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# The Second Largest Number Of Maximal Independent Sets In Quasi-Unicyclic Graphs

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***ABSTRACT.*** *Let $G = (V, E)$ be a simple undirected graph. An independent set is a subset $S$ of $V(G)$ such that no two vertices in $S$ are adjacent. A maximal independent set is an independent set that is not a proper subset of any other independent set. A graph is said to be unicyclic if it contains exactly one cycle. A graph $G$ with vertex set $V(G)$ is called a quasi-unicyclic graph, if there exists a vertex $x \in V(G)$ such that $G - x$ is a unicyclic graph. In this paper, we determine the second largest number of maximal independent sets among all quasi-unicyclic graphs. We also characterize those extremal graphs achieving these values.*

***Keywords**: independent set; maximal independent sets; unicyclic graphs; quasi-unicyclic graphs.*

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## I. Introduction

Let$G = (V, E)$ be a simple undirected graph. An *independent set* is a subset $S$ of $V(G)$ such that no two vertices in $S$ are adjacent. A *maximal independent set* is an independent set that is not a proper subset of any other independent set. The cardinality of the set of all maximal independent sets of a graph $G$ is denoted by $mi(G)$.

The problem of determining the largest number of $mi(G)$ in a general graph of order $n$ and those graphs achieving the largest number was proposed by Erdös and Moser, and solved by Moon and Moser [11]. It was then studied for various families of graphs, including trees, forests, (connected) graphs with at most one cycle, (connected) triangle-free graphs, ($k$-)connected graphs, bipartite graphs; for a survey see [5]. Jin and Li [2] investigated the second largest number of $mi(G)$ among all graphs of order $n$; Jou and Lin [6] further explored the same problem for trees and forests.

A graph is said to be *unicyclic* if it contains exactly one cycle. Jou and Chang [4] settled the largest number of $mi(G)$ for the family of (connected) unicyclic graphs. Lin and Jou [9] investigated the second and the third largest numbers of $mi(G)$ among all (connected) unicyclic graphs of order $n$. A graph $G$ with vertex set $V(G)$ is called a *quasi-unicyclic graph*, if there exists a vertex $x \in V(G)$ such that $G - x$ is a unicyclic graph. The concept of quasi-unicyclic graphs was first introduced in [1]. The problem of determining the largest numbers of $mi(G)$ among all connected quasi-unicyclic graphs and quasi-unicyclic graphs of order $n$ was solved by Lin and Jou [10]. The purpose of this paper is to determine the second largest number of maximal independent sets among all quasi-unicyclic graphs. Additionally, extremal graphs achieving these values are also given.

## II. Preliminary

In this section, we describe some notations and preliminary results. Let $G = (V, E)$ be a graph. The *neighborhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of vertices adjacent to $v$ in $G$ and the *closed neighborhood* $N_G[v]$ is $v \cup N_G(v)$. The *degree* of $v$ is the cardinality of $N_G(v)$, denoted by $\deg_G(v)$. For a set $A \subseteq V(G)$, the *deletion* of $A$ from $G$ is the graph $G - A$ obtained from $G$ by removing all vertices in $A$ and their incident edges. Two graphs $G_1$ and $G_2$ are *disjoint* if $V(G_1) \cap V(G_2) = \emptyset$. The *union* of two disjoint graphs $G_1$ and $G_2$ is the graph $G_1 \cup G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. $nG$ is the short notation for the union of $n$ copies of disjoint graphs isomorphic to $G$. Denote by $C_n$ a *cycle* with $n$ vertices, $P_n$ a *path* with $n$ vertices and $K_n$ a *complete graph* with $n$ vertices. Throughout this paper, for simplicity, let $r = \sqrt{2}$.

**Lemma 2.1.** ([3]) *For any vertex $x$ in a graph $G$, $mi(G) \leq mi(G-x) + mi(G - N_G[x])$.*

**Lemma 2.2.** ([8]) *For positive integers $m, p, q, s$ and $t$, if $f(x) = p\,r^x + qr^{m-x}$ for $s \leq x \leq t$, then $f(x)$ has a maximum value at $x = s$ or $t$.*

**Theorem 2.3.** ([4]) *If $G$ is a connected unicyclic graph of order $n \geq 3$, then $mi(G) \leq u_1(n)$, where*

$$u_1(n) = \begin{cases} r^{n-1}+1, & \text{if } n \geq 3 \text{ is odd,} \\ 3r^{n-4}, & \text{if } n \geq 4 \text{ is even.} \end{cases}$$

*Furthermore, $mi(G) = u_1(n)$ if and only if $G \in U_1(n)$, where $U_1(n)$ is shown in Figure 1.*

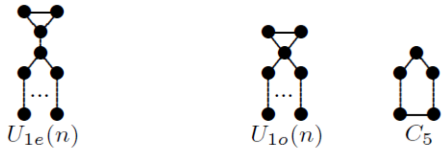



Figure 1: The graph $U_1(n)$

**Theorem 2.4.** ([4]) *If $G$ is a graph with at most one cycle of order $n \geq 2$, then $mi(G) \leq h_1'(n)$, where*

$$h_1'(n) = \begin{cases} 3r^{n-3}, & \text{if } n \geq 3 \text{ is odd,} \\ r^n, & \text{if } n \geq 2 \text{ is even.} \end{cases}$$

*Furthermore, $mi(G) = h_1'(n)$ if and only if $G \in H_1'(n)$, where $H_1'(n)$ is shown in Figure 2.*


Figure 2: The graph $H_1'(n)$

**Theorem 2.5.** ([7]) *If $G$ is a graph with at most one cycle of order $n \geq 4$ having $G \notin H_1'(n)$, then $mi(G) \leq h_2'(n)$, where*

$$h_2'(n) = \begin{cases} 5r^{n-5}, & \text{if } n \geq 5 \text{ is odd,} \\ 3r^{n-4}, & \text{if } n \geq 4 \text{ is even.} \end{cases}$$

*Furthermore, $mi(G) = h_2'(n)$ if and only if $G \in H_2'(n)$, where $H_2'(n)$ is shown in Figure 3.*

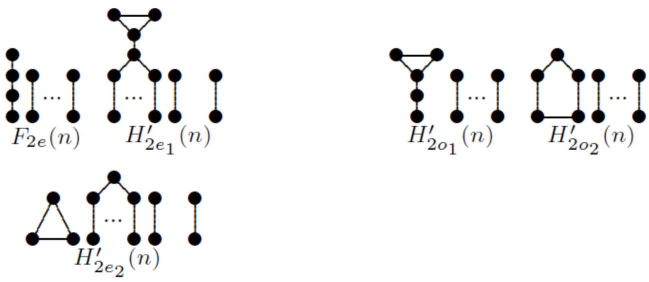



Figure 3: The graph $H_2'(n)$

**Theorem 2.6.** ([10]) *If $G$ is a quasi-unicyclic graph of order $n \geq 5$, then $mi(G) \leq qu_1'(n)$, where*

$$qu_1'(n) = \begin{cases} 3r^{n-3}, & if\ n \geq 5\ is\ odd, \\ 9r^{n-6}, & if\ n \geq 6\ is\ even. \end{cases}$$

*Furthermore, $mi(G) = qu_1'(n)$ if and only if $G \in QU_1'(n)$, where $QU_1'(n)$ is shown in Figure* 4.


Figure 4: The graph $QU_1'(n)$

## III. Main results

In this section, we determine the second largest values of $mi(G)$ among all quasi-unicyclic graphs of order $n \geq 5$, respectively. Moreover, the extremal graphs achieving these values are also determined.

**Theorem 3.1.** If $Q$ is a quasi-unicyclic graph of even order $n \geq 6$ with $Q \neq QU'_{1e}(n)$, then $mi(Q) \leq r^n$. Furthermore, the equality holds if and only if $Q \in \left\{K_4 \cup \frac{n-4}{2}P_2, QU_{1e}(6) \cup \frac{n-6}{2}P_2\right\}$.

Proof. It is straightforward to check that $mi\left(K_4 \cup \frac{n-4}{2}P_2\right) = mi\left(QU_{1e}(6) \cup \frac{n-6}{2}P_2\right) = r^n$. Suppose that $Q$ is a unicyclic graph, by Lemma 2.5, we obtain that $mi(Q) \leq mi\left(H'_{2e_1}(n)\right) = mi\left(H'_{2e_2}(n)\right) = 3r^{n-4} < r^n$. Then we assume that $Q$ contains at least two cycles. Let $x$ be the vertex such that $Q - x$ is a unicyclic graph. Then $x$ is on some cycle of $Q$, it follows that $\deg_Q(x) \geq 2$. We distinguish two cases to consider.

Case 1: $\deg_Q(x) = 2$. By Lemmas 2.1 and 2.4, we have that $mi(Q - x) \geq mi(Q) - mi(Q - N_Q[x]) \geq r^n - 3r^{(n-3)-3} = 5r^{(n-1)-5}$. It follows that $Q - x \in \{H'_{1o}(n-1), H'_{2o_1}(n-1), H'_{2o_2}(n-1)\}$. Note that $Q \neq QU'_{1e}(n)$.

*Subcase* 1.1. $Q - x = H'_{1o}(n-1)$, then $Q = D \cup \frac{n-4}{2}P_2$, where $D$ is the graph obtained from a $K_4$ by removing an arbitrary edge. According to a straightforward computation, we have that $mi(Q) = 3r^{n-4} < r^n$.

*Subcase* 1.2. $Q - x \in \{H'_{2o_1}(n-1), H'_{2o_2}(n-1)\}$, by Lemmas 2.1, 2.4 and 2.5, we have that $mi(Q) \leq mi(Q - x) + mi(Q - N_Q[x]) \leq 5r^{(n-1)-5} + 3r^{(n-3)-3} = r^n$. Furthermore, the equalities holding imply that $Q - x = H'_{2o_1}(n-1)$ and $Q - N_Q[x] = H'_{1o}(n-3)$. In conclusion, $Q = QU_{1e}(6) \cup \frac{n-6}{2}P_2$.

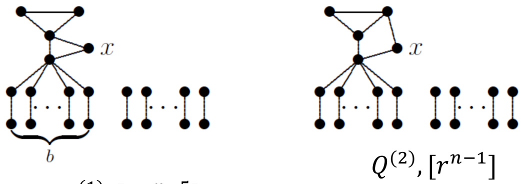
Case 2: $\deg_Q(x) \geq 3$. By Lemmas 2.1, 2.4 and 2.6, we have that $mi(Q) \leq mi(Q - x) + mi(Q - N_Q[x]) \leq 3r^{(n-1)-3} + \max\{r^{n-4}, 3r^{(n-5)-3}\} = r^n$. Furthermore, the equalities holding imply that $Q - x = H'_{1o}(n-1)$ and $Q - N_Q[x] = F_{1e}(n-4)$. In conclusion, $Q = K_4 \cup \frac{n-4}{2}P_2$ or $QU_{1e}(6) \cup \frac{n-6}{2}P_2$.

**Theorem 3.2.** If $Q$ is a quasi-unicyclic graph of odd order $n \geq 5$ having $Q \neq H'_{1o}(n)$, then $mi(Q) \leq 5r^{n-5}$. Furthermore, the equality holds if and only if $Q \in \left\{H'_{2o_1}(n), H'_{2o_2}(n), W \cup \frac{n-5}{2}P_2\right\}$, where $W$ is a bow, that is, two triangles $C_3$ having one common vertex.

Proof. It is straightforward to check that $mi\left(H'_{2o_1}(n)\right) = mi\left(H'_{2o_2}(n)\right) = mi\left(W \cup \frac{n-5}{2}P_2\right) = 5r^{n-5}$. Suppose that $Q$ is a unicyclic graph, by Lemma 2.3, it follows that $Q \in \{H'_{2o_1}(n), H'_{2o_2}(n)\}$. Now we assume that $Q$ contains at least two cycles. Let $x$ be the vertex such that $Q - x$ is a unicyclic graph. Then $x$ is on some cycle of $Q$ and $Q - x \neq F_{1e}(n)$, it follows that $\deg_Q(x) \geq 2$ and $mi(Q - x) \leq 3r^{(n-1)-4}$. By Lemmas 2.1, 2.4 and 2.5, we have that

$$mi(Q) \leq mi(Q - x) + mi(Q - N_Q[x]) \leq 3r^{(n-1)-4} + \max\{r^{n-3}, 3r^{(n-4)-3}\} = 3r^{n-5} + r^{n-3} = 5r^{n-5}.$$

Furthermore, the equalities holding imply that $\deg_Q(x) = 2$, $Q - x = H'_{2e_1}(n-1)$ and $Q - N_Q[x] = F_{1e}(n-3)$. There are two possibilities for graph $Q$. See Figure 5. The number inside the brackets in figure indicates the largest number of maximal independent sets of the corresponding graphs. According to a straightforward computation and Lemma 2.2, we have that $mi(Q) \leq 5r^{n-5}$. Also, the equality holding imply that $b = 0$ in $Q^{(1)}$. Hence we obtain that $Q = W \cup \frac{n-5}{2} P_2$, where $W$ is a bow, that is, two triangles $C_3$ having one common vertex.




$Q^{(1)}, [5r^{n-5}]$ $Q^{(2)}, [r^{n-1}]$

Figure 5. The possible graphs $Q^{(1)}$ and $Q^{(2)}$

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