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Boolean representations of finite De Morgan algebras

Wu Wan[1]
College of Science, China Three Gorges University, Yichang, Hubei, 443002, China

Abstract
In this paper, by considering the Boolean matrix on binary Boolean algebra, the definition of inverse-preserving incidence mapping of Boolean matrix is proposed, and the Boolean representations of Ockham algebra and finite De Morgan algebra are obtained.
Keywords: *Boolean matrix; row column space lattice; De Morgan algebra; Ockham algebra;*

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I. Introduction

Lattice is a new algebraic system introduced with the development of classical logic algebra and universal algebra. In recent years, lattice theory has also been widely used in many other disciplines such as combinatorial mathematics, graph theory, logic and computer science. Among them, distributive lattice is a particularly important lattice, and De Morgan algebra is a particularly important distributive lattice. With the introduction of fuzzy set theory and fuzzy logic, fuzzy phenomena have been incorporated into people's research scope. The study of De Morgan algebra has received widespread attention and has flourished. In the 1950s, A. Kalman, A. Bialyniki-Birula and H. Rasiowa et al. began to study De Morgan algebras. The topological representation theorem given by R. Cignol et al. deepened the study of congruence relations.

In this paper, by using the Boolean matrix on the binary Boolean algebra, the definition of the inverse preserving incidence mapping of the Boolean matrix is proposed, and the relationship between the Ockham algebra and the finite De Morgan algebra and the partial order relation matrix is obtained.

II. Preliminaries

Let β_0 denote the Boolean algebra $\{0,1\}$ with three binary operations $\vee$, $\wedge$, $^{-}$, which are defined as $0\vee 0=0\wedge 1=1\wedge 0=0\wedge 0=0$, $1\vee 0=0\vee 1=1\vee 1=1\wedge 1=1$, $\overline{0}=1$ and $\overline{1}=0$. The vector or matrix is the vector or matrix on β_0 and the matrix on β_0 is called a Boolean matrix. Let B_{mn} be the set of all $m\times n$-order Boolean matrices. If $m=n$, write it B_n. The i row of matrix A is represented by A_{i*}, and the j column is represented by A_{*j}. In this paper, the set $\{1,2,\cdots,n\}$ is represented by $\underline{n}$.

Definition 2.1 Error! Reference source not found. Let V_n denotes the set of all n-tuples $(a_1,a_2,\cdots,a_n)$ on β_0. An element of V_n is called an n-dimensional Boolean row vector. If any element $(a_1,a_2,\cdots,a_n)$ of V_n has only $a_i=1$, then this element is denoted by e_i. Let $V^n=\{v^T \mid v\in V_n\}$, where v^T is an n-dimensional Boolean column vector.

Definition 2.2 Error! Reference source not found. A subspace of V_n is a subset of V_n which contains zero vectors and is closed to vector addition. The generating space of a vector set W is the intersection of all subspaces containing W, denoted by $\langle W\rangle$. Let $A\in \mathrm{B}_{mn}$, the space generated by the set of all column vectors of matrix A is the column space of A. Similarly, the row space of matrix A can be defined. The row (column) space of matrix A is denoted by $C(A)(R(A))$.

Definition 2.3 Error! Reference source not found. Let $A=(a)_{ij\,n\times n}\in B_n$, if for each i, $a_{ii}=1$, A is said to be reflexive.

[1] E-mail addresses: ww1213102022@163.com.

If $a_{ij}=1$, $a_{jk}=1$, then $a_{ik}=1$, A is said to be transitive. If $a_{ij}=1$ and $i\neq j$, then $a_{ji}=0$, A is antisymmetric. If matrix A is reflexive, transitive and antisymmetric, then A is called partial order relation matrix.

Definition 2.4 Error! Reference source not found. Let $(L;\wedge,\vee,0,1)$ be a bounded distributive lattice and f be a unary operation on L. If

(1) $f(0)=1, f(1)=0$,

(2) $\forall x,y\in L, f(x\wedge y)=f(x)\vee f(y), f(x\vee y)=f(x)\wedge f(y)$,

then $(L;\wedge,\vee,0,1)$ is called an Ockham algebra.

Definition 2.5[0] Let $(L;\wedge,\vee,0,1)$ be a bounded distributive lattice and f be a unary operation on L. If

(1) $f(0)=1, f(1)=0$

(2) $\forall x,y\in L, f(x\wedge y)=f(x)\vee f(y), f(x\vee y)=f(x)\wedge f(y)$;

(3) $f^2(1)=1$

then $(L;\wedge,\vee,0,1,f)$ is called an Ockham algebra.

III. Boolean representations of finite DeMorgan algebras

Theorem 3.1 Error! Reference source not found. Let A be a reflexive and antisymmetric Boolean matrix. Then is regular if and only if A is transitive.

It can be seen from the above theorem that the partial order relation matrix must be a regular matrix, then according to [3], A is regular if and only if the column space $C(A)$ is a distributive lattice, so the following lemma is obvious.

Lemma 3.1 Let A be a n-order partial order relation matrix if and only if the column space $C(A)$ is a distributive lattice.

Definition 3.1 Let $A=(a_{ij})_{n\times n}\in B_n$, $B=(b_{ij})_{m\times m}\in B_m$, then the mapping $f:\underline{n}\to\underline{m}$ is called a preserving incidence from matrix A to matrix B, if for any $i,j\in\underline{n}$, $a_{ij}=1$, there must be $b_{f(i)f(j)}=1$.

Definition 3.2 Let $A=(a_{ij})_{n\times n}\in B_n$, $B=(b_{ij})_{m\times m}\in B_m$, then the mapping $f:\underline{n}\to\underline{m}$ is called an inverse preserving incidence from matrix A to matrix B, if for any $i,j\in\underline{n}$, $a_{ij}=1$, there must be $b_{f(j)f(i)}=1$.

Theorem 3.2 A is a partial order relation matrix with inverse preserving incidence if and only if $C(A)$ is an Ockham algebra.

Proof: If A is a partial order relation matrix with inverse preserving incidence, then $C(A)$ is a distributive lattice according to Lemma 2.1.

Proof necessity, let the inverse preserving correlation of partial order relation matrix A be $g:\underline{n}\to\underline{n}$.

Define the mapping $f:C(A)\to C(A)$, for $\forall\alpha=(a_1,a_2,\cdots,a_n)^T\in C(A)$, $f(\alpha)=(b_1,b_2,\cdots,b_n)^T$, when $i\in\underline{n}\setminus g^{-1}(S(\alpha))$, one has $b_i=1$, otherwise $b_i=0$.

Let f satisfy De Morgan law. For $\forall a,b\in C(A)$, $f(a\vee b)=f(c)=(c_1,c_2,\cdots,c_n)^T$. When $i\in\underline{n}\setminus g^{-1}(S(c))$, one has $c_i=1$, otherwise $c_i=0$, that is, when $i\in\underline{n}\setminus g^{-1}(S(a\vee b))=\underline{n}\setminus g^{-1}(S(a)\bigcup S(b))$, one has $c_i=1$, otherwise $c_i=0$.

$f(a)\wedge f(b)=(d_1,d_2,\cdots,d_n)^T$, when $i\in\underline{n}\setminus g^{-1}(S(a))$ and $i\in\underline{n}\setminus g^{-1}(S(b))$, one has $d_i=1$, otherwise $d_i=0$, that is, when $i\in\underline{n}\setminus(g^{-1}(S(a)\bigcup S(b)))$, one has $d_i=1$, otherwise $d_i=0$. So $c_i=d_i$, $f(a\vee b)=f(a)\wedge f(b)$. Similarly, $f(a\wedge b)=f(a)\vee f(b)$.

If $C(A)$ is a distributive lattice and the unary operation f on $C(A)$ satisfies the De Morgan law, then

$C(A)$ is an Ockham algebra.

Proof sufficiency. Let f be an unary operation on Ockham algebra $C(A)$, and let the mapping $g:\underline{n}\to\underline{n}$, $\forall i\in\underline{n}$, $g(i)=\max\{j\in\underline{n}\mid f(\alpha)=\beta\}$, where $\alpha,\beta\in C(A)$, $\alpha=(a_1,a_2,\cdots,a_j=0,\cdots,a_n)^T$, $\beta=(b_1,b_2,\cdots,b_i=1,\cdots,b_n)^T$.

The lower proof the mapping g is an inverse preserving association.

If the mapping $g:\underline{n}\to\underline{n}$ is not an inverse preserving relation, then there exist $x,y\in\underline{n}$ and $a_{xy}=1$, one has

$$a_{g(y)g(x)}=a_{ji}=0.$$

And $g(x)=i, g(y)=j$, according to the definition may be let $f(a)=b$, $f(c)=d$. And because the mapping f satisfies the De Morgan law, namely $f(a\vee b)=f(a)\wedge f(b)$, then $f(A_{*i}\vee c)=f(A_{*i})\wedge f(c)$.

According to the definition of the mapping g, one can know $A_{*i}\geq c$, so $f\left(A_{*i}\vee c\right)=f\left(A_{*i}\right)$, namely $f(A_{*i})=f(A_{*i})\wedge f(c)$, so $A_{*i}\leq c$, namely $A_{*i}=c$.

Obviously, the value range of A_{*i},c is different, that is, contradiction. Then g is an inverse preserving relation.

Lemma 3.2 For any partial order relation matrix A, $C(A)$ must be pseudo-complemented.

Proof: For $\forall\alpha\in C(A)$, $\alpha=(a_1,a_2,\cdots,a_n)^T$, define

$$\alpha^*=\max\left\{(y_1,y_2,\cdots,y_n)^T\,\middle|\,(a_1,a_2,\cdots,a_n)^T(y_1,y_2,\cdots,y_n)^T=(0,0,\cdots,0)^T\right\}$$

and $\alpha^*\in C(A)$.

Theorem 3.3 Let $A=(a_{ij})_{n\times n}\in B_n$ be a partial order relation matrix and $C(A)$ be a De Morgan algebra if and only if there exists an inverse-preserving relation g with $\forall x\in\underline{n}$ for $g^2(x)=x$.

Proof: Proof necessity, let the inverse preserving correlation of partial order relation matrix A be $g:\underline{n}\to\underline{n}$, and $g^2(x)=x$, define the mapping $f:C(A)\to C(A)$, for $\forall\alpha=(a_1,a_2,\cdots,a_n)^T\in C(A)$, $f(\alpha)=(b_1,b_2,\cdots,b_n)^T$, when $i\in\underline{n}\setminus g^{-1}(S(\alpha))$, one has $b_i=1$, otherwise $b_i=0$.

The following proves that the unary operation f satisfies the De Morgan law. For $\forall a,b\in C(A)$, $f(a\vee b)=f(c)=(c_1,c_2,\cdots,c_n)^T$, when $i\in\underline{n}\setminus g^{-1}(S(c))$, one has $c_i=1$, otherwise $c_i=0$, namely, when $i\in\underline{n}\setminus g^{-1}(S(a\vee b))=\underline{n}\setminus g^{-1}(S(a)\bigcup S(b))$, one has $c_i=1$, otherwise $c_i=0$.

$f(a)\wedge f(b)=(d_1,d_2,\cdots,d_n)^T$, when $i\in\underline{n}\setminus g^{-1}(S(a))$ and $i\in\underline{n}\setminus g^{-1}(S(b))$, one has $d_i=1$, otherwise $d_i=0$, namely, when $i\in\underline{n}\setminus(g^{-1}(S(a)\bigcup S(b)))$, one has $d_i=1$, otherwise $d_i=0$. So $c_i=d_i$, $f(a\vee b)=f(a)\wedge f(b)$. Similarly, $f(a\wedge b)=f(a)\vee f(b)$.

And for $\forall a\in C(A)$, let $f^2(a)=f(b)=c$, when $i\in\underline{n}\setminus g^{-1}(S(b))$, one has $c_i=1$, otherwise $c_i=0$. And $S(b)=\underline{n}\setminus g^{-1}(S(a))$, so when $i\in\underline{n}\setminus g^{-1}(\underline{n}\setminus g^{-1}(S(a)))$, one has $c_i=1$, otherwise $c_i=0$, one has $\underline{n}\setminus g^{-1}(\underline{n}\setminus g^{-1}(S(a)))=g^{-1}(g^{-1}(S(a)))=S(a)$, namely $f^2(a)=a$.

In summary, $C(A)$ is a distributive lattice and the unary operation f on $C(A)$ satisfies De Morgan law, $f^2(a)=a$, then $C(A)$ is a De Morgan algebra.

Proof sufficiency. Let f be an unary operation on De Morgan algebra $C(A)$, and let the mapping

$g: \underline{n} \to \underline{n}$, $\forall i \in \underline{n}$, $g(i) = \max\{ j \in \underline{n} \mid f(\alpha) = \beta \}$, where $\alpha, \beta \in C(A)$, $\alpha = (a_1, a_2, \cdots, a_j = 0, \cdots, a_n)^T$, $\beta = (b_1, b_2, \cdots, b_i = 1, \cdots, b_n)^T$.

The following proves that the mapping g is an inverse preserving association. In reduction to absurdity, if the mapping $g: \underline{n} \to \underline{n}$ is not an inverse preserving association, then there exist $x, y \in \underline{n}$, $a_{xy} = 1$, one has

$$a_{g(y)g(x)} = a_{ji} = 0,$$

where $g(x) = i, g(y) = j$.

According to the definition may be let $f(a) = b$, $f(c) = d$. Because the mapping f satisfies the De Morgan law, namely $f(a \vee b) = f(a) \wedge f(b)$, then $f(A_{*i} \vee c) = f(A_{*i}) \wedge f(c)$.

According to the definition of the mapping g, one can see $A_{*i} \geq c$, so $f(A_{*i} \vee c) = f(A_{*i})$, namely $f(A_{*i}) = f(A_{*i}) \wedge f(c)$, and $A_{*i} \leq c$, that is $A_{*i} = c$.

Obviously, the value range of A_{*i}, c is different, that is, contradiction. Then g is an inverse preserving relation.

For $\forall x \in \underline{n}$, one has $g^2(x) = x$. Proof by contradiction, if $g^2(x) \neq x$, let $g^2(i) = g(j) = z$.

According to the definition of mapping g, there is a vector $a, b, c, d \in C(A)$ that satisfies the condition such that $f(a) = b, f(c) = d$, where $a = (a_1, \cdots, a_j = 0, \cdots, a_n)^T$, $b = (a_1, \cdots, a_i = 1, \cdots, a_n)^T$, $c = (a_1, \cdots, a_z = 0, \cdots, a_n)^T$, $d = (a_1, \cdots, a_j = 1, \cdots, a_n)^T$.

Because the mapping f satisfies the De Morgan law, namely $f(a \vee b) = f(a) \wedge f(b)$, then $f(a \wedge c) = f(a) \vee f(c)$, namely

$$f(a_1, \cdots, a_j = 0, \cdots, a_z = 0 \cdots, a_n)^T = (a_1, \cdots, a_i = 1, \cdots, a_n)^T \vee (a_1, \cdots, a_j = 1, \cdots, a_n)^T,$$

$$f(a_1, \cdots, a_j = 0, \cdots, a_z = 0 \cdots, a_n)^T = (a_1, \cdots, a_i = 1, \cdots, a_j = 1, \cdots, a_n)^T,$$

According to the definition of mapping g, one has $g(i) = z$, it is contradictory to $g(i) = j$. So $g^2(x) = x$. Then g is an inverse preserving association and for $\forall x \in \underline{n}$ has $g^2(x) = x$.

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