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Analytical methods for solving linear and nonlinear fuzzy integro-differential equations

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Abstract
In this article, some analytical methods, viz, the homotopy perturbation method (HPM) and the new iterative method (NIM) are proposed with reliable algorithms to solve linear and nonlinear fuzzy integro-differential equations in parametric form. Numerical examples are solved and finally comparison between the results obtained by the two methods which confirm efficiency and power of these methods in solving linear and nonlinear fuzzy integro-differential equations.

Keywords: *Fuzzy integro-differential equation; Homotopy perturbation method; New iterative method.*

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I. Introduction

The concept of fuzzy numbers and fuzzy arithmetic, introduced first by Zadeh in 1975 and then by Dubois and Prade [1], used widely in a large and various class of problems such as fuzzy linear systems [2,3], fuzzy differential equations [4,5], fuzzy integral equations [6-8] and fuzzy integro-differential equations [9-12]. More details about this concept can be found in [13,14]. In recent years, several numerical methods were suggested to solve integro-differential equations, for example, sine-cosine wavelets were used by Tavassoli, et al, to obtain a solution of linear integro-differential equations [15]. In [12] fuzzy integro-differential equation of first order derivative is solved by Abbasbandy and Hashemi using the variational iteration method (VIM). Some other worthwhile can be found in [17-21].

HPM, proposed first by Ji-Huan He [20-21], for solving differential and integral equations, linear and nonlinear, has been the subject of extensive analytical and numerical studies. The method, which is a coupling of the traditional perturbation method and homotopy in topology, deforms continuously to a simple problem which is easily solved. This method which does not require a small parameter in an equation, has a significant advantage in that it provides an analytical approximate solution to a wide range of linear and nonlinear problems in applied sciences[22-24].

NIM, proposed first by Daftardar-Gejji and Jafari [25-28], this method has proven useful for solving a variety of linear and nonlinear equations such as algebraic equations, integral equations, ordinary and partial differential equations and systems of equations as well. The NIM is simple to understand and easy to implement using computer packages and yields better results [33] than the existing Adomian decomposition method (ADM) [30] or VIM [31].

The aim of this work is to extend the analysis of the HPM and NIM with reliable algorithms to solve linear and nonlinear fuzzy integro-differential equations with comparing the obtained results.

II. Preliminaries

In this section, we set up the basic definitions of fuzzy numbers and fuzzy functions.
Definition 1 [2,3,12]: A fuzzy number in parametric form is an ordered pair of functions
$\left(\underline{u}(r), \overline{u}(r)\right)$, $0 \leq r \leq 1$ which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded left continuous nondecreasing function over [0; 1];
2. $\overline{u}(r)$ is a bounded left continuous nonincreasing function over [0; 1];
3. $\left(\underline{u}(r), \overline{u}(r)\right), 0 \leq r \leq 1$

The set of all fuzzy numbers, as given by definition 1, is denoted by E: One of the definitions for parametric form of a fuzzy number is given by Kaleva [32].
Definition 2 [33]: Let u = $\left(\underline{u}, \overline{u}\right)$ and v = $\left(\underline{v}, \overline{v}\right)$ be fuzzy numbers, then the operations of addition, subtraction

and multiplication are defined as follows:
Addition:

$$u \oplus v = (\underline{u}, \overline{u}) \oplus (\underline{v}, \overline{v}) = (\underline{u} + \underline{v}, \overline{u} + \overline{v}), \tag{1a}$$

Subtraction:

$$u \ominus v = (\underline{u}, \overline{u}) \ominus (\underline{v}, \overline{v}) = (\underline{u} - \underline{v}, \overline{u} - \overline{v}), \tag{1b}$$

Multiplication:

$$u \odot v = (\underline{u}, \overline{u}) \odot = (min(\underline{u} \cdot \underline{v}, \underline{u} \cdot \overline{v}, \overline{u} \cdot \underline{v}, \overline{u} \cdot \overline{v}), \max(\underline{u} \cdot \underline{v}, \underline{u} \cdot \overline{v}, \overline{u} \cdot \underline{v}, \overline{u} \cdot \overline{v})), \tag{1c}$$

Definition 3 [33,34]: The integral of a fuzzy function was defined in [34] by using the Riemann integral concept. Let $f : [a, b] \rightarrow E$; for each partition $p = \{t_0, \dots, t_n\}$ of [a;b] and for arbitrary $\xi_i \in [t_{i-1}, t_i], 1 \leq i \leq n$; suppose that $R_P = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})$ and $\Delta := max\{|t_i - t_{i-1}|, i = 1, \dots, n\}$ .The definite integral of f(t) over [a;b] is $\int_a^b f(t)dt = \lim_{\Delta \to 0} R_p$ provided that this limit exists in the metric D. If the fuzzy function f(t) is continuous in the metric D; its definite integral exists and also $\underline{\left(\int_a^b f(t,r)\, dt\right)} = \int_a^b \underline{f}(t,r)\, dt$ and $\overline{\left(\int_a^b f(t,r)\, dt\right)} = \int_a^b \overline{f}(t,r)\, dt$.

Definition 4 [8,12]: Let $f : [a, b] \rightarrow R_F$ and $t_0 \in (a, b)$. We say that f is first-order Hukuhara differentiable at t_0 if there exists an element $f^{/}(t_0) \in R_F$; such that for all h > 0 sufficiently small, $\exists\, f(t_0 + h) \ominus_H f(t_0), f(t_0) \ominus_H f(t_0 - h)$ and

$$\lim_{h \to 0} \frac{f(t_0+h) \ominus_H f(t_0)}{h} = \lim_{h \to 0} \frac{f(t_0) \ominus_H f \quad (t_0 - h)}{h} = f^{/}(t_0) \tag{2}$$

where $\ominus_H$ is Hukuhara difference, R_F is the set of all fuzzy functions and the limit is in the metric D.
In parametric form, if $f = (\underline{f}, \overline{f})$; then $f^{/} = (\underline{f}^{/}, \overline{f}^{/})$, where $\underline{f}^{/}, \overline{f}^{/}$ are the first-order Hukuhara differentiable of $\underline{f}^{/}, \overline{f}^{/}$ respectively.

Definition 5 [12]: Let $f^{/} : (a, b) \rightarrow R_F$ and $t_0 \in (a, b)$ where $f^{/}$ is the first-order Hukuhara differentiable of f at t_0: We say that $f^{/}$ is second-order Hukuhara differentiable at t_0 if there exists an element $f^{//}(t_0) \in R_F$; such that for all $h > 0$ sufficiently small, $\exists\, f^{/}(t_0 + h) \ominus_H f^{/}(t_0),\ f^{/}(t_0) \ominus_H f^{/}(t_0 - h)$ and

$$\lim_{h \to 0} \frac{f^{/}(t_0+h) \ominus_H f^{/}(t_0)}{h} = \lim_{h \to 0} \frac{f^{/}(t_0) \ominus_H f^{/}(t_0+h)}{h} = f^{//}(t_0) \tag{3}$$

In parametric form, if $f = (\underline{f}, \overline{f})$; then $f^{//} = (\underline{f}^{//}, \overline{f}^{//})$, where $\underline{f}^{//}, \overline{f}^{//}$ are the second-order Hukuhara differentiable of $\underline{f}^{//}, \overline{f}^{//}$ respectively.

III. Fuzzy integro-differential equation

Consider the second-order derivative integro-differential equation:

$$u^{//}(t) + u(t) + \lambda \int_0^T k(s,t)\, u^{/}(s) ds = g(t) \tag{4}$$

where $t \in [0, T]; \lambda \in R$; $g(t)$ is a known function and the kernel $k(s;t) > 0$ with the initial conditions:

$$u(0) = h_0, u^{/}(0) = h_1,\ h_i \in R, i = 0,1 \tag{5}$$

In the fuzzy case; that is u and g be fuzzy functions, let:

$$u(t,r) = (\underline{u}(t,r); \overline{u}(t,r)), g(t,r) = (\underline{g}(t,r); \overline{g}(t,r)),$$

$$u^{(n)}(t,r) = (\underline{u}^{(n)}(t,r), \overline{u}^{(n)}(t,r)), n = 1,2 \tag{6}$$

where all derivatives are with respect to t; be fuzzy functions. Therefore, related fuzzy integro-differential equation, Eq. (4) can be written in the form:

$$\underline{u}^{//}(t,r) + \underline{u}(t,r) + \lambda \int_0^T k(s,t)\, \underline{u}^{/}(s,r) ds = \underline{g}(t,r) \tag{7a}$$

$$\overline{u}^{//}(t,r) + \overline{u}(t,r) + \lambda \int_0^T k(s,t)\, \overline{u}^{/}(s,r) ds = \overline{g}(t,r) \tag{7b}$$

IV. The analysis of the considered methods

In this section we analyze the considered methods for solving fuzzy integro-differential equations.

4.1. HPM

To illustrate the basic idea of the HPM, we consider the following nonlinear differential equation [19-23]:

$$L(u) + N(u) = f(t);\ t \in \Omega \tag{8}$$

with the boundary conditions:

$$B\left(u, \frac{\partial u}{\partial t}\right) = 0, t \in \Gamma \tag{9}$$

Where L is a linear operator, N is a nonlinear operator, B is a boundary operator, $f(t)$ is a known analytic function and Γ is the boundary of the domain Ω.

By the homotopy technique, we construct a homotopy $v(t,p) : \Omega \times [0,1] \to R$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(t)] = 0 \tag{10a}$$

$$H(v,p) = \left[L(v) - L(u_0) + pL(u_0) + p[N(v) - f(t)]\right] = 0 \tag{10b}$$

Where $t \in \Omega, p \in [0,1]$ is an impeding parameter and u_0 is an initial approximation which satisfies the boundary conditions. Obviously, from (10), we have:

$$H(v,0) = L(v) - L(u_0) = 0, H(v,1) = L(v) + N(v) - f(t) = 0 \tag{11}$$

The change process of p from zero to unity is just that of $v(t, p)$ from u_0 to u. In topology, this is called deformation, $L(v) - L(u_0)$ and $L(v) + N(v) - f(t)$ are called homotopic. The basic assumption is that the solution of Eq. (10) can be expressed as a power series in p :

$$v = v_0 + pv_1 + p^2v_2 + \cdots \tag{12}$$

Setting $p = 1$; the approximate solution of Eq. (8) is given by:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{13}$$

The convergence of the series (13) has been proved in [20].

4.2. Reliable algorithm of HPM

According to the above homotopy technique, we introduce the following reliable algorithm for solving Eq. (4):

$$u^{//}(t) + u(t) - g(t) = p\left[-\lambda \int_0^\lambda k(s,t)u^{/}(s)ds\right] \tag{14a}$$

or

$$u^{//}(t) - g(t) = p\left[-u(t) - \lambda \int_0^\lambda k(s,t)u^{/}(s)ds\right] \tag{14b}$$

Where $p \in [0;1]$. The homotopy parameter p always changes from zero to unity. In case $p = 0$; Eq. (14) becomes the differential equation:

$$u^{//}(t) = g(t) - u(t) \tag{15a}$$

Or

$$u^{//}(t) = g(t) \tag{15b}$$

and when $p = 1$; Eq. (14) turns out to be the original integro-differential Eq. (4). The basic assumption is that the solution of Eq. (14) can be written as a power series in p :

$$u = u_0 + pu_1 + p^2u_2 + \cdots, \tag{16}$$

and therefore, we approximate the solution $u(t)$ by:

$$u(t) = \sum_{i=0}^{\infty} u_i(t) \tag{17}$$

In view of the above algorithm, we can construct the following algorithm for solving the two fuzzy integro-differential Eqs. (7)

$$\underline{u}^{//}(t,r) + \underline{u}(t,r) - \underline{g}(t,r) = p\left[-\lambda \int_0^T k(s,t)\underline{u}^{/}(s,r)ds\right], \tag{18a}$$

$$\overline{u}^{//}(t,r) + \overline{u}(t,r) - \overline{g}(t,r) = p\left[-\lambda \int_0^T k(s,t)\overline{u}^{/}(s,r)ds\right], \tag{18b}$$

Or

$$\underline{u}^{//}(t,r) - \underline{g}(t,r) = p\left[-\underline{u}(t,r) - \lambda \int_0^T k(s,t)\underline{u}^{/}(s,r)ds\right], \tag{19a}$$

$$\overline{u}^{//}(t,r) - \overline{g}(t,r) = p\left[-\overline{u}(t,r) - \lambda \int_0^T k(s,t)\overline{u}^{/}(s,r)ds\right] \tag{19b}$$

4.3. NIM

Also, to illustrate the basic idea of the NIM, we consider the following general functional equation [25-28]:

$$u(t) = f(t) + N\big(u(t)\big) \qquad (20)$$

where N is a nonlinear operator from a Banach space $B \to B$ and $f(t)$ is a known function. We are looking for a solution u of Eq. (20) having the series form:

$$u(t) = \sum_{i=0}^{\infty} u_i(t) \qquad (21)$$

The nonlinear operator N can be decomposed as:

$$N\left(\sum_{i=0}^{\infty} u_i(t)\right) = N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\} \qquad (22)$$

From Eqs. (21) and (22), Eq. (20) is equivalent to:

$$\sum_{i=0}^{\infty} u_i = f(t) + N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\} \qquad (23)$$

The required solution or Eq. (20) can be obtained recurrencely from the recurrence relation:

$$\begin{cases} u_0 = f \\ u_1 = N(u_0) \\ u_{n+1} = N\left(\sum_{i=0}^{n} u_i\right) - N\left(\sum_{i=0}^{n-1} u_i\right), n = 1,2,\dots. \end{cases} \qquad (24)$$

Then

$$\sum_{i=0}^{n+1} u_i = N\left(\sum_{i=0}^{n} u_i\right), n = 0,1,2,\dots.. \qquad (25)$$

And

$$\sum_{i=0}^{\infty} u_i = f + N\left(\sum_{i=0}^{\infty} u_i\right) \qquad (26)$$

The *n-term* approximate solution for Eq. (9) is given by: $u(t) = \sum_{i=0}^{\infty} u_i$.

If N is a contraction, i.e. $\|N(x) - N(y)\| \le k\|x - y\|, 0 < k < 1$, thgen:

$$\|u_{n+1}\| \le k^{n+1}\|u_0\|, n = 0,1,2,\dots.. \qquad (27)$$

and the series $\sum_{i=0}^{\infty} u_i$ absolutely and uniformly converges to a solution of Eq. (20) [35], which is unique in view of the Banach xed point theorem [36]. The convergence of the NIM has been proved in [27,28].

4.4. Reliable algorithm of NIM

According to NIM, we introduce the following reliable algorithm for solving Eq. (4):

$$u(t) = \sum_{k=0}^{1} h_k \frac{t^k}{k!} + I_t^2[g(t)] + I_t^2\left[-u(t) - \lambda \int_0^T k(s,t)u^{/}(s)ds\right] = f + N(u) \quad (28)$$

Where I_t^2 is a *second-order (2-fold)* integral operator, $f = \sum_{k=0}^{1} h_k \frac{t^k}{k!} + I_t^2[g(t)]$ and $N(u) = I_t^2\left[-u(t) - \lambda \int_0^T k(s,t)u^{/}(s)ds\right]$. We get the solution of (28) by employing the recurrence relation (24).

In view of the above algorithm, we can construct the following algorithm for solving the two fuzzy integro-differential Eqs. (7):

$$\underline{u}(t,r) = \sum_{k=0}^{1} \underline{h}_k(0,r)\frac{t^k}{k!} + I_t^2\left[\underline{g}(t,r)\right] + I_t^2\left[-\underline{u}(t,r) - \lambda \int_0^T k(s,t)\underline{u}^{/}(s,r)ds\right] = \underline{f} + N\big(\underline{u}\big) \quad (29a)$$

$$\overline{u}(t,r) = \sum_{k=0}^{1} \overline{u}_k(0,r)\frac{t^k}{k!} + I_t^2[\overline{g}(t,r)] + I_t^2\left[-\overline{u}(t,r) - \lambda \int_0^T k(s,t)\overline{u}^{/}(s,r)ds\right] = \overline{f} + N(\overline{u}) \quad (29b)$$

Where $\underline{f} = \sum_{k=0}^{1} \underline{h}_k(0,r)\frac{t^k}{k!} + I_t^2\left[\underline{g}(t,r)\right]$, $N\big(\underline{u}\big) = I_t^2\left[-\underline{u}(t,r) - \lambda \int_0^T k(s,t)\underline{u}^{/}(s,r)ds\right]$, $\overline{f} = \sum_{k=0}^{1} \overline{u}_k(0,r)\frac{t^k}{k!} + I_t^2[\overline{g}(t,r)]$, $N(\overline{u}) = I_t^2\left[-\overline{u}(t,r) - \lambda \int_0^T k(s,t)\overline{u}^{/}(s,r)ds\right]$ and $\underline{h}_k(0,r), \overline{h}_k(0,r), k = 0,1$ are the initial values for the fuzzy integro-differential equations (7).

Remark 1: when the general functional (20) is linear, the recurrence relation (24) can be simplified in the form:

$$\begin{cases} u_0 = f, \\ u_{n+1} = N(u_n), n = o,1,\dots. \end{cases} \qquad (30)$$

proof: See [41].

The main advantage of the new modifications of the two methods, as shown in the following section, is that they can be applicable simply to a wide class of linear and nonlinear fuzzy integro-differential equations.

V. Illustrative examples

In this section one example are solved in order to demonstrate the simplicity and efficiency of the two methods.

Example Consider the linear fuzzy integro-differential equation:

$$u^{//}(t,r) + \lambda \int_0^1 k(t,r)u(t,r)ds = g(t,r), 0 \le t, r \le 1, \quad (31a)$$

with the initial conditions:

$$u(0,r) = (0,6), u^{/}(0,r) = (3r, -r). \quad (31b)$$

Where $u(t,r) = \left(\underline{u}(t,r), \overline{u}(t,r)\right), \lambda = 1, k(s,t) = s^2 t, g(t,r) = \left(\underline{g}(t,r), \overline{g}(t,r)\right) = \left(\frac{3rt}{4}, 2t - rt/4\right)$.

HPM: According to Eq. (14), the homotopy for (31) becomes.

$$u^{//}(t,r) - g(t,r) = -p\left[t\int_0^1 s^2 u(s,r)ds\right],$$

and the homotopy for $\underline{u}$ and $\overline{u}$; according to Eq. (19), becomes:

$$\underline{u}^{//}(t,r) - \underline{g}(t,r) = -p.\left[t.\int_0^1 s^2 \underline{u}(s,r)ds\right], \quad (32a)$$

$$\overline{u}^{//}(t,r) - \overline{g}(t,r) = -p\left[t\int_0^1 s^2 \overline{u}(s,r)ds\right], \quad (32b)$$

Substituting: $\underline{u} = \underline{u}_0 + p\underline{u}_1 + p^2\underline{u}_2 + \cdots, \overline{u} = \overline{u}_0 + p\overline{u}_1 + p^2\overline{u}_2 + \cdots$ and the initial values: $\underline{u}(0,r) = 0, \underline{u}^{/} = (0,r) = 3r, \overline{u}(0,r) = 6 \text{ and } \overline{u}^{/}(0,r) = -r$ into (32) and equating the terms with equal powers of p; we can obtain the following two sets of integro-differential equations for $\underline{u}$ and $\overline{u}$:

$$p^0: \underline{u}_0^{//} = \frac{3rt}{4}, \underline{u}_0(0.r) = 0, \underline{u}_0^{/}(0,r) = 3r,$$

$$\overline{u}_0^{//} = 2t - \frac{rt}{4}, \overline{u}_0(0,r) = 6, \overline{u}_0{}^{/}\ (0,r) = -r',$$

$$p^1: \underline{u}_1^{//} = -t\int_0^1 s^2 \underline{u}_0(s,r)ds, \underline{u}_1(0,r) = 0, \underline{u}_1^{/}(0,r) = 0,$$

$$\underline{u}_1^{//} = -t\int_0^1 s^2 \underline{u}_0(s,r)ds, \qquad \underline{u}_1(0,r) = 0, \qquad \underline{u}_1^{/}(0,r) = 0,$$

$$p^2: \underline{u}_2^{//} = -t\int_0^1 s^2 \underline{u}_1(s,r)ds, \underline{u}_2(0,r) = 0, \underline{u}_2^{/}(0,r) = 0,$$

$$\underline{u}_2^{//} = -t\int_0^1 s^2 \underline{u}_1(s,r)ds, \underline{u}_2(0,r) = 0, \underline{u}_2^{/}(0,r) = 0,$$

Consequently, by solving the above two sets of integro-differential equations, the rst few components of the homotopy perturbation solutions for $\underline{u}$ and $\overline{u}$ are obtained in the forms:

$$\underline{u}_0(t,r) = 3rt + \frac{rt^3}{8}, \overline{u}_0(t,r) = 6 - rt + \frac{t^3}{3} - \frac{rt^3}{24},$$

$$\underline{u}_1(t,r) = -\frac{37rt^3}{288}, \overline{u}_1(t,r) = -\frac{37t^3}{108} + \frac{37rt^3}{864},$$

$$\underline{u}_2(t,r) = \frac{37rt^3}{10368}, \overline{u}_2(t,r) = \frac{37t^3}{3888} - \frac{37rt^3}{31104},$$

and so on. The 6-term homotopy perturbation solution for (31) is given by:

$$\underline{u}(t,r) = \sum_{i=0}^{5} \underline{u}_i\,(t,r) = 3rt - \frac{rt^3}{483729408}, \quad (33a)$$

$$\overline{u}(t,r) = \sum_{i=0}^{5} \overline{u}_i = 6 - rt - \frac{t^3}{181398528} + \frac{rt^3}{1451188224}, \quad (33b)$$

which converge to the exact solution, i.e. $\underline{u}(t,r) = \sum_{i=0}^{\infty} \underline{u}_i(t,r) = 3rt$ and $\overline{u}(t,r) = \sum_{i=0}^{\infty} \overline{u}_i(t,r) = 6 - rt$.

NIM: According to (7), Eq. (31) gives the integro-differential equations:

$$\underline{u}^{//}(t,r) + t\int_0^1 s^2 \underline{u}(s,r)ds = \underline{g}(t,r),\ \underline{u}(0,r) = 0,\ \underline{u}^{/}(0,r) = 3r, \quad (34a)$$

$$\overline{u}^{//}(t,r) + t\int_0^1 s^2 \overline{u}(s,r)ds = \overline{g}(t,r),\ \overline{u}(0,r) = 6,\ \overline{u}^{/}(0,r) = -r, \quad (34b)$$

Therefore, from (29), Eqs (34) are equivalent to the two integral equations:

$$\underline{u}(t,r) = 3rt + \frac{rt^3}{8} - I_t^2\left[t\int_0^1 s^2 \underline{u}(s,r)ds\right], \quad (35a)$$

$$\overline{u}(t,r) = 6 - rt + \frac{t^3}{3} - \frac{rt^3}{24} - I_t^2\left[t\int_0^1 s^2 \overline{u}(s,r)ds\right] \quad (35b)$$

Let $N(\underline{u}) = -I_t^2\left[t\int_0^1 s^2 \underline{u}(s,r)ds\right], N(\overline{u}) = -I_t^2\left[t\int_0^1 s^2 \overline{u}(s,r)ds\right]$. Therefore, from (30), we can obtain the following first few components of the new iterative solution for $\underline{u}$ and $\overline{u}$ for (31):

$$\underline{u}_0(t,r) = 3rt + \frac{rt^3}{8}, \qquad \overline{u}_0(t,r) = 6 - rt + \frac{t^3}{3} - \frac{rt^3}{24},$$

$$\underline{u}_1(t,r) = -\frac{37rt^3}{288}, \qquad \overline{u}_1(t,r) = -\frac{37t^3}{108} + \frac{37rt^3}{864},$$

$$\underline{u}_2(t,r) = \frac{37rt^3}{10368}, \qquad \overline{u}_2(t,r) = \frac{37t^3}{3888} - \frac{37rt^3}{31104},$$

and so on. The *6-term* new iterative solution for (31) is given by:

$$\underline{u}(t,r) = \sum_{i=0}^{5} \underline{u}_i\,(t,r) = 3rt - \frac{rt^3}{483729408}, \quad (36a)$$

$$\overline{u}(t,r) = \sum_{i=0}^{5} \underline{u}_i(t,r) = 6 - rt - \frac{t^3}{181398528} + \frac{rt^3}{1451188224}, \quad (36b)$$

which is the same result as obtained by HPM in (33).

Let us dene the absolute value of the *nth-term* error by: $|e_n(t,r)| = |u^*(t,r) - u_n(t,r)|$, where $u^*(t,r)$ is the exact solution and $u_n(t,r)$ is the *nth-term* approximate solution obtained by HPM or NIM. It is clear from (33) and (36) that the smallest value of the absolute *nth-term* error is at *t = r = 0*; that is $|e_n(0,0)| = |\underline{e}_n(0,0) - \overline{e}_n(0,0)| = (0,0), n = 0,1,..,5$ and the largest value is at *t = r = 1* as shown in Table 1 and that $|e_n| \to 0\ as\ n \to \infty$. It is clear from the table that as the number of iterations becomes large, the absolute error becomes small. In Fig. 1, we have plotted the approximate solution for (31), obtained by HPM and NIM in (33) and (36) and the corresponding exact solution for $\underline{u}$ and $\overline{u}$ at *r = 1* : It is clear, from the Figs. that the two solutions are almost equal.

Table 1

n	0	1	2	3	4	5
$\lvert\underline{e}_n(1,1)\rvert$	1.25^{-1}	3.472^{-3}	9.645^{5}	2.6792^{-6}	7.4422^{-8}	2.0673^{-9}
$\lvert\overline{e}_n(1,1)\rvert$	2.9167^{-1}	8.1019^{-3}	2.2505^{-4}	6.2514^{-6}	1.7365^{-7}	4.8236^{-9}

VI. Conclusion

In this work, the HPM and the NIM are successfully applied with reliable algorithms for solving linear and nonlinear fuzzy integro-differential equations. The obtained results show that the two methods are easy, simple and equal. Therefore, these two methods are very powerful and efficient techniques for solving linear and nonlinear fuzzy integro-differential equations. In conclusion, the HPM and the NIM may be considered as a nice refinement in existing numerical techniques and might find the wide applications.

Conflict of interests
The authors declare that there is no conflict of interests regarding the publication of this article.

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