ea(1 - e) + 1 - 2e$. Hence $ea(1 - e) = 0$ for any $a \in R$, it follows that R is Abel. □

Theorem 3.2. *R is an Abel ring if and only if every idempotent of R can be written the product of uniquely idempotent and an involution.*

Proof. $\Longrightarrow$ Assume that R is an Abel ring and $e \in E(R)$. Then $e = e(2e - 1)$, where $2e - 1$ is an involution. Now let $e = gu$, where $g^2 = g$, $u^2 = 1$. Then $ge = g^2u = gu = e$. Since $eu = gu^2 = g1 = g$, $eg = e^2u = eu = g$. Noting that R is an Abel ring. Then $e \in C(R)$ and $g = eg = ge = e$.
$\Longleftarrow$ Suppose that $e \in E(R)$. For $a \in R$. Set $g = e - ea(1 - e)$. Then $eg = g, ge = e, g^2 = g$. Since $(2e - 1 - ea(1 - e))^2 = 1$, so $2e - 1 - ea(1 - e)$ is involution. Since $g = e(2e - 1 - ea(1 - e))$ and $g = g(2g - 1)$, where $2g - 1$ is involution. By hypothesis, we have $g = e$. Hence $ea(1 - e) = 0$ for any $a \in R$, it follows that R is Abel. □

Let R be a ring and $u \in R$. If there exists an integer $n \geq 1$ such that $u^n = 1$, then u is called a generalized involution of R.

Theorem 3.3. *R is an Abel ring if and only if every idempotent of R can be written uniquely the sum of a generalized involution and an idempotent .*

Proof. $\Longrightarrow$ Assume that R is an Abel ring and $e \in E(R)$. Then $e = (1 - e) + (2e - 1)$, where $1 - e$ is idempotent with $(2e - 1)^2 = 1$. Now let $e = g + u$, where $g^2 = g$, $u^n = 1$. Then $g + u = e = e^2 = (g + u)^2 = g + 2gu + u^2$, that is $(1 - 2g)u = u^2$, thus $(1 - 2g)u^{n-1} = u^n = 1$. Hence $u^{n-1} = 1 - 2g$ and $1 = u^n = (1 - 2g)u$. Thus $u = 1 - 2g$, $e = g + u = g + 1 - 2g = 1 - g$, $g = 1 - e$, $u = 1 - 2(1 - e) = 2e - 1$.
$\Longleftarrow$ Suppose that $e \in E(R)$. For any $a \in R$. Set $g = e - ea(1 - e)$. Then $(1 - g) + (2g - 1) = g = (1 - e) + (2e - 1 - ea(1 - e))$ is two decompositions of g. Thus

$$1 - g = 1 - e$$

$$2g - 1 = 2e - 1 - ea(1 - e)$$

. Hence $ea(1 - e) = 0$ for any $a \in R$, it follows that R is Abel. □

Let R be a ring and $e, g \in E(R)$. Assume that $e + g = 1$ and $eg = ge = 0$, then e, g are called a pair of orthogonal idempotents of R.

Theorem 3.4. *R is an Abel ring if and only if for any e, g of a pair of orthogonal idempotents and $x \in R$, When $x^2 = 0$ and $(e + x)(g - x) = 0$, there is $x = 0$.*

Proof. $\Longrightarrow$ Since $0 = (e + x)(g - x) = eg - ex + xg - x^2 = -ex + xg$, that is $ex = xg$. By hypothesis, R is Abel, so $ex = e(ex) = (ex)e = xge = 0$. Hence $xg = 0$ and $x = x1 = x(e + g) = xe + xg = ex + xg = 0$.
$\Longleftarrow$ Suppose that $e \in E(R)$, $a \in R$ and $(e + ea(1 - e))(1 - e - ea(1 - e)) = 0$. e and $1 - e$ are a pair of orthogonal idempotents of R with $(ea(1 - e))^2 = 0$. By hypothesis, $ea(1 - e) = 0$, it follows that R is Abel. □

4. Solutions of equation

Theorem 4.1. *R is an Abel ring if and only for any $e, g \in E(R)$ and $c \in R$, when $exg = c$ has a solution, there is $c = gce$.*

Proof. $\Longrightarrow$ Assume that $exg = c$ has a solution $x = d$, then $c = edg$, hence $ecg = c$. By hypothesis, R is Abel, so $e, g \in C(R)$. Hence $c = ecg = gce$.
$\Longleftarrow$ Suppose that $e \in E(R), a \in R$. Set $c = ea(1 - e)$, then the equation $ex(1 - e) = c$ has a solution $x = a$. By hypothesis, $c = (1 - e)ce = (1 - e)ea(1 - e)e = 0$. Hence $ea(1 - e) = 0$ for any $a \in R$, it follows that R is Abel. □

Corollary 4.2. *R is an Abel ring if and only for any $e, g, f \in E(R)$, when $exg = f$ has a solution, there is $f = gfe$.*

Proof. $\Longrightarrow$ It follows from Theorem 4.1.
$\Longleftarrow$ Suppose that $e \in E(R), a \in R$. Set $g = e + ea(1 - e)$, then $eg = g$, $ge = e$ and $g^2 = g$. Since the equation $exg = g$ has a solution $x = e$, by hypothesis, $g = gge = ge = e$. Hence $ea(1 - e) = 0$ for any $a \in R$, it follows that R is Abel. □

Theorem 4.3. *R is an Abel ring if and only if $xy - yx \in ZE(R)$ for any $x, y \in R$.*

Proof. $\Longrightarrow$ Assume that R is an Abel ring, then $ZE(R) = R$, hence $xy - yx \in ZE(R)$ for any $x, y \in R$.
$\Longleftarrow$ Suppose that $e \in E(R)$ and $a \in R$, then $ea(1 - e) = e(a(1 - e)) - (a(1 - e))e \in ZE(R)$. Hence $ea(1 - e) = e(ea(1 - e)) - ea(1 - e)e = 0$ for any $a \in R$, it follows that R is Abel. □

Theorem 4.4. *R is an Abel ring if and only if for $e \in E(R)$ and $x \in R$, there exists an integer $n \geq 1$ such that $ex^n - x^n e = 0$.*

Proof. $\Longrightarrow$ It is obvious.
$\Longleftarrow$ Suppose that $e \in E(R)$. For any $a \in R$, set $g = e - ea(1 - e)$. Then $eg = g, ge = e, g^2 = g$. By hypothesis, there exists an integer $n \geq 1$ such that $eg^n - g^n e = 0$. Hence $g = eg = eg^n = g^n e = ge = e$. Thus $ea(1 - e) = 0$ for any $a \in R$, it follows that R is Abel. □

Let R be a ring and $a \in R$. If there exists $b \in R$ such that $a = aba$, then a is called a regular element of R. Set $a(1) = \{x \in R | axa = a\}$. Suppose that $e \in E(R)$, then e is a regular element of R and $e(1) = \{c - ece + e | c \in R\}$.

Theorem 4.5. *R is an Abel ring if and only if for $e \in E(R)$, $e(1) = \{c - ec + e | c \in R\}$.*

Proof. $\Longrightarrow$ It is obvious.
$\Longleftarrow$ Suppose that $e \in E(R)$. For any $a \in R$, set $g = e + ea(1 - e)$. Then $eg = g, ge = e, g^2 = g$. Since $ege = ge = e$, then $g \in e(1)$. By hypothesis, $e(1) = \{c - ec + e | c \in R\}$, that is $g = c - ec + e$, where $c \in R$. Hence $g = eg = e(c - ec + e) = e$. Thus $ea(1 - e) = 0$ for any $a \in R$, it follows that R is Abel. □

Theorem 4.6. *R is an Abel ring if and only if for any $e, g \in E(R)$, when $\begin{pmatrix} e \\ g \end{pmatrix}$ is a regular element, then $ge = eg$.*

Proof. $\Longrightarrow$ Assume that $(x, y) \in \begin{pmatrix} e \\ g \end{pmatrix}(1)$. Then

$$\begin{pmatrix} e \\ g \end{pmatrix} = \begin{pmatrix} e \\ g \end{pmatrix} (x \quad y) \begin{pmatrix} e \\ g \end{pmatrix} = \begin{pmatrix} exe + eyg \\ gxe + gyg \end{pmatrix}$$

Thus

$$e = exe + eyg$$

$$g = gxe + gyg$$

$$eg = exeg + eyg$$

By hypothesis, R is an Abel ring, then $e, g \in C(R)$. Therefore

$$eg = gxe + eyg$$

$$ge = gxe + eyg$$

Thus

$$eg = ge$$

$\Leftarrow$= Suppose that $e \in E(R)$. For any $a \in R$, set $g = e + (1 - e)ae$. Then

$$eg = e$$

$$ge = g$$

$$g^2 = g$$

$$\begin{pmatrix} e \\ g \end{pmatrix} (e \quad 0) \begin{pmatrix} e \\ g \end{pmatrix} = \begin{pmatrix} e \\ ge \end{pmatrix} = \begin{pmatrix} e \\ g \end{pmatrix}$$

By hypothesis, $ge = eg$, thus $g = e$, this gives $(1 - e)ae = 0$ for any $a \in R$. Hence R is Abel. □

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