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STRONG COMPLETE PERMUTATION POLYNOMIAL OF THE FORM $x^{1+\frac{q^s-1}{q-1}}+bx$ IN $\mathbb{F}_{q^s}, s=4$

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ABSTRACT. We will be observing polynomial $x^{1+\frac{q^s-1}{q-1}}+bx$ for $s=4$, analogously with cases for $s=2$ and $s=3$ in [3] provided that it is permutation polynomial, complete polynomial and strong complete permutation polynomial. These polynomials, as well as other similar polynomials, are being applied in cryptography, theory of codes, combinatorial design, and especially for Knut Vik design and solution of the n queens problem. For the construction of such polynomials, we have used the existing complete permutation polynomial and we have added additional conditions so that we have received strong complete permutation polynomial.

Keywords: Finite field, permutation polynomial, complete permutation polynomial, strong complete permutation polynomial, problem of n queens.

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1. INTRODUCTION

Let p be a prime number, let n be a positive integer and $q=p^n$. Suppose $\mathbb{F}_q$ is finite field of order q. Polynomial $f(x)$ in $\mathbb{F}_q[x]$ is called permutation polynomial (PP) in $\mathbb{F}_q$ if it is induced with mapping $c \mapsto \theta(c)$ from $\mathbb{F}_q$ in $\mathbb{F}_q$, which is permutation from $\mathbb{F}_q$, that is bijective function. Permutation polynomial $f(x)$ is called complete permutation polynomial (CPP) from $\mathbb{F}_q$ if $f(x)+x$ is a permutation polynomial. Permutation polynomial $f(x)$ is called orthomorphism from $\mathbb{F}_q$ if $f(x)-x$ is permutation polynomial. Permutation polynomial $f(x)$ is called **Strong Complete Polynomial** (SCP) over $\mathbb{F}_q$ if $f(x)+x$ and $f(x)-x$ are permutation polynomials from $\mathbb{F}_q$. These polynomials are induced by strong complete mapping at additive group $(\mathbb{F}_q,+)$.

The number of strong complete polynomials from $\mathbb{F}_q$ is smaller than the number of permutation polynomials, complete permutation polynomials and the number of orthomorphism because strong complete polynomial does not exist when q is an even number and because of several conditions due to which some polynomial is strong complete polynomial.

More details regarding permutation polynomials, complete permutation polynomials and strong permutation polynomials over finite field are available in [1], [3], [6], [7], [8] [16], [21], [22], [23], [25], [28], [29], [31] [32],[33], [34].

n queen problem is the placement of n queens on chessboard of dimensions $n \times n$, or at torus so that queens cannot be attacked, or that maximum one queen is placed in every row, column and diagonal. This problem is generalization of the 8 queens problem that was set by M.Bezzel in 1848, but the problem continues to be solved even today.

Standard (simple) n queens problems is the placement of n queen on chessboard, in format $n \times n$, so that neither pair of queens is mutually attacked, that is, they do not belong to the same row, column and diagonal. (Picture 1).

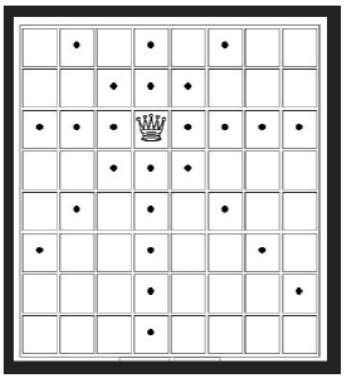

FIGURE 1. Standard n queen problem

Modular n queen problem is placement of n queen on modular board of dimension $n \times n$, that is, modular table or torus (specifically, the surface in three-dimensional space that is obtained by joining two opposite edges of the square of dimensions $n \times n$), so that two queens at torus $n \times n$ do not belong to the same row, column and diagonal.

N.J.Fine in [24]] showed that modular solution of the problem n queen does not exist for n even number.

T. Kløve in ([14], Theorem 1) demonstrated that modular n queen problem has solution if and only if $gcd(n, 6) = 1$, where gcd is the greatest common divisor.

More information concerning the n queen problem is available in [5], [12], etc.

With strong complete polynomials in the finite field $\mathbb{F}_q$ can be solved modular n queen problem, where $n = q = p^m$, m is a natural number, p is a prime odd number and $gcd(n, 6) = 1$. Likewise, with strong complete polynomials, Knut Vik design can be created (see [10], [11]).

For polynomial of the form $x^{\frac{q-1+n}{n}} + bx \in \mathbb{F}_q[x]$ is important criterion that demonstrates when it is permutation polynomial.

Lemma 1.1 (Lemma 1, [23]). *If $n \geq 2$ is integer number such that $q \equiv 1 \pmod{n}$, then $x^{\frac{q-1+n}{n}} + bx \in \mathbb{F}_q[x]$ is permutation polynomial from $\mathbb{F}_q$ if and only if the following conditions are met:*

(i) $(-b)^n \neq 1$

(ii) $\Psi_n((b+\omega^i)(b+\omega^j)^{-1}) \neq \omega^{j-i}$ *for alle* $0 \leq i < j < n$, *where* $\Psi_n(x) = x^{\frac{q-1}{n}}$ *n-th character and* ω *fixed primitive n-th root of the unit in* $\mathbb{F}_q$.

In this paper, we are examining polynomial $x^{1+\frac{q^s-1}{q-1}} + bx$ for $s = 4$, analogously with cases for $s = 2$ and $s = 3$ in [3]. Furthermore, we are examining for which conditions this polynomial is complete permutation polynomial and strong complete permutation polynomial.

2. Strong Complete Permutation Polynomial of the form $x^{1+\frac{q^s-1}{q-1}}+bx$ in $\mathbb{F}_{q^s}, s=4$

L.A.Bassalygo i V.A.Zinoviev [3] were studying the polynomial of the following form

$$x^{1+\frac{q^s-1}{q-1}}+bx,$$

over $\mathbb{F}_{q^s}$ for $s=2$ (see alternatively [27], [25]) and $s=3$ (vidi alternativno [28], [32]), that is, polynomials with form $x^{q+2}+bx$ and $x^{q^2+q+2}+bx$ and they examined conditions to be permutation polynomials and complete polynomials. At the end of the article [3], autors suggested to continue examining polynomial with form $x^{1+\frac{q^s-1}{q-1}}+bx$, for $s\geq 4$ over $\mathbb{F}_{q^s}$. Also Wu G., Li N., Helleseth T. and Zhang Y. ([33], Theorem 4.1, Theorem 4.13) examined the polynomial with form $x^{1+\frac{q^s-1}{q-1}}+bx, s\geq 4$ over $\mathbb{F}_{q^s}$ for $s=4$ and partially for $s=6$ (for $p=3$ and $p=5$). In [32] they found three classes of complete polynomials $x^{1+\frac{q^s-1}{q-1}}+bx$ for $p=2$ and for $s=4, s=6, s=10$. Methods used in [25], [27], [28], [32] are different compared to [3].

Let us observe polynomial $x^{1+\frac{q^s-1}{q-1}}+bx$ for $s=4$, analogously with cases for $s=2$ and $s=3$ in [3].

Theorem 2.1. *Polynomial $x^{q^3+q^2+q+2}+bx$ is permutation polynomial over $\mathbb{F}_{q^4}$ if and only if $b\in\mathbb{F}_{q^4}\backslash\mathbb{F}_q$ and equation*

$$x^4+x^3y+x^2y^2+xy^3+y^4+(x^3+x^2y+xy^2+y^3)B_1+(x^2+xy+y^2)B_2+$$
$$+(x+y)B_3+B_4=0 \tag{2.1}$$

does not have a solution in $\mathbb{F}_q$ for $x\neq 0, y\neq 0, x\neq y$, where:

$$B_1=b^{q^3}+b^{q^2}+b^q+b,$$
$$B_2=b^{q^3+q^2}+b^{q^3+q}+b^{q^3+1}+b^{q^2+q}+b^{q^2+1}+b^{q+1},$$
$$B_3=b^{q^3+q^2+q}+b^{q^3+q^2+1}+b^{q^3+q+1}+b^{q^2+q+1},$$
$$B_4=b^{q^3+q^2+q+1}.$$

Proof. Analogously to Lemma 1.1, we consider the field $\mathbb{F}_{q^4}$, for $n=q-1$. Then, condition (i) in Lemma 1.1, which is $(-b)^n\neq 1$ equivalent with $b\in\mathbb{F}_{q^4}\backslash\mathbb{F}_q$, because for $b\in\mathbb{F}_q$ we have $(-b)^n=(-b)^{q-1}=1$.

If we replace $x=\omega^i, y=\omega^j$, then inequation (ii) in Lemma 1.1 takes the form:

$$((b+x)(b+y)^{-1})^{q^3+q^2+q+1}\neq yx^{-1},$$

which we can write in the form

$$x(b+x)^{q^3+q^2+q+1}\neq y(b+y)^{q^3+q^2+q+1},$$

for each $x,y\in\mathbb{F}_q$, so that $x\neq 0, y\neq 0, x\neq y$.

Now, polynomial $x^{q^3+q^2+q+2}+bx$ is permutation polynomial over $\mathbb{F}_{q^4}$ if and only if $b\in\mathbb{F}_{q^4}\backslash\mathbb{F}_q$ and the equation

$$x(b+x)^{q^3+q^2+q+1}=y(b+y)^{q^3+q^2+q+1} \tag{2.2}$$

does not have a solution in $\mathbb{F}_q$ for every $x,y\in\mathbb{F}_q$ for which it is $x\neq 0, y\neq 0, x\neq y$.

Since in finite field $\mathbb{F}_q$, except $a^q=a$, it is ([18], Theorem 1.46):

$$(a+b)^{q^t} = a^{q^t} + b^{q^t}, a, b \in \mathbb{F}_q, t \in \mathbb{N},$$

then we have:

$$(b+x)^{q^3+q^2+q+1} = (b^{q^3}+x)(b^{q^2}+x)(b^q+x)(b+x) \tag{2.3}$$

$$(b+y)^{q^3+q^2+q+1} = (b^{q^3}+y)(b^{q^2}+y)(b^q+y)(b+y) \tag{2.4}$$

If we replace relations (2.3) and (2.4) in equation (2.2), then we get:

$$(x-y)[x^4+x^3y+x^2y^2+xy^3+y^4+(x^3+x^2y+xy^2+y^3)(b^{q^3}+b^{q^2}+b^q+b)+$$

$$+(x^2+xy+y^2)(b^{q^3+q^2}+b^{q^3+q}+b^{q^3+1}+b^{q^2+q}+b^{q^2+1}+b^{q+1})+$$

$$+(x+y)(b^{q^3+q^2+q}+b^{q^3+q^2+1}+b^{q^3+q+1}+b^{q^2+q+1})+b^{q^3+q^2+q+1}]=0$$

Since $x \neq y$, then we divide the above equation with $x-y$ so that remains the equation (2.1) that does not have a solution in $\mathbb{F}_q$, for every $x, y \in \mathbb{F}_q$, for $x \neq 0, y \neq 0, x \neq y$, where B_1, B_2, B_3, B_4 are values as seen in Theorem. □

It is noticeable that in finite filed with characteristic 2 it is meaningless to consider strong complete polynomial $f(x) \in \mathbb{F}_q[x]$. In order for $f(x))$ to be strong complete polynomial it is necessary that $f(x), f(x)+x, f(x)-x$ to be permutation polynomials. However, if $p=2$, $f(x)-x = f(x)+x$ so that strong complete polynomial comes down to complete polynomial. This is why we take p prime number , $p \geq 3$ and finite field $\mathbb{F}_q$ with $q = p^m$ elemennts.

In that case, for every $x, y \in \mathbb{F}_q$ applies equation $4xy = (x+y)^2-(x-y)^2$, and if parts of equation (2.1) are transformed as follows:
$x^2+xy+y^2 = (x+y)^2 - xy = (x+y)^2 - \frac{1}{4}((x+y)^2-(x-y)^2) =$
$= \frac{1}{4}(3(x+y)^2+(x-y)^2),$
$x^3+x^2y+xy^2+y^3 = (x+y)^3 - 2xy(x+y) = (x+y)((x+y)^2-$
$-\frac{1}{2}((x+y)^2-(x-y)^2)) =$
$= \frac{1}{2}(x+y)((x+y)^2+(x-y)^2),$
$x^4+x^3y+x^2y^2+xy^3+y^4 = (x+y)^4 - 3xy(x+y)^2+(xy)^2 =$
$= \frac{5}{16}(x+y)^4 + \frac{5}{8}(x+y)^2(x-y)^2 + \frac{1}{16}(x-y)^4,$
then equation (2.1) is equivalent to equation:

$$5(x+y)^4+10(x+y)^2(x-y)^2+(x-y)^4+8(x+y)((x+y)^2+$$

$$+(x-y)^2)B_1+4(3(x+y)^2+(x-y)^2)B_2+16(x+y)B_3+16B_4=0 \tag{2.5}$$

If we introduce change $x+y=z, x-y=u$, in equation (2.5), then we have equation

$$u^4+(10z^2+8zB_1+4B_2)u^2+5z^4+8z^3B_1+$$

$$+12z^2B_2+16zB_3+16B_4=0 \tag{2.6}$$

Instead of conditions $x \neq 0, y \neq 0, x \neq y$ which are in Theorem 2.1, now we have $z \in \mathbf{F}_q, u \neq 0, u \neq \pm z$. Indeed, from $x \neq y$ it ensues $u \neq 0$. Besides, if $u = \pm z$, then by replacement in equation (2.6), we obtain

$$z^4+B_1z^3+B_2z^2+B_3z+B_4 = (z+b)^{q^3+q^2+q+1} = 0,$$

where it is not possible for $z \in \mathbb{F}_q$ and $b \in \mathbb{F}_{q^4} \backslash \mathbb{F}_q$. So, new conditions remain $z \in \mathbb{F}_q$, $u \in \mathbb{F}_q^* = \mathbb{F}_q \backslash \{0\}$.

Finally, we have

Proposition 2.2. *Suppose $q = p^m$, where $p \geq 3$. Polynomial $x^{q^3+q^2+q+2}+bx$ is permutation polynomial over finite field F_{q^4} if and only if $b \in \mathbb{F}_{q^4}\backslash\mathbb{F}_q$ and equation (2.6) does not have a solution in $\mathbb{F}_q$, for $u \in \mathbb{F}_q^* = \mathbb{F}_q\backslash\{0\}$, $z \in \mathbb{F}_q$, where*

$B_1 = b^{q^3} + b^{q^2} + b^q + b,$
$B_2 = b^{q^3+q^2} + b^{q^3+q} + b^{q^3+1} + b^{q^2+q} + b^{q^2+1} + b^{q+1},$
$B_3 = b^{q^3+q^2+q} + b^{q^3+q^2+1} + b^{q^3+q+1} + b^{q^2+q+1},$
$B_4 = b^{q^3+q^2+q+1}.$

Since the function

$$F(u,z) = u^4+(10z^2+8zB_1+4B_2)u^2+5z^4+8z^3B_1+12z^2B_2+16zB_3+16B_4$$

(left side of equation (2.6)), is even function by variable u, that means $F(u,z) = F(-u,z)$ and that is why we calculate twice smaller values of function $F(u,z)$. Since equation (2.6) is biquadratic equation by variable u, then that equation can come down to the form:

$$(u^2 + \frac{A}{2})^2 = (\frac{A}{2})^2 - B, \tag{2.7}$$

where

$$A = 10z^2 + 8zB_1 + 4B_2$$

$$B = 5z^4 + 8z^3B_1 + 12z^2B_2 + 16zB_3 + 16B_4.$$

Equation (2.7) can be interpreted so that $(\frac{A}{2})^2 - B$ a reduced square from $u^2 + \frac{A}{2}$ in $\mathbb{F}_{q^4}$. Since in finite field not all elements are reduced squares, then we can formulate as

Proposition 2.3. *Suppose that $q = p^m$, where $p \geq 3$. Polynomial $x^{q^3+q^2+q+2}+bx$ is permutation polynomial over finite field $\mathbb{F}_{q^4}$ if and only if $b \in \mathbb{F}_{q^4}\backslash\mathbb{F}_q$ and $(\frac{A}{2})^2 - B$ is not a reduced square of $u^2 + \frac{A}{2}$ in $\mathbb{F}_{q^4}$, for eache $u \in \mathbb{F}_q^*, z \in \mathbb{F}_q$, where*

$A = 10z^2 + 8zB_1 + 4B_2,$
$B = 5z^4 + 8z^3B_1 + 12z^2B_2 + 16zB_3 + 16B_4,$
$B_1 = b^{q^3} + b^{q^2} + b^q + b,$
$B_2 = b^{q^3+q^2} + b^{q^3+q} + b^{q^3+1} + b^{q^2+q} + b^{q^2+1} + b^{q+1},$
$B_3 = b^{q^3+q^2+q} + b^{q^3+q^2+1} + b^{q^3+q+1} + b^{q^2+q+1},$
$B_4 = b^{q^3+q^2+q+1}.$

Now, we can present a method of finding elements $b \in \mathbb{F}_{q^4} \backslash \mathbb{F}_q$ provided that equation (2.6) does not have solution in $\mathbb{F}_q$, for $u \in \mathbb{F}_q^*, z \in \mathbb{F}_q$.

For every arranged pair $(u, z) \in \mathbb{F}_q^* \times \mathbb{F}_q$, from the condition that equation (2.6) does not have a solution in $\mathbb{F}_q$ we need to find, $b_i = \alpha^i \in \mathbb{F}_{q^4} \backslash \mathbb{F}_q$, for $1 \leq i \leq q^4 - 1$. If we mark

$$T = \{1 \leq i \leq q^4 - 1 : b = b_i = \alpha^i \in F_{q^4} \backslash F_q\},$$

so that polynomial $x^{q^3+q^2+q+2} + bx$ is permutation polynomial. If $gcd(q^3 + q^2 + q + 2, q^4 - 1) = 1$, then for $b = b_i, i \in T$ polynomial $b^{-1}x^{q^3+q^2+q+2}$ 2 is complete polynomial. In this case $d = q^3 + q^2 + q + 2$ is called CP exponent of polynomial $b^{-1}x^{q^3+q^2+q+2}$.

For $b = -\alpha^i, i \in T$, polynomial $x^{q^3+q^2+q+2} - bx$ is also permutation polynomial, so that if $gcd(q^3 + q^2 + q + 2, q^4 - 1) = 1$, then polynomial $b^{-1}x^{q^3+q^2+q+2}$ is strong permutation polynomial. Now in the rest of the paper, we can seek, if there exists $i \in T$ so that except $f(x) = x^{q^3+q^2+q+2}+bx$ bx is permutation polynomial, likewise $f(x) + x, f(x) - x$ is permutation polynomial. For such $b = b_i = \alpha^i, i \in T$ polynomial $f(x) = x^{q^3+q^2+q+2} + bx$ is strong complete permutation polynomial. In such cases, we obtain b-strong complete polynomial.

We will take two examples, for $p = 3, q = 3^1 = 3$ and $p = 5, q = 5^1 = 5$.

Example 2.4. Let $p = 3, q = p^1 = 3$. Then, equation (2.6) takes the form:

$$u^4 + (z^2 + 2zB_1 + B_2)u^2 + 2z^4 + 2z^3B_1 + zB_3 + B_4 = 0 \qquad (2.8)$$

For $u \in \mathbb{F}_3^* = \{1, 2\}, z \in F_3 = \{0, 1, 2\}$ we have $u^2 = u^4 = 1, z^2 = z^4 = 1, z^3 = z$, so that equation (2.8) becomes:

$$(B_1 + B_3)z + 1 + B_2 + B_4 = 0 \qquad (2.9)$$

where:

$B_1 = b^{27} + b^9 + b^3 + b,$
$B_2 = b^{36} + b^{30} + b^{28} + b^{12} + b^{10} + b^4,$
$B_3 = b^{39} + b^{37} + b^{31} + b^{13},$
$B_4 = b^{40}.$

Equation (2.9), as linear equation by z does not have a solution in $\mathbb{F}_3$ if

$$B_1 + B_3 = 0, \qquad 1 + B_2 + B_4 \neq 0 \qquad (2.10)$$

In [18], we take irreducible polynomial $x^4 + x + 2 \in \mathbb{F}_3[x]$ and let $\alpha \in \mathbb{F}_{3^4}$, be primitive element, that means that
$\alpha^4 + \alpha + 2 = 0$, or $\alpha^4 = 2\alpha + 1$.

By using software package (Wolfram Mathematica 12.1), considering $b \in \mathbb{F}_{q^4} \backslash \mathbb{F}_q = \mathbb{F}_{3^4} \backslash \mathbb{F}_3$, meets conditions (2.10) for $b = b_i = \alpha^i$, for
$i \in \{4, 5, 7, 10, 11, 12, 15, 17, 19, 20, 21, 23, 25, 28, 29, 30, 33, 35, 36, 44, 45, 47, 50, 51, 52, 55, 57, 59, 60, 61, 63, 65, 68, 69, 70, 73, 75, 76\} = T$.

In total, we get 38 $b-CP$ exponent, for which polynomial $x^{41}+bx$ is permutation polynomial over $\mathbb{F}_{3^4}$.

Since $gcd(41,3^4-1)=gcd(41,80)=1$, then for each $b=\alpha^i, i\in T$, polynomial $b^{-1}x^{41}$ is complete polynomial over $\mathbb{F}_{3^4}$.

Considering that $\{\alpha^i, i\in T\}=\{-\alpha^i, i\in T\}$ and that conditions have been met (2.10) except for $b=\alpha^i, i\in T$, likewise for $-b=-\alpha^i, i\in T$. That is why, for every $-b=-\alpha^i, i\in T$ polynomial $x^{41}-bx$ is permutation

polynomial and $b^{-1}x^{41}$ is strong complete polynomial over $\mathbb{F}_{3^4}$.

Going further, by seeking for which $b\in\mathbb{F}_{3^4}\backslash\mathbb{F}_3$ polynomial $x^{41}+bx$ is strong complete polynomial over $\mathbb{F}_{3^4}$?
We get that for

$$b=\alpha^i, i\in\{7,10,20,21,23,25,29,30,35,47,50,60,61,63,65,69,70,75\}$$

polynomials $x^{41}+bx$ are b-strong complete polynomials over $\mathbb{F}_{3^4}$.

As it is seen, the number of strong complete polynomials is 18, that number matches if we calculate the number of strong complete polynomials from formula ([6], Theorem 2.5.1)

Example 2.5. Suppose $p=5, q=p^1=5$. Equation (2.6) takes the form:

$$(3zB_1+4B_2)u^2+1+3z^3B_1+2z^2B_2+zB_3+B_4=0, \tag{2.11}$$

Where

$B_1=b^{125}+b^{25}+b^5+b,$
$B_2=b^{150}+b^{130}+b^{126}+b^{30}+b^{26}+b^6,$
$B_3=b^{155}+b^{151}+b^{131}+b^{31},$
$B_4=b^{156}.$

For $u\in\{1,4\}$, we have $u^2=1$, so that equation (2.11) becomes

$$(3z+3z^3)B_1+(4+2z^2)B_2+zB_3+B_4+1=0 \tag{2.12}$$

For $u\in\{2,3\}$, we have $u^2=4$, so that equation (2.11) becomes

$$(2z+3z^3)B_1+(1+2z^2)B_2+zB_3+B_4+1=0 \tag{2.13}$$

We replace $z\in\{0,1,2,3,4\}$ in both equations (2.12) and (2.13) and seek $b\in\mathbb{F}_{5^4}\backslash\mathbb{F}_5$ so that equations do not have solution in $\mathbb{F}_5$. For such b, polynomial $x^{157}+bx$ is PP over $\mathbb{F}_{5^4}$ and since $gcd(157,5^4-1)=gcd(157,624)=1$, polynomial $b^{-1}x^{157}$ is complete polynomial over $\mathbb{F}_{5^4}$. We use polynomial x^4+x^2+2x+2 that is irreducible in field $\mathbb{F}_5$ and let α be primitive element in field $\mathbb{F}_{5^4}$. We seek $b=\alpha^i, 1\le i\le 624,$, for which values polynomial $x^{157}+bx$ is permutation polynomial over $\mathbb{F}_{5^4}$. As a result we obtain that for $b=\alpha^i, i\in\{1,5,8,25,39,40,44,64,78,87,111,117,123,125,147,$

$157, 161, 164, 181, 195, 196, 200, 220, 234, 243, 267, 273, 279, 281, 303, 313,$
$317, 320, 337, 351, 352, 356, 376, 390, 399, 423, 429, 435, 437, 459, 469, 473,$
$476, 493, 507, 508, 512, 532, 546, 555, 579, 585, 591, 593, 615\} = S, (CardS = 60)$
polynomial $x^{157} + bx$ is permutation polynomial, and polynomial $b^{-1}x^{157}$ is complete polynomial over $\mathbb{F}_{5^4}$.

For irreducible polynomial $x^4 + x^2 + 2x + 2 \in \mathbf{F}_5[x]$, which polynomial generates the finite field $\mathbb{F}_{5^4}$ and α primitive root of $\mathbb{F}_{5^4}$, means that $\alpha^4 + \alpha^2 + 2\alpha + 2 = 0$, the equation holds $\{\alpha^i : i \in S\} = \{-\alpha^i : i \in S\}$, then polynomials $x^{157} - bx$ are permutation polynomials for every $b = \alpha^i, i \in S$, wherefrom it ensues that polynomials $b^{-1}x^{157}$ are complete permutation polynomials and strong complete permutation polynomials over $\mathbb{F}_{5^4}$.

Polynomial $x^{157} + bx$ remains only a permutation polynomial, for $b = \alpha^i, i \in S$, which means that it is not complete polynomial and orthomorpshism, that is, it is not strong complete polynomial for any $b \in \mathbb{F}_{5^4} \backslash \mathbb{F}_5$.

Remark 2.6. . It was examined polynomial $x^{q^3+q^2+q+2} + bx$, for $p = 3, q = p^2 = 3^2$ over $\mathbb{F}_{9^4} = \mathbb{F}_{6561}$ and for $p = 7, q = p^1 = 7$ over $\mathbb{F}_{7^4} = F_{2401}$. Also for these two cases we get certain values $b \in \mathbb{F}_{q^4} \backslash \mathbb{F}_q$, so that polynomial $x^{q^3+q^2+q+2} + bx$ is permutation polynomial, polynomial $b^{-1}x^{q^3+q^2+q+2}$ is complete polynomial and strong complete polynomial over $\mathbb{F}_{q^4}$.

Open question. To enumerate alle when the polynomial $x^{1+\frac{q^s-1}{q-1}} + bx, s \geq 5$ over $\mathbb{F}_{q^s}$ is b-complete or b-strong complete permutation polynomial. In particular, for case $q = 2^m$, this question coincides with the question asked in [32].

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