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A Study On Pell And Modified Pell Numbers

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Abstract: *This paper presents some facts about Pell and Modified Pell numbers. Pell numbers can be changed to Modified Pell numbers and Vice Versa using matrices. The identities satisfied by Pell and Modified Pell numbers with proofs are also discussed in this paper.*
Keywords: *Recurrence relation, Pell numbers, Modified Pell numbers, Binet formula.*

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I. Introduction

The Pell numbers are named after English mathematician John Pell (1611-1685) . The details about Pell numbers are found in [1, 2, 3]. In [3] the author has shown that Pell and Modified Pell numbers can be represented by matrices. The identities satisfied by Pell and Modified Pell numbers are also mentioned in [3].Both the Pell numbers and Modified Pell numbers can be calculated by recurrence relations.

The sequence of Pell numbers $\{P_n\}$ is defined by recurrence relation

$$P_n = 2P_{n-1} + P_{n-2} \text{ for } n \geq 2 \text{ with } P_0 = 0 \text{ and } P_1 = 1 \quad (1)$$

Where P_n denotes nth Pell number. The sequence of Pell numbers starts with 0 and 1 and then each number is the sum of twice its previous number and the number before its previous number.

The sequence of Modified Pell numbers $\{q_n\}$ is defined by recurrence relation

$$q_n = 2q_{n-1} + q_{n-2} \text{ for } n \geq 2 \text{ with } q_0 = q_1 = 1 \quad (2)$$

Where q_n denotes nth Modified Pell number. In the sequence of Modified Pell numbers each of the first two numbers is 1 and then each number is the sum of twice its previous number and the number before its previous number.

The first few Pell and Modified Pell numbers calculated from (1) and (2) are given in the following Table no.1.

Table no.1: First few Pell and Modified Pell numbers

n	0	1	2	3	4	5	6	7	8	9	10
P_n	0	1	2	5	12	29	70	169	408	985	2378
q_n	1	1	3	7	17	41	99	239	577	1393	3363

The rest of the paper is organized as follows. Some facts about Pell and Modified Pell numbers are mentioned in Section-II. Matrix representations of Pell and Modified Pell numbers are given in Section-III. The identities satisfied by Pell and Modified Pell numbers are stated in Section-IV. Finally conclusion is given in Section-V.

II. Some facts about Pell and Modified Pell numbers

1. The Pell numbers $\{P_n\}$ are either even or odd but Modified Pell numbers $\{q_n\}$ are all odd.
2. *Binet formula*:
   The Binet formulas satisfied by Pell and Modified Pell numbers are given by

$$P_n = \frac{a^n - b^n}{a-b} \quad (3)$$

$$q_n = \frac{a^n + b^n}{a+b} \quad (4)$$

Where a and b are the roots of quadratic equation $x^2 - 2x - 1 = 0$. Solving this equation, we get

$$a = 1 + \sqrt{2} \quad (5)$$

and

$$b = 1 - \sqrt{2} \quad (6)$$

Then

$$a + b = 2 \quad (7)$$

$$a - b = 2\sqrt{2} \quad (8)$$
$$ab = -1 \quad (9)$$

3. *Powers of a and b :*
The powers of $'a'$ in Binet formula (3) or (4) can be expressed in terms of Pell numbers$\{P_n\}$ and Modified Pell numbers$\{q_n\}$.
$a = 1 + \sqrt{2} = q_1 + P_1\sqrt{2}$ [By (5)]
$a^2 = \left(1 + \sqrt{2}\right)^2 = 3 + 2\sqrt{2} = q_2 + P_2\sqrt{2}$
$a^3 = \left(1 + \sqrt{2}\right)^3 = 7 + 5\sqrt{2} = q_3 + P_3$
$a^4 = \left(1 + \sqrt{2}\right)^4 = \left(3 + 2\sqrt{2}\right)\left(3 + 2\sqrt{2}\right) = 17 + 12\sqrt{2} = q_4 + P_4\sqrt{2}$
Noting the forms of above four expressions, one can write the n^{th}power of $'a'$ as given below.
$$a^n = q_n + P_n\sqrt{2} \quad (10)$$
Similarly, the powers of $'b'$ in Binet formula (3) or (4) can be expressed in terms of Pell and Modified Pell numbers.
$b = 1 - \sqrt{2} = q_1 - P_1\sqrt{2}$ [By (6)]
$b^2 = \left(1 - \sqrt{2}\right)^2 = 3 - 2\sqrt{2} = q_2 - P_2\sqrt{2}$
$b^3 = \left(1 - \sqrt{2}\right)^3 = 7 - 5\sqrt{2} = q_3 - P_3\sqrt{2}$
$b^4 = \left(1 - \sqrt{2}\right)^4 = \left(3 - 2\sqrt{2}\right)\left(3 - 2\sqrt{2}\right) = 17 - 12\sqrt{2} = q_4 - P_4\sqrt{2}$
Looking at the forms of above four expressions, one can write the n^{th}power of 'b' as given below
$$b^n = q_n - P_n\sqrt{2} \quad (11)$$
[Refer Table no-1 for values of Pell and Modified Pell numbers]

4. Let us now write Modified Pell numbers in terms of Pell numbers.
$q_2 = 3 = 5 - 2 = P_3 - P_2$
$q_3 = 7 = 12 - 5 = P_4 - P_3$
$q_4 = 17 = 29 - 12 = P_5 - P_4$
Noting the forms of above expressions, one can write the Modified Pell number $q_n$in terms of Pell numbers $P_n$as given below.
$$q_n = P_{n+1} - P_n \quad \text{for } n = 0,1,2, \ldots .etc. \quad (12)$$
Replacing n by $(n + 1)$ in the above expression, we get
$$\begin{aligned} q_{n+1} &= P_{n+2} - P_{n+1} \\ &= (2P_{n+1} + P_n) - P_{n+1} \quad \text{[By (1)]} \\ \Rightarrow q_{n+1} &= P_{n+1} + P_n \quad (13) \end{aligned}$$
Adding (12) & (13), we obtain
$$q_n + q_{n+1} = 2P_{n+1} \quad (14)$$
Modified Pell numbers are expressed in terms of Pell numbers by (12), (13) and (14).

III. Matrix representations of Pell and Modified Pell numbers

1. Replacing n by $(n + 2)$ in (12), we have
$$\begin{aligned} q_{n+2} &= P_{n+3} - P_{n+2} \\ &= (2P_{n+2} + P_{n+1}) - P_{n+2} \quad \text{[By (1)]} \\ &= P_{n+2} + P_{n+1} \\ &= (2P_{n+1} + P_n) + P_{n+1} \quad \text{[By (1)]} \\ \Rightarrow q_{n+2} &= 3P_{n+1} + P_n \quad (15) \end{aligned}$$
Writing (13) and (15) in matrix form we obtain
$$\begin{pmatrix} q_{n+2} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} P_{n+1} \\ P_n \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} q_{n+2} \\ q_{n+1} \end{pmatrix} = M \begin{pmatrix} P_{n+1} \\ P_n \end{pmatrix} \quad (16)$$
Where
$$M = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \quad (17)$$
The Pell and Modified Pell numbers are related by the matrix equation (16), which transforms Pell numbers into Modified Pell numbers.

2. Replacing n by $(n + 1)$ in (14), we have
$$q_{n+1} + q_{n+2} = 2P_{n+2}$$
$$\Rightarrow P_{n+2} = \frac{1}{2}(q_{n+1} + q_{n+2})$$

$$= \frac{1}{2}\{q_{n+1} + (2q_{n+1} + q_n)\} \quad \text{[By (2)]}$$

$$\Rightarrow P_{n+2} = \frac{1}{2}(3q_{n+1} + q_n) \qquad (18)$$

Expressing (14) and (18) in matrix form, we get

$$\begin{pmatrix} P_{n+2} \\ P_{n+1} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} q_{n+1} \\ q_n \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} P_{n+2} \\ P_{n+1} \end{pmatrix} = \frac{1}{2}M\begin{pmatrix} q_{n+1} \\ q_n \end{pmatrix} \qquad \text{[By (17)]} \qquad (19)$$

Thus Modified Pell numbers can be changed to Pell numbers by matrix M using the above relation (19).

3. Expressing (12) and (13) in matrix form we get

$$\begin{pmatrix} q_{n+1} \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} P_{n+1} \\ P_n \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} q_{n+1} \\ q_n \end{pmatrix} = G\begin{pmatrix} P_{n+1} \\ P_n \end{pmatrix} \qquad (20)$$

Where

$$G = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad (21)$$

Pell numbers can be changed to Modified Pell numbers by matrix$G$ using the above relation (20).

4. Subtracting (12) from (13), we get

$$q_{n+1} - q_n = [P_{n+1} + P_n] - [P_{n+1} - P_n]$$
$$= 2P_n$$

$$\Rightarrow P_n = \frac{1}{2}(q_{n+1} - q_n) \qquad (22)$$

From (14) we get

$$P_{n+1} = \frac{1}{2}(q_{n+1} + q_n) \qquad (23)$$

Writing the above expressions (22) and (23) in matrix form we obtain

$$\begin{pmatrix} P_{n+1} \\ P_n \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} q_{n+1} \\ q_n \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} P_{n+1} \\ P_n \end{pmatrix} = \frac{1}{2}G\begin{pmatrix} q_{n+1} \\ q_n \end{pmatrix} \quad \text{[By (21)]} \qquad (24)$$

Modified Pell numbers can be changed to Pell numbers by matrix Gusing the above relation (24).

5. *Even Powers of matrix M*:

The even powers of matrix Mgiven by (17) can be written in terms of Pell numbers as given below.

$$M^2 = M \times M = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ 4 & 2 \end{pmatrix} = 2\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \qquad (25)$$

Writing M^2 in terms of Pell numbers we obtain

$$M^2 = 2\begin{pmatrix} P_3 & P_2 \\ P_2 & P_1 \end{pmatrix} \qquad (26)$$

Let us now find out the value of M^4.

$$M^4 = M^2 \times M^2 = 4\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{[By (25)]}$$

$$= 2^2\begin{pmatrix} 29 & 12 \\ 12 & 5 \end{pmatrix} = 2^2\begin{pmatrix} P_5 & P_4 \\ P_4 & P_3 \end{pmatrix} \qquad (27)$$

Noting the above expressions (26) and (27), one can write in general

$$M^n = 2^{n/2}\begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix} \text{ if } n \text{ even} \qquad (28)$$

6. *Odd Powers of matrix M*:

The odd powers of matrix M can be written in terms of Modified Pell numbers as given below.

$$M = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} q_2 & q_1 \\ q_1 & q_0 \end{pmatrix} \quad \text{[By (17)]} \qquad (29)$$

$$M^3 = M^2 \times M = 2\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{[By (17) \& (25)]}$$

$$= 2\begin{pmatrix} 17 & 7 \\ 7 & 3 \end{pmatrix} \qquad (29a)$$

Writing M^3 in terms of Modified Pell numbers we get

$$M^3 = 2\begin{pmatrix} q_4 & q_3 \\ q_3 & q_2 \end{pmatrix} \qquad (30)$$

The above expressions (29) and (30) in general can be written as

$$M^n = 2^{(n-1)/2}\begin{pmatrix} q_{n+1} & q_n \\ q_n & q_{n-1} \end{pmatrix} \text{ if } n \text{ odd} \qquad (31)$$

7. *Negative Powers of matrix M*:

The determinant of matrix M given by (17) is

$$\det M = \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 2 \tag{32}$$

Since $\det M \neq 0$, the matrix M is an invertible matrix.

Let us now find out a matrix B whose elements are the cofactors of matrix M.

$$B = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix} = B^T \qquad \text{[Refer (17)]} \tag{33}$$

As $B = B^T$, the matrix B is a symmetric matrix.

The inverse of matrix M is calculated using the following relation.

$$M^{-1} = \frac{1}{\det M} B^T = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix} \quad \text{[Using (32) \& (33)]} \tag{34}$$

Writing M^{-1} in terms of modified Pell numbers we get

$$M^{-1} = \frac{1}{2}\begin{pmatrix} q_0 & -q_1 \\ -q_1 & q_2 \end{pmatrix} \tag{35}$$

Let us now find out the matrices M^{-2}, M^{-3} and M^{-4}.

$$M^{-2} = M^{-1} \times M^{-1} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix} \quad \text{[By (34)]}$$

$$= \frac{1}{4}\begin{pmatrix} 2 & -4 \\ -4 & 10 \end{pmatrix}$$

$$\Rightarrow M^{-2} = \frac{1}{2}\begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} \tag{36}$$

In terms of Pell numbers, the above matrix can be expressed as

$$M^{-2} = \frac{1}{2}\begin{pmatrix} P_1 & -P_2 \\ -P_2 & P_3 \end{pmatrix} \tag{37}$$

$$M^{-3} = M^{-1} \times M^{-2} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} \quad \text{[By (34) \& (36)]}$$

$$= \frac{1}{4}\begin{pmatrix} 3 & -7 \\ -7 & 17 \end{pmatrix}$$

$$\Rightarrow M^{-3} = \frac{1}{2^2}\begin{pmatrix} q_2 & -q_3 \\ -q_3 & q_4 \end{pmatrix} \tag{38}$$

$$M^{-4} = M^{-2} \times M^{-2} = \frac{1}{4}\begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} \quad \text{[By (36)]}$$

$$= \frac{1}{4}\begin{pmatrix} 5 & -12 \\ -12 & 29 \end{pmatrix}$$

$$\Rightarrow M^{-4} = \frac{1}{2^2}\begin{pmatrix} P_3 & -P_4 \\ -P_4 & P_5 \end{pmatrix} \tag{39}$$

Considering the forms of matrices (37) and (39), one can write in general

$$M^{-n} = \frac{1}{2^{n/2}}\begin{pmatrix} P_{n-1} & -P_n \\ -P_n & P_{n+1} \end{pmatrix} \; if\ n\ even \tag{40}$$

Similarly considering the forms of matrices (35) and (38), one can write in general

$$M^{-n} = \frac{1}{2^{(n+1)/2}}\begin{pmatrix} q_{n-1} & -q_n \\ -q_n & q_{n+1} \end{pmatrix} \; if\ n\ odd \tag{41}$$

8. Let

$$S = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \tag{42}$$

The matrices $S, S^2 \& S^3$ are expressed in terms of Pell numbers as given below.

$$S = \begin{pmatrix} P_2 & P_1 \\ P_1 & P_0 \end{pmatrix} \tag{43}$$

$$S^2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} P_3 & P_2 \\ P_2 & P_1 \end{pmatrix} \quad \text{[By (42)]} \tag{44}$$

$$S^3 = S^2 \times S = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 12 & 5 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} P_4 & P_3 \\ P_3 & P_2 \end{pmatrix} \tag{45}$$

[By (42) & (44)]

The above three matrices (43),(44) and (45) in general can be written as

$$S^n = \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix} for\ n = 1,2,3, \ldots \tag{46}$$

9. The matrices $GS, GS^2 \& GS^3$ are expressed in terms of modified Pell numbers as given below.

$$GS = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{[Refer (21) \& (42)]}$$

The above matrix in terms of modified Pell numbers is written as

$$GS = \begin{pmatrix} q_2 & q_1 \\ q_1 & q_0 \end{pmatrix} \tag{47}$$

Similarly we can calculate $GS^2 \& GS^3$.

$$GS^2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 3 & 1 \end{pmatrix} \quad \text{[Refer (21) \& (44)]}$$

$$\Rightarrow GS^2 = \begin{pmatrix} q_3 & q_2 \\ q_2 & q_1 \end{pmatrix} \quad (48)$$

$$GS^3 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 12 & 5 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 17 & 7 \\ 7 & 3 \end{pmatrix} \text{ [Refer (21) \& (45)]}$$

$$\Rightarrow GS^3 = \begin{pmatrix} q_4 & q_3 \\ q_3 & q_2 \end{pmatrix} \quad (49)$$

The above three matrices (47),(48) and (49) in general can be written as

$$GS^n = \begin{pmatrix} q_{n+1} & q_n \\ q_n & q_{n-1} \end{pmatrix} \quad for\ n = 1,2,3, \ldots \quad (50)$$

10. The matrices G and S satisfy the following relation.

$$G^2S^n = 2S^n \quad (51)$$

Where

$$G = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \text{ [By (21) \& (42)]}$$

$Proof$:

$$G^2S^n = G \times GS^n = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} q_{n+1} & q_n \\ q_n & q_{n-1} \end{pmatrix} \text{ [By (50)]}$$

$$= \begin{pmatrix} q_{n+1} + q_n & q_n + q_{n-1} \\ q_{n+1} - q_n & q_n - q_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} 2P_{n+1} & 2P_n \\ q_{n+1} - q_n & q_n - q_{n-1} \end{pmatrix} \text{ [By (14)]} \quad (52)$$

Now the elements in the second row of above matrix are modified as given below.

$$q_{n+1} - q_n = (2q_n + q_{n-1}) - q_n \text{ [By (2)]}$$

$$= q_n + q_{n-1} = 2P_n \text{ [By (14)]} \quad (53)$$

and

$$q_n - q_{n-1} = (2q_{n-1} + q_{n-2}) - q_{n-1} \text{ [By (2)]}$$

$$= q_{n-1} + q_{n-2} = 2P_{n-1} \text{ [By (14)]} \quad (54)$$

Substituting (53) and (54) in the elements of second row of matrix (52), we obtain

$$G^2S^n = \begin{pmatrix} 2P_{n+1} & 2P_n \\ 2P_n & 2P_{n-1} \end{pmatrix} = 2\begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix}$$

$$\Rightarrow G^2S^n = 2S^n \text{ [By (46)]}$$

Hence (51) is proved.

11. Let

$$W = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \quad (55)$$

and

$$R_n = \begin{pmatrix} P_{n+1} & q_n \\ P_n & q_{n-1} \end{pmatrix} \quad (56)$$

These matrices satisfy the relation

$$R_n.W = S^n \quad (57)$$

The matrix S^n in the above relation (57) is given by (46).

Proof:

$$R_n.W = \begin{pmatrix} P_{n+1} & q_n \\ P_n & q_{n-1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} P_{n+1} & P_{n+1} - q_n \\ P_n & P_n - q_{n-1} \end{pmatrix} \quad (58)$$

Using (12) we get

$$P_n = P_{n+1} - q_n \quad (59)$$

Replacing n by $(n-1)$ in (12) we obtain

$$q_{n-1} = P_n - P_{n-1}$$

$$\Rightarrow P_{n-1} = P_n - q_{n-1} \quad (60)$$

Using (59) and (60) in the elements of second column of matrix (58) we have

$$R_n.W = \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix}$$

$$\Rightarrow R_n.W = S^n \text{ [By (46)]}$$

Hence (57) is proved.

12. The matrices S^n, W & R_n receptively given by (46), (55) and (56) satisfy the relation

$$S^n.W = R_n \quad (61)$$

Proof:

$$S^n.W = \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \text{ [By (46) \& (55)]}$$

$$= \begin{pmatrix} P_{n+1} & P_{n+1} - P_n \\ P_n & P_n - P_{n-1} \end{pmatrix}$$

Using (12) and (60) in the elements of second column of above matrix we get

$$S^n.W = \begin{pmatrix} P_{n+1} & q_n \\ P_n & q_{n-1} \end{pmatrix}$$

$$\Rightarrow S^n.W = R_n \quad \text{[By (56)]}$$

Hence (61) is proved.

13. *Eigen values of M^n*:

The Eigen values of n^{th} power of matrix M given by (17) can be expressed in terms of Pell and modified Pell numbers.

(a) $n\ even$:

For $n = 4$, we have from (27)

$$M^4 = 2^2 \begin{pmatrix} 29 & 12 \\ 12 & 5 \end{pmatrix} = \begin{pmatrix} 116 & 48 \\ 48 & 20 \end{pmatrix}$$

The characteristic equation of M^4 is

$$\begin{vmatrix} 116 - \omega & 48 \\ 48 & 20 - \omega \end{vmatrix} = 0$$

Where ω is the Eigen value of M^4. Solving the above determinant we get

$$(116 - \omega)(20 - \omega) - 48 \times 48 = 0$$

$$\Rightarrow \omega^2 - 136\omega + 16 = 0$$

$$\Rightarrow \omega = \frac{136 \pm \sqrt{18496 - 64}}{2}$$

$$\Rightarrow \omega = 4\left(17 \pm 12\sqrt{2}\right)$$

Writing the above expression in terms of Pell and modified Pell numbers we have

$$\omega = 2^{4/2}\left(q_4 \pm P_4\sqrt{2}\right)$$

The above expression gives two Eigen values of M^4. Hence in general the Eigen values ω_1 and ω_2 of M^n for n even are given by

$$\left.\begin{aligned} \omega_1 &= 2^{n/2}\left(q_n + P_n\sqrt{2}\right) \\ \omega_2 &= 2^{n/2}\left(q_n - P_n\sqrt{2}\right) \end{aligned}\right\} \text{ if } n \text{ even} \qquad (62)$$

(b) $n\ odd$:

For $n = 3$, we have from (29a)

$$M^3 = \begin{pmatrix} 34 & 14 \\ 14 & 6 \end{pmatrix}$$

The characteristic equation of M^3 is

$$\begin{vmatrix} 34 - \omega & 14 \\ 14 & 6 - \omega \end{vmatrix} = 0$$

Where ω denotes the Eigen value of M^3. Solving the above determinant we have

$$(34 - \omega)(6 - \omega) - 14 \times 14 = 0$$

$$\Rightarrow \omega^2 - 40\omega + 8 = 0$$

$$\Rightarrow \omega = \frac{40 \pm \sqrt{1600 - 32}}{2}$$

$$\Rightarrow \omega = 2\left(2 \times 5 \pm 7\sqrt{2}\right)$$

Writing the above expression in terms of Pell and modified Pell numbers we have

$$\omega = 2^{(3-1)/2}\left(P_3 \pm q_3\sqrt{2}\right)$$

The above expression gives two Eigen values of M^3. Hence in general the Eigen values ω_1 and ω_2 of M^n for n odd are given by

$$\left.\begin{aligned} \omega_1 &= 2^{(n-1)/2}\left(P_n + q_n\sqrt{2}\right) \\ \omega_2 &= 2^{(n-1)/2}\left(P_n - q_n\sqrt{2}\right) \end{aligned}\right\} \text{ if } n \text{ odd} \qquad (63)$$

IV. Identities satisfied by Pell and Modified Pell numbers

The Pell and Modified Pell numbers satisfy the following identities.

1. *Simpson formula*:

The Pell numbers satisfy *Simpson's formula* given by

$$P_{n+1}P_{n-1} - P_n^2 = (-1)^n \qquad (64)$$

Proof: Using Binet formula (3), we get

$$P_{n+1}P_{n-1} - P_n^2 = \frac{(a^{n+1} - b^{n+1})(a^{n-1} - b^{n-1})}{(a-b)^2} - \frac{(a^n - b^n)^2}{(a-b)^2}$$

$$= \frac{(a^{2n} - a^{n+1}b^{n-1} - b^{n+1}a^{n-1} + b^{2n}) - (a^{2n} + b^{2n} - 2a^n b^n)}{(a-b)^2}$$
$$= \frac{-a^{n-1}b^{n-1}(a^2+b^2-2ab)}{(a-b)^2}$$
$$= -(ab)^{n-1} = -(-1)^{n-1} \quad \text{[By (9)]}$$
$$\Rightarrow P_{n+1}P_{n-1} - P_n^2 = (-1)^n$$

Thus the Simpson formula (64) is proved.

2. $P_{n+1}P_{n-1} + P_n^2 = \frac{1}{2}\{q_n + (-1)^n\}$ (65)

Proof: Using Binet formula (3), we obtain

$$P_{n+1}P_{n-1} + P_n^2 = \frac{(a^{n+1} - b^{n+1})(a^{n-1} - b^{n-1})}{(a-b)^2} + \frac{(a^n - b^n)^2}{(a-b)^2}$$
$$= \frac{(a^{2n} - a^{n+1}b^{n-1} - b^{n+1}a^{n-1} + b^{2n}) + (a^{2n} + b^{2n} - 2a^n b^n)}{(a-b)^2}$$
$$= \frac{2(a^{2n} + b^{2n}) - a^{n-1}b^{n-1}(a^2+b^2+2ab)}{(a-b)^2}$$
$$= \frac{2(a^{2n} + b^{2n}) - (ab)^{n-1}(a+b)^2}{(a-b)^2}$$
$$= \frac{2q_{2n}(a+b) - (ab)^{n-1}(a+b)^2}{(a-b)^2} \quad \text{[By (4)]}$$
$$= \frac{2q_{2n} \times 2 - (-1)^{n-1} \times 2^2}{(2\sqrt{2})^2} \quad \text{[By (7),(8) \&(9)]}$$
$$= \frac{1}{2}\{q_{2n} + (-1)^n\}$$

Thus the identity (65) is proved.

3. $q_{n+1}q_{n-1} - q_n^2 = 2(-1)^{n+1}$ (66)

Proof: Using Binet formula (4), we get

$$q_{n+1}q_{n-1} - q_n^2 = \frac{(a^{n+1} + b^{n+1})(a^{n-1} + b^{n-1})}{(a+b)^2} - \frac{(a^n + b^n)^2}{(a+b)^2}$$
$$= \frac{(a^{2n} + a^{n+1}b^{n-1} + b^{n+1}a^{n-1} + b^{2n}) - (a^{2n} + b^{2n} + 2a^n b^n)}{(a+b)^2}$$
$$= \frac{a^{n-1}b^{n-1}(a^2+b^2-2ab)}{(a+b)^2}$$
$$= (ab)^{n-1}\frac{(a-b)^2}{(a+b)^2}$$
$$= (-1)^{n-1}\frac{(2\sqrt{2})^2}{2^2} \quad \text{[By (7),(8) \&(9)]}$$
$$= 2(-1)^{n-1} = 2(-1)^{n-1}(-1)^2$$
$$\Rightarrow q_{n+1}q_{n-1} - q_n^2 = 2(-1)^{n+1}$$

Hence the identity (66) is proved.

4. $q_{n+1}q_{n-1} + q_n^2 = q_{2n} + (-1)^{n+1}$ (67)

Proof: Using Binet formula (4), we obtain

$$q_{n+1}q_{n-1} + q_n^2 = \frac{(a^{n+1} + b^{n+1})(a^{n-1} + b^{n-1})}{(a+b)^2} + \frac{(a^n + b^n)^2}{(a+b)^2}$$
$$= \frac{(a^{2n} + a^{n+1}b^{n-1} + b^{n+1}a^{n-1} + b^{2n}) + (a^{2n} + b^{2n} + 2a^n b^n)}{(a+b)^2}$$
$$= \frac{2(a^{2n} + b^{2n}) + a^{n-1}b^{n-1}(a^2+b^2+2ab)}{(a+b)^2}$$
$$= \frac{2q_{2n}(a+b) + (ab)^{n-1}(a+b)^2}{(a+b)^2} \quad \text{[By (4)]}$$
$$= \frac{2q_{2n}}{(a+b)} + (ab)^{n-1}$$
$$= \frac{2q_{2n}}{2} + (-1)^{n-1} \quad \text{[By (7) \& (9)]}$$
$$= q_{2n} + (-1)^{n-1}(-1)^2$$
$$\Rightarrow q_{n+1}q_{n-1} + q_n^2 = q_{2n} + (-1)^{n+1}$$

Thus the identity (67) is proved.

5. $P_n q_{m-1} - q_{n-1}P_m = (-1)^{m+1}P_{n-m}$ (68)

Proof: Using (3) and (4) we have

$$P_n q_{m-1} - q_{n-1}P_m = \left(\frac{a^n - b^n}{a-b}\right)\left(\frac{a^{m-1} + b^{m-1}}{a+b}\right) - \left(\frac{a^{n-1} + b^{n-1}}{a+b}\right)\left(\frac{a^m - b^m}{a-b}\right)$$

$$= \frac{(a^{n+m-1} + a^n b^{m-1} - b^n a^{m-1} - b^{n+m-1}) - (a^{n+m-1} - a^{n-1}b^m + b^{n-1}a^m - b^{n+m-1})}{(a-b)(a+b)}$$
$$= \frac{a^n b^{m-1} - b^n a^{m-1} + a^{n-1}b^m - b^{n-1}a^m}{(a-b)(a+b)}$$

$$= \frac{a^{n-1}b^{m-1}(a+b)-b^{n-1}a^{m-1}(b+a)}{(a-b)(a+b)}$$
$$= \frac{a^{n-1}b^{m-1}-b^{n-1}a^{m-1}}{(a-b)}$$
$$= \frac{a^{m-1}b^{m-1}}{(a-b)}\left(\frac{a^{n-1}}{a^{m-1}}-\frac{b^{n-1}}{b^{m-1}}\right)$$
$$= (ab)^{m-1}\left(\frac{a^{n-m}-b^{n-m}}{a-b}\right)$$
$$= (-1)^{m-1}P_{n-m} \quad \text{[By (3) \& (9)]}$$
$$= (-1)^{m-1}(-1)^2P_{n-m}$$
$$= (-1)^{m+1}P_{n-m}$$

Hence the identity (68) is proved.

6. $P_nq_{m+1} - q_mP_{n+1} = (-1)^{m+1}q_{n-m}$ (69)

Proof: Using (3) and (4) we get

$$P_nq_{m+1} - q_mP_{n+1} = \left(\frac{a^n-b^n}{a-b}\right)\left(\frac{a^{m+1}+b^{m+1}}{a+b}\right) - \left(\frac{a^m+b^m}{a+b}\right)\left(\frac{a^{n+1}-b^{n+1}}{a-b}\right)$$

$$= \frac{(a^{n+m+1}+a^nb^{m+1}-b^na^{m+1}-b^{n+m+1})-(a^{m+n+1}-a^mb^{n+1}+b^ma^{n+1}-b^{m+n+1})}{(a-b)(a+b)}$$
$$= \frac{a^nb^{m+1}-b^na^{m+1}+a^mb^{n+1}-b^ma^{n+1}}{(a-b)(a+b)}$$
$$= \frac{-a^nb^m(a-b)-b^na^m(a-b)}{(a-b)(a+b)}$$
$$= \frac{-a^nb^m-b^na^m}{(a+b)}$$
$$= \frac{-a^mb^m}{(a+b)}\left(\frac{a^n}{a^m}+\frac{b^n}{b^m}\right)$$
$$= -(ab)^m\left(\frac{a^{n-m}+b^{n-m}}{a+b}\right)$$
$$= -(-1)^mq_{n-m} \quad \text{[By (4) \& (9)]}$$
$$= (-1)^{m+1}q_{n-m}$$

Thus the identity (69) is proved.

7. $q_nq_{m+1} - q_{n+1}q_m = 2(-1)^{m+1}P_{n-m}$ (70)

Proof: Using (4) we have

$$q_nq_{m+1} - q_{n+1}q_m = \left(\frac{a^n+b^n}{a+b}\right)\left(\frac{a^{m+1}+b^{m+1}}{a+b}\right) - \left(\frac{a^{n+1}+b^{n+1}}{a+b}\right)\left(\frac{a^m+b^m}{a+b}\right)$$

$$= \frac{(a^{n+m+1}+a^nb^{m+1}+b^na^{m+1}+b^{n+m+1})-(a^{n+m+1}+a^{n+1}b^m+b^{n+1}a^m+b^{n+m+1})}{(a+b)^2}$$
$$= \frac{a^nb^{m+1}+b^na^{m+1}-a^{n+1}b^m-b^{n+1}a^m}{(a+b)^2}$$
$$= \frac{-a^nb^m(a-b)+b^na^m(a-b)}{(a+b)^2}$$
$$= \frac{(a-b)(b^na^m-a^nb^m)}{(a+b)^2}$$
$$= \frac{-(a-b)a^mb^m}{(a+b)^2}\left(\frac{a^n}{a^m}-\frac{b^n}{b^m}\right)$$
$$= -\left(\frac{a-b}{a+b}\right)^2(ab)^m\left(\frac{a^{n-m}-b^{n-m}}{a-b}\right)$$
$$= -\left(\frac{2\sqrt{2}}{2}\right)^2(-1)^mP_{n-m} \quad \text{[By (3), (7), (8) \& (9)]}$$
$$= 2(-1)^{m+1}P_{n-m}$$

Hence the identity (70) is proved.

8. $2P_nq_n = P_{2n}$ (71)

Proof: Using (3) and (4) we get

$$2P_nq_n = 2\left(\frac{a^n-b^n}{a-b}\right)\left(\frac{a^n+b^n}{a+b}\right)$$
$$= \frac{2}{(a+b)}\left(\frac{a^{2n}-b^{2n}}{a-b}\right)$$
$$= P_{2n} \quad \text{[By (3) and (7)]}$$

Thus the identity (71) is proved.

V. Conclusion

Pell and Modified Pell numbers can be represented by matrices. The identities satisfied by Pell and Modified Pell numbers can be derived using Binet formula. This study on Pell and Modified Pell numbers will inspire curious mathematicians to explore it further.

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