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Boundary Conditions In Orthogonal Coordinates

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ABSTRACT
Our main aim here is to a necessary and sufficient Absorbing Boundary Conditions for the Wave equation in Orthogonal Coordinates. Absorbing Boundary Conditions are obtained for Cartesian Coordinates and Cylindircal Coordinates . We present Cartesian and Cylindrical results for the one medium and two media. These two seismograms show acceptaple results.

Key words : *Absorbing Boundary Conditions, Orthogonal Coordinates, Cauchy equation, Seismogram.*

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I. INTRODUCTION

In this section Absorbing boundary conditions (ABCs) are examined for the wave equation. The wave propagation problems are normally solved for an infinite medium, but because of the finite core of computers and finite time available the FDM solution can only be obtained at a finite number of points, thus it is necessary to introduce boundaries to obtain a finite model. Traditional boundary conditions, such as Dirichlet and Neumann boundary conditions will generate reflecting waves, and pollute numerical wave solutions. When we solve our problem in a finite medium, we need ABC*s* due to edge reflection since ABC*s* reduce edge reflection. We give boundary conditions which greatly reduce this edge reflection and compare our boundary conditions with the Dirichlet and Neumann boundary conditions. Most of the ABCs developed are for use in FDM of the wave equation.

In the FDM, ABCs for the wave equation have been developed by several scientists such as (Clayton & Engquist 1977), with modifications by (Fuyuki & Matsumoto 1980), and by (Emerman & Stephen 1983). (Emerman & Stephen 1983) found that (Clayton & Engquist 1977) ABCs are stable when $v_s/v_p > 0.46$. Whilst (Emerman & Stephen 1983) suggest $v_s/v_p > 0$. Numerical experimental evidence of this was shown by (Higdon 1991). (Lindman 1975) who developed a remarkably effective solution for 2-D acoustic waves, based on a series expansion of $1/\cos\theta$. (Randall 1988,1989) decomposed the wavefield at the boundary into compressional and shear components by a potential method, and then applied Lindman's technique to each one separately. (Long & Liow 1990) also decomposed the incident wavefield into dilational and rotational strains and applied a one way equation to the two components independently

The rigorous analysis of stability and accuracy for boundary conditions is very difficult, because of the many approximations typically involved in their formulation. (Renaut & Peterson (1989) have made some limited progress. (Randall 1989) indicates that his approach is applicable in 3-D but gives only 2-D examples.. Also (Stacey 1988) was improved ABCs for the elastic wave equation. We concentrated on Reynolds and Clayton-Engquist ABCs. (Reynolds 1978) approaches have the advantages of being reliable, easy to understand, and fairly easy to implement. We are modified Reynolds ABCs (Demir 1998). Reynolds ABCs are ideal in a staggered grid approach because it uses only values on lines normal to the absorbing surface.

II. ABSORBING BOUNDARY CONDITIONS IN ORTHOGONAL COORDİNATES

Below is an advancement of the Cartesian only approach of Reynolds [**11**]. Following Reynolds with $\tau = v_p t$ in orthogonal coordinates x_1, x_2, x_3 with line elements h_1, h_2, h_3 , and defining $H = h_1 h_2 h_3$ the 3-D wave equation is

$$\frac{\partial^2 u}{\partial \tau^2} = L[u] = L_1[u] + L_2[u] + L_3[u] \tag{1}$$

where,

$$L_i[u] = \frac{1}{h_i^2}\left(\frac{\partial^2 u}{\partial x_i^2} + \frac{\partial}{\partial x_i}\left(\ln\left(\frac{H}{h_i^2}\right)\right)\frac{\partial u}{\partial x_i}\right) \tag{2}$$

We shall "complete the square" on $L_i[u]$ with an operator

$$M_i[u] = \frac{1}{h_i}\left(\frac{\partial u}{\partial x_i} + A_i u\right)$$

so that

$$M_i^2[u] = \frac{1}{h_i^2}\left(\frac{\partial^2 u}{\partial x_i^2} + \left(2A_i - \frac{\partial}{\partial x_i}\left(\ln\left(h_i\right)\right)\right)\frac{\partial u}{\partial x_i}\right) + \frac{1}{h_i^2}\left(\frac{\partial A_i}{\partial x_i} + A_i^2 - A_i\frac{\partial}{\partial x_i}\left(\ln\left(h_i\right)\right)\right)$$

Writing $L_i[u] = M_i^2[u] + B_i u$ we obtain

$$2A_i - \frac{\partial}{\partial x_i}\left(\ln\left(h_i\right)\right) = \frac{\partial}{\partial x_i}\left(\ln\left(\frac{H}{h_i^2}\right)\right)$$

$$B_i = \frac{1}{h_i^2}\left(A_i\frac{\partial}{\partial x_i}\left(\ln\left(h_i\right)\right) - A_i^2 - \frac{\partial A_i}{\partial x_i}\right)$$

so $$A_i = \frac{\partial}{\partial x_i}\left(\ln\sqrt{\frac{H}{h_i}}\right),$$

The wave equation can be written as

$$\left(\frac{\partial}{\partial \tau} - \sqrt{L}\right)\cdot\left(\frac{\partial}{\partial \tau} + \sqrt{L}\right)[u] = 0 \qquad (3)$$

Thus an obvious boundary condition would be

$$\left(\frac{\partial}{\partial \tau} - \sqrt{L}\right)[u] = 0 \text{ on } x_i = -a \quad \text{and} \quad \left(\frac{\partial}{\partial \tau} + \sqrt{L}\right)[u] = 0 \text{ on } x_i = +a$$

We can use this equation to obtain relevant ABCs as follows. Consider

$$\left(\frac{\partial}{\partial \tau} \pm \sqrt{L}\right)[u] = 0 \text{ on } x_i = \pm a \qquad (4)$$

and pre-operate by M_i to obtain $\left(\frac{\partial M_i}{\partial \tau} \pm M_i\sqrt{L}\right)[u] = 0$

In the spirit of Reynolds approximate, $M_i\sqrt{L}$ by $M_i^2 + \frac{p}{p+1}B_i$ where p=1 is usual. It is assumed , possibly incorrectly, that M_i and B_i , and M_i and $\sqrt{L}$ commute.

Since $$\frac{\partial^2 u}{\partial \tau^2} = \left(M_i^2 + B_i\right)[u] \quad \text{and} \quad B_i[u] = \frac{\partial^2 u}{\partial \tau^2} - M_i^2[u]$$

approximate $M_i\sqrt{L}$ by

$$M_i^2 + \frac{p}{p+1}\left(\frac{\partial^2}{\partial \tau^2} - M_i^2\right) = \frac{p}{p+1}\frac{\partial^2}{\partial \tau^2} + \frac{1}{p+1}M_i^2$$

Then the boundary conditions are

$$\left(\frac{\partial M_i}{\partial \tau} \pm \frac{1}{p+1}\left(p\frac{\partial^2}{\partial \tau^2} + M_i^2\right)\right)[u] = 0$$

or (6)

$$\left(p\frac{\partial}{\partial \tau} \pm M_i\right)\left(\frac{\partial}{\partial \tau} \pm M_i\right)[u] = 0$$

Writing $\tau = v_p t$ in orthogonal co-ordinates the approximate boundary condition would be

$$\frac{p}{v_p^2}\frac{\partial^2 u}{\partial t^2} \pm \frac{p+1}{v_p}\frac{\partial}{\partial t} M_i[u] + M_i^2[u] = 0. \tag{7}$$

ABCs for Cartesian Coordinates

The Cartesian coordinate ABCs may be found from equation (7) by setting:

$$h_1 = h_2 = h_3 = 1 \quad \text{and} \quad H = 1$$

$$A_1 = A_2 = A_3 = 0 \quad , \quad M_1 = \frac{\partial u}{\partial x}, M_2 = \frac{\partial u}{\partial y}, M_3 = \frac{\partial u}{\partial z}$$

Thus at $x = \pm a$:

$$\frac{p}{v_p^2}\frac{\partial^2 u}{\partial t^2} \pm \frac{p+1}{v_p}\frac{\partial^2 u}{\partial t \partial x} + \frac{\partial^2 u}{\partial x^2} = 0 \tag{8}$$

Whilst at $y = \pm a$:

$$\frac{p}{v_p^2}\frac{\partial^2 u}{\partial t^2} \pm \frac{p+1}{v_p}\frac{\partial^2 u}{\partial t \partial y} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{9}$$

and at $z = \pm a$:

$$\frac{p}{v_p^2}\frac{\partial^2 u}{\partial t^2} \pm \frac{p+1}{v_p}\frac{\partial^2 u}{\partial t \partial z} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{10}$$

These equations are given in (Reynolds 1978).

ABCs for Cylindrical Coordinates

We now obtain cylindrical coordinates boundary conditions from equations (7). On setting

$$h_1 = h_3 = 1 \quad , h_2 = r \quad \text{and} \quad H = r$$

$$A_1 = \frac{1}{2r} \quad , A_2 = A_3 = 0 \quad , \quad M_1 = \frac{\partial u}{\partial r} + \frac{u}{2r}, M_2 = \frac{1}{r}\frac{\partial u}{\partial \theta}, M_3 = \frac{\partial u}{\partial z}$$

We find

At $r = a$

$$\frac{p}{v_p^2}\frac{\partial^2 u}{\partial t^2} \pm \frac{p+1}{v_p}\left(\frac{\partial^2 u}{\partial r \partial t} + \frac{1}{2r}\frac{\partial u}{\partial t}\right) + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{4r^2} = 0 \tag{11}$$

At $\theta = \pm a$

$$\frac{p}{v_p^2}\frac{\partial^2 u}{\partial t^2} \pm \frac{p+1}{v_p}\frac{1}{r}\frac{\partial^2 u}{\partial t \partial \theta} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{12}$$

At $z = \pm a$

$$\frac{p}{v_p^2}\frac{\partial^2 u}{\partial t^2} \pm \frac{p+1}{v_p}\frac{\partial^2 u}{\partial t \partial z} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{13}$$

We have modified the Reynolds ABCs for cylindrical coordinates.

III. CARTESIAN AND CYLINDRICAL RESULTS

In this section, we compare the Cartesian and Cylndrical results. Figure (1) shows the seismogram with Cartesian and Cylindrical results for one media. Figure (2) presents the seismogram calculated for two media. There is an excelent similarity between the two results. This similarities means that Absorbing boundary conditions appreciated that Cartesian and Cylindrical coordinates.

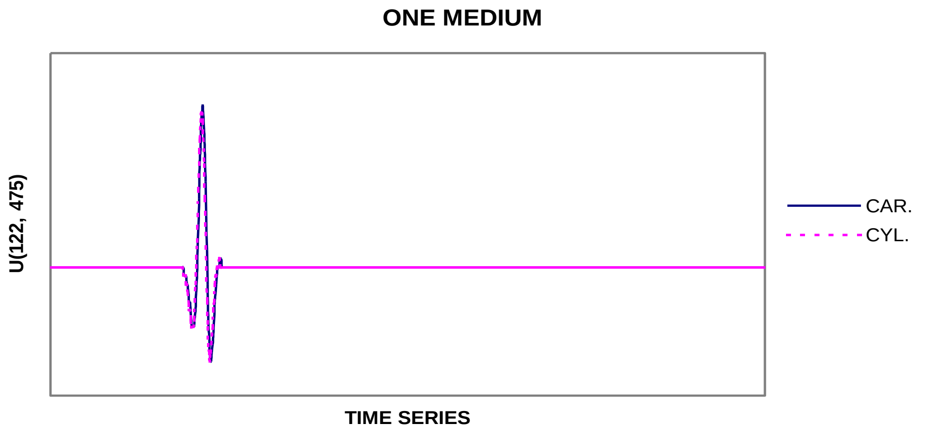

Figure 1. Seismograms obtained from the Cartesian and Cylindrical solutions for one medium.

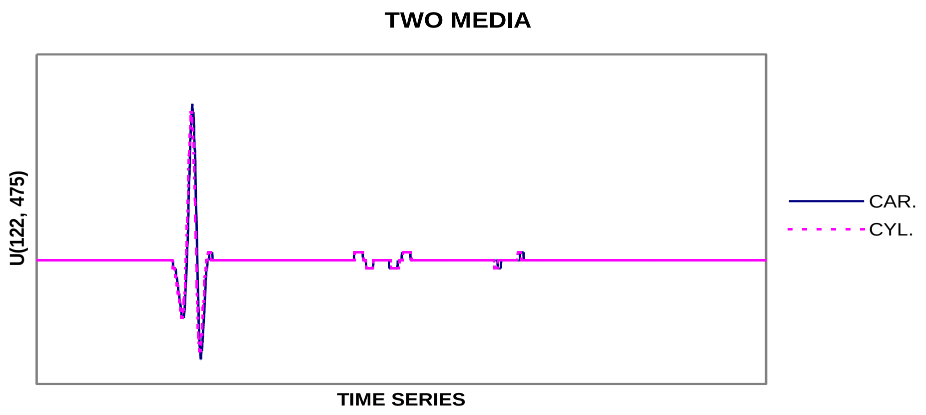

Figure 2. Seismograms obtained from the Cartesian and Cylindrical solutions for two media.

IV. CONCLUSION

We solved the 2-D acoustic wave propagation problem numerically with absorbing boundary conditions in Cartesian and Cylindrical coordinates for one medium and two media. These two seismogram was found to be most effective. Absorbing boundary conditions in Orthogonal coordinates will be used for different coordinates system.

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