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Some Facts about Silver Ratio and its Relation with Pell Numbers and Silver Rectangle

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Abstract: *In this paper, the relation of Silver Ratio with Pell numbers and Silver Rectangle is discussed. The value $\left(1+\sqrt{2}\right)$ is known as silver ratio. It can be expressed as a trigonometric function and also in terms of an infinite continued fraction. A rectangle whose length and breadth are in the ratio $(1+\sqrt{2}): 1$is known as Silver Rectangle. Silver rectangle can be constructed within a regular octagon. The area of an octagon depends on silver ratio. Silver rectangles of gradually decreasing size can be formed successively within a rectangle whose sides are in the ratio $\sqrt{2}: 1$.The ratio $\sqrt{2}: 1$is used in the construction of temples and towers in Japan.*
Keywords: *Recurrence relation, Pell numbers, Silver ratio, Octagon, Silver rectangle*

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I. Introduction

The Pell numbers are named after English mathematician John Pell (1611-1685). The details about Pell numbers can be found in [1, 2, 3]. The information about Silver Ratio is given in [4]. The sequence of Pell numbers $\{P_n\}$ is defined by recurrence relation

$$P_n = 2P_{n-1} + P_{n-2} \text{ for } n \geq 2 \text{ with } P_0 = 0 \text{ and } P_1 = 1 \qquad (1)$$

Where P_n denotes nth Pell number. The sequence of Pell numbers starts with 0 and 1 and then each number is the sum of twice its previous number and the number before its previous number.The first few Pell numbers calculated from (1) are given in the following Table no.1.

Table no.1: First few Pell numbers

n	0	1	2	3	4	5	6	7	8	9	10
P_n	0	1	2	5	12	29	70	169	408	985	2378

Let a and b are the roots of quadratic equation

$$x^2 - 2x - 1 = 0. \qquad (2)$$

Solving this equation, we get

$$a = 1 + \sqrt{2} \qquad (3)$$

and

$$b = 1 - \sqrt{2} \qquad (4)$$

The root $a = 1 + \sqrt{2} \approx 2.4142135623$ of quadratic equation (2) is called 'Silver *Ratio'*.
Definition: Silver Ratio is defined as an irrational mathematical constant, whose value is equal to one plus the square root of 2 and approximately 2.4142135623.

A rectangle whose length and breadth are in the ratio $\left(1+\sqrt{2}\right): 1$is known as Silver Rectangle. Silver rectangle can be constructed within a regular octagon. Also, Silver rectangles of gradually decreasing size can be formed successively within a rectangle whose sides are in the ratio $\sqrt{2}: 1$.

The rest of the paper is organized as follows. The relations between Silver Ratio and Pell numbers are mentioned in Section-II. Some other facts about silver ratio and its relation with silver rectangle are given in Section-III. The use of the ratio $\sqrt{2}: 1$in Japanese architecture is mentioned in Section-IV. Finally conclusion is given in Section-V.

II. Relations between Silver Ratio and Pell Numbers

1. Silver ratio can be expressed as the ratio of two consecutive Pell numbers P_n and P_{n+1}as $n \to \infty$.

$$a = \lim_{n\to\infty} \frac{P_{n+1}}{P_n} \qquad (5)$$

For example, $\frac{P_9}{P_8} = \frac{985}{408} \approx 2.41421568$ and $\frac{P_{10}}{P_9} = \frac{2378}{985} \approx 2.41421319$. These values show that as n increases, the ratio P_{n+1}/P_n is more and more close to silver ratio.

2. Powers of Silver Ratio:

Now let us find out the square and cube of silver ratio $'a'$.

$$a^2 = \left(1+\sqrt{2}\right)^2 = 1+2+2\sqrt{2} = 1+2\left(1+\sqrt{2}\right) = P_1 + P_2 a \qquad (6)$$

$$a^3 = \left(1+\sqrt{2}\right)^3 = 1+3\sqrt{2}+6+2\sqrt{2} = 2+5\left(1+\sqrt{2}\right)$$

$$= P_2 + P_3 a \qquad (7)$$

[Refer Table no.1 for values of Pell numbers]

Looking at the form of (6) and (7), one can write the n^{th} power of silver ratio in general by the following relation.

$$a^n = P_{n-1} + P_n a \text{for } n = 1,2,3, \ldots etc. \qquad (8)$$

3. The Pell number can be written in terms of silver ratio by the following relation.

$$P_n = \lim_{n\to\infty} \frac{a^n}{\sqrt{8}} \qquad (9)$$

Examples:

(a) For $n = 2$, we have

$$\frac{a^2}{\sqrt{8}} = \frac{\left(1+\sqrt{2}\right)^2}{2\sqrt{2}} = \frac{3+2\sqrt{2}}{2\sqrt{2}} = \frac{3+2\times 1.414}{2\times 1.414} = \frac{5.828}{2.828} = 2.060 \approx 2 = P_2 \qquad (10)$$

(b) For $n = 4$, we get

$$\frac{a^4}{\sqrt{8}} = \frac{a^2\times a^2}{2\sqrt{2}} = \frac{5.828\times 5.828}{2.828} = 12.010 \approx 12 = P_4 \qquad (11)$$

The above two examples show that the value of $a^n/\sqrt{8}$ is becoming more and more close to P_n as n increases. Hence the relation (9) is verified.

4. Consider a function $f(x)$ given by

$$f(x) = x^2 - 2x - 1 \qquad (12)$$

Taking some initial value for x, the next roots of the above function $f(x)$ can be calculated using the following Newton's or Newton-Raphson's method formula.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (13)$$

Where $f'(x_n)$ is the first derivative of the function $f(x)$.The above formula is named after Sir Isaac Newton (1642 -1727) and English mathematician Joseph Raphson (1648-1715). In numerical analysis this formula is used to find the next roots of a real valued function $f(x)$ taking some initial value for x.

The first derivative of the function $f(x)$ is given by

$$f'(x) = 2x - 2 \qquad (14)$$

Let

$$x_1 = 2 = \frac{P_2}{P_1} \qquad (15)$$

is the initial root of function $f(x)$. Then the next roots $x_2, x_3 \& x_4$ of $f(x)$ are calculated as given below.

Putting $n = 1$ in (13) we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{(x_1^2 - 2x_1 - 1)}{(2x_1 - 2)} \quad \text{[By (12) \& (14)]} \qquad (16)$$

Using (15) in (16) we obtain

$$x_2 = 2 - \frac{(4-2\times 2-1)}{(2\times 2-2)} = \frac{5}{2} = \frac{P_3}{P_2} \qquad (17)$$

Similarly, putting $n = 2$ in (13) we have

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{\left(x_2^2 - 2x_2 - 1\right)}{(2x_2 - 2)} \quad \text{[By (12) \& (14)]}$$

$$= \frac{5}{2} - \frac{\left\{\left(\frac{5}{2}\right)^2 - 2\times\frac{5}{2} - 1\right\}}{\left(2\times\frac{5}{2} - 2\right)} \quad \text{[By (17)]}$$

$$\Rightarrow x_3 = \frac{29}{12} = \frac{P_5}{P_4} \qquad (18)$$

Similarly, taking $n = 3$ in (13) we get

$$x_4 = \frac{985}{408} = \frac{P_9}{P_8} \qquad (19)$$

Looking at the forms of (15), (17), (18) & (19), one can write in general that

$$x_{n+1} = \frac{P_{2^n+1}}{P_{2^n}} \quad \text{for } n = 0,1,2,3, \ldots etc. \qquad (20)$$

Let $k = 2^n$.Then the above expression can be written as

$$x_{n+1} = \frac{P_{k+1}}{P_k}$$

Taking the limits of both sides of above expression, we have

$$\lim_{n\to\infty} x_{n+1} = \lim_{k\to\infty} \frac{P_{k+1}}{P_k}$$

$$= \text{Silver Ratio } (\alpha) \text{ [By (5)]}$$

$$\Rightarrow \text{Silver Ratio } (\alpha) = \lim_{n\to\infty} x_{n+1} \qquad (21)$$

Thus when the initial root of the function $f(x)$ given by (12) is 2, its higher roots tend to Silver Ratio.

III. Some other facts about Silver Ratio and its relation with Silver Rectangle

(1) *Silver Ratio as a trigonometric function:*

We know that

$$\cos 2\theta = 1 - 2\,sin^2\theta$$

$$\Rightarrow \sin\theta = \sqrt{\frac{1-\cos\ 2\theta}{2}}$$

Putting $\theta = \frac{\pi}{8} = 22.5^o$ in the above relation, we get

$$\sin\frac{\pi}{8} = \sqrt{\frac{1-\cos\ 45}{2}} = \sqrt{\frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)} = \frac{\sqrt{2-\sqrt{2}}}{2} \qquad (22)$$

Similarly,

$$\cos 2\theta = 2\,cos^2\theta - 1$$

$$\Rightarrow \cos\theta = \sqrt{\frac{1+\cos\ 2\theta}{2}}$$

Substituting $\theta = \frac{\pi}{8} = 22.5^o$ in the above relation, we obtain

$$\cos\frac{\pi}{8} = \sqrt{\frac{1+\cos\ 45}{2}} = \sqrt{\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)} = \frac{\sqrt{2+\sqrt{2}}}{2} \qquad (23)$$

Then,

$$\cot\frac{\pi}{8} = \frac{\cos\frac{\pi}{8}}{\sin\frac{\pi}{8}} = \frac{\frac{\sqrt{2+\sqrt{2}}}{2}}{\frac{\sqrt{2-\sqrt{2}}}{2}} = \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2+\sqrt{2}}} \quad \text{[By (22) \& (23)]}$$

$$\Rightarrow \cot\frac{\pi}{8} = 1+\sqrt{2}$$

$$= \text{Silver Ratio } (\alpha) \text{ [By (3)]}$$

$$\Rightarrow \text{Silver Ratio } (\alpha) = \cot\frac{\pi}{8} = \cot 22.5^o$$

Since, $\cot\frac{\pi}{8} = \cot\left(\frac{\pi}{2}-\frac{3\pi}{8}\right) = \tan\frac{3\pi}{8}$, the above relation can also be written as

$$\text{Silver Ratio } (\alpha) = \cot\frac{\pi}{8} = \tan\frac{3\pi}{8} \qquad (24)$$

(2) *Silver Ratio as an infinite continued fraction:*

The infinite continued fraction expansion of $\sqrt{2}$ is given by

$$\sqrt{2} = 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}} \qquad (25)$$

Using (25), we get

$$\text{Silver Ratio} = \alpha = 1+\sqrt{2} = 2 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}} = [2; 2,2,2, \ldots.] \qquad (26)$$

The above expression is the infinite continued fraction expansion of Silver Ratio.

(3) *Silver Rectangle:*

Definition: A rectangle whose sides are in the ratio $1+\sqrt{2}$: 1is known as *Silver Rectangle.*

(a) Silver Rectangle with in a regular octagon:

A silver rectangle can be formed within a regular octagon. Consider a regular octagon ABCDEFGH and divide it into two isosceles trapeziums (ABCD and EFGH) and one rectangle (ADEH) as shown Figure-1.Draw perpendiculars BI and CJ on AD.

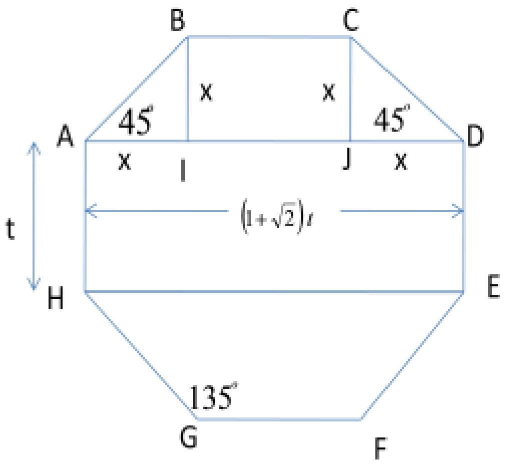

Figure 1: Silver Rectangle ADEH within a regular octagon

Let

$t =$ Side of the octagon

and

BI = CJ $= x$

Then $AI = JD = x$ as triangles ABI and CJD are right angled isosceles triangles.

In the right angled isosceles triangle ABI,

$$AI^2 + BI^2 = AB^2$$

$$\Rightarrow x^2 + x^2 = t^2$$

$$\Rightarrow x = \frac{t}{\sqrt{2}} \qquad (27)$$

Length of the rectangle $ADEH = AD$

$$= AI + IJ + JD$$

$$= x + t + x = 2x + t$$

$$= 2 \times \frac{t}{\sqrt{2}} + t \quad \text{[By (27)]}$$

$$= t(1+\sqrt{2}) = a\,t \quad \text{[By (3)]} \qquad (28)$$

Where a =Silver ratio.

The ratio of length and breadth of the rectangle ADEH= $AD{:}AH = at{:}t = a{:}1$. Hence this rectangle whose sides are in the ratio $a{:}1 = 1+\sqrt{2}{:}1$ is a *Silver Rectangle*.

Area of octagon:

The area of the octagon shown in Figure-1 can be calculated in terms of silver ratio.

Area of the octagon = Area of silver rectangle ADEH + 2× Area of the trapezium ABCD

$$= AD \times AH + 2 \times \frac{1}{2} \times BI \times (AD + BC)$$

$$= at \times t + x(at + t) \quad \text{[By (28)]}$$

$$= at^2 + \frac{t}{\sqrt{2}}(at + t) \quad \text{[By (27)]}$$

$$= at^2 + \frac{t^2}{\sqrt{2}}(a + 1)$$

$$= at^2 + \frac{t^2}{\sqrt{2}}(2+\sqrt{2}) \quad \text{[By (3)]}$$

$$= at^2 + t^2(1+\sqrt{2})$$

$$= 2at^2 \quad \text{[By (3)]} \qquad (29)$$

(b) Silver Rectangles with in a rectangle whose sides are in the ratio $\sqrt{2}:1$*:*

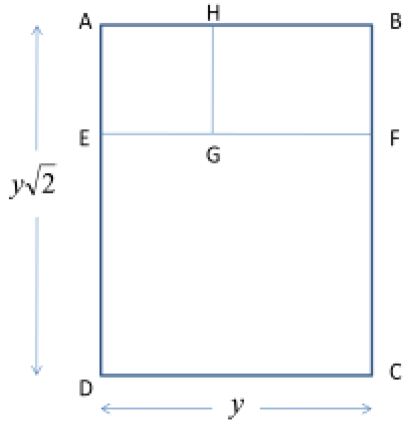

Figure 2: Silver rectangles within a rectangle whose sides are in the ratio $\sqrt{2}:1$*:*

Consider a rectangle $ABCD$, whose sides are in the ratio $\sqrt{2}:1$as shown in Figure-2. The length and breadth of this rectangle are

$$AD = BC = y\sqrt{2}$$

and

$$AB = CD = y$$

Now consider a a square $CDEF$ with in the given rectangle .Removal of this square from rectangle $ABCD$leaves the rectangle $ABFE$. The length and breadth of the new rectangle are

$$AB = EF = y \qquad (30)$$

and

$$AE = BF = AD - ED = y\sqrt{2} - y = y(\sqrt{2} - 1) \qquad (31)$$

Hence, the length and breadth of the rectangle $ABFE$ are in the ratio,$y: y(\sqrt{2}-1) = 1:\sqrt{2}-1 = \sqrt{2}+1:1 = a:1$.

Where $\sqrt{2} + 1 = a =$ Silver ratio.

Hence the rectangle $ABFE$ is a Silver Rectangle as per its definition.

Similarly now consider the removal of the square $AEGH$ from the rectangle, $ABFE$. Removal of this square leaves the rectangle $BFGH$ within silver rectangle$ABFE$. The length and breadth of this rectangle are

$$HB = AB - AH = AB - AE = y - y(\sqrt{2}-1) \quad \text{[By (30) \& (31)]}$$

$$= y(2 - \sqrt{2}) \qquad (32)$$

and

$$BF = AE = y(\sqrt{2}-1) \quad \text{[By (31)]} \qquad (33)$$

The ratio of length and breadth of rectangle $BFGH$

$$= HB:BF$$
$$= y(2-\sqrt{2}):y(\sqrt{2}-1) \quad \text{[By (32) \& (33)]}$$
$$= (2-\sqrt{2}):(\sqrt{2}-1)$$
$$= (2-\sqrt{2})(\sqrt{2}+1):1$$
$$= \sqrt{2}:1$$

This ratio is same as the ratio of sides of the original rectangle $ABCD$. So, removal of a square of side BF from the rectangle $BFGH$ leaves a silver rectangle as explained above. Thus silver rectangles of gradually decreasing size can be formed one after the other within $ABCD$ byrepetition of above procedure.

IV. The ratio $\sqrt{2}:1$ in Japanese architecture

Japanese believe that the use of the ratio$1.414:1$ $(or\ \sqrt{2}:1)$ in the construction of a building makes it look beautiful. This ratio was used in the design of Horyu-Ji temples and Tokyo Sky Tree in Japan.

Horyu-Ji temples are the oldest wooden temples of the world. In one Horyu-Ji temple there are two floors and in another there are four floors. In the first Horyu-Ji temple, the length of ground floor is 1.414 times the length of first floor. In the second temple, the length of ground floor is 1.414 times the length of top third floor.

Tokyo Sky Tree is one of the world's tallest towers. It has two observatories and a digital broadcasting antenna. The distance between the floor and the top of the tower is 1.414 times the distance of the floor to the second observatory.

V. Conclusion

This paper presents the relation of silver ratio with Pell numbers and silver rectangle. Silver rectangle can be formed within a regular octagon. The area of an octagon can be expressed in terms of silver ratio. Silver rectangles of gradually decreasing size can be formed successively within a rectangle whose sides are in the ratio$\sqrt{2}:1$.The ratio $1.414:1$ $\left(or\ \sqrt{2}:1\right)$ finds application in Japanese architecture. This study on Silver Ratio, Pell numbers and Silver Rectangle will inspire mathematicians to explore it further.

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