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Infinite Series Identities Involving Central Binomial Coefficients

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Abstract:
By introducing parameters into the hypergeometric summation formula, and applying differential method to the summation formula, two new infinite summation formulae are obtained, which are closely related to central binomial coefficients and harmonic numbers. More similar identities can be obtained by applying this method, which illustrates the important role of hypergeometric series summation formulas in solving the problem of infinite series summation involving central binomial coefficients and harmonic numbers.
Key Word*: Hypergeometricseries; Central binomials coefficients; Harmonic numbers; Identities.*

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I. Introduction

Central binomial coefficients play an important role in the fields of number theory, probability and statistics, and combinatorics. In general, central binomial coefficientsare closely related to Catalan number. Readers can refer to the paper[1]. The study on harmonic numbers has a long history. In recent years, a large number of identities involving central binomial coefficients and harmonic numbers have been derived by means of integral, difference and computer algebra[2-9]. In [2], by introducing particular parameters into some classical hypergeometric series summation formulas, and then applying the method of differentiation, many beautiful identities are obtained.Inspired by this work, this paper obtains two identities with central binomial coefficients based on a hypergeometric series summation formula.

Define generalized harmonic functions by

$$H_0^{(s)}=0, \qquad H_k^{(s)}(z)=\sum_{j=1}^{k}\frac{1}{(j+z)^s}, \qquad z\in\mathbb{C}/\mathbb{Z}^-, k,s=1,2,\ldots.$$

When $s=1$, $H_k^{(s)}$ reduce to the classical harmonic numbers H_k. Generalized hypergeometric series are defined by[10]

$${}_{1+p}F_p\left(\begin{matrix}a_0,a_1,\ldots,a_p\\ b_1,\ldots,b_p\end{matrix};z\right)=\sum_{n=0}^{\infty}\frac{(a_0)_n(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_p)_n}\frac{z^n}{n!}.$$

The definition of shifted factorial $(x)_n$ in the formulaabove is defined by

$$(x)_0\equiv 1, \qquad (x)_n=x(x+1)\cdots(x+n-1), \qquad n\in\mathbb{N}, x\in\mathbb{C}/\mathbb{Z}^-.$$

In view of the definition above, $(x)_n$ obviously meets the following properties

$$(x+1)_n=\frac{x+n}{x}(x)_n, \qquad \left(\frac{1}{2}\right)_n=\frac{n!}{4^n}\binom{2n}{n}.$$

For convenience, we introduce some related properties of the $\Gamma(x)$ and $\psi(x)$.

$$\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin(\pi x)}, \qquad \Gamma(x+1)=x\Gamma(x), \qquad \psi(x+1)=\psi(x)+\frac{1}{x},$$

$$\psi\left(\frac{1}{2}\right)=\psi(1)-2\ln 2, \qquad \psi\left(\frac{1}{4}\right)=\psi(1)-\frac{\pi+6\ln 2}{2}, \qquad \psi\left(\frac{3}{4}\right)=\psi(1)+\frac{\pi-6\ln 2}{2}.$$

II. Three MainLemmas

In order to express the results of this paper more clearly, the following lemmas need to be introduced.

Lemma 2.1 (Generalized Kummer Theorem)[10, p. 8-11]When $c\neq -1,-2,\ldots$, then

$$ {}_2F_1\left(\begin{matrix}a,1-a\\c\end{matrix};\frac{1}{2}\right)=\frac{\Gamma\left(\frac{1}{2}c\right)\Gamma\left(\frac{1}{2}c+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}c+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}c-\frac{1}{2}a\right)}. \tag{1}$$

Lemma 2.2 When $\lambda,u,v,p,q>0$, we can obtain

$$\frac{d}{dx}\frac{1}{(\lambda-x)_n}\bigg|_{x=0}=\frac{1}{(\lambda)_n}H_n(\lambda), \tag{2}$$

$$\frac{d}{dx}\left(\frac{\Gamma(u+\theta x)\Gamma(v+\theta x)}{\Gamma(p+\theta x)\Gamma(q+\theta x)}\right)\bigg|_{x=0}=\theta\frac{\Gamma(u)\Gamma(v)}{\Gamma(p)\Gamma(q)}\big(\psi(u)+\psi(v)-\psi(p)-\psi(q)\big). \tag{3}$$

The proof of Lemma 2.2 can refer to the proof of Theorem 2.1 and Theorem 2.2 in paper [2].

Lemma 2.3 When $\lambda>0$, we have

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{32^n}\frac{(1)_n}{(\lambda)_n}=\frac{\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(\frac{\lambda+1}{2}\right)}{\Gamma^2\left(\frac{\lambda}{2}+\frac{1}{4}\right)}, \tag{4}$$

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{32^n}\frac{(1)_n}{(\lambda)_n}H_n(\lambda)=\frac{\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(\frac{\lambda+1}{2}\right)}{2\Gamma^2\left(\frac{\lambda}{2}+\frac{1}{4}\right)}\left(2\psi\left(\frac{\lambda}{2}+\frac{1}{4}\right)-\psi\left(\frac{\lambda}{2}\right)-\psi\left(\frac{\lambda+1}{2}\right)\right). \tag{5}$$

ProofSetting $a=\frac{1}{2},c=\lambda-x$ in Lemma 2.1, we can obtain

$$ {}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\\lambda-x\end{matrix};\frac{1}{2}\right)=\frac{\Gamma\left(\frac{\lambda}{2}-\frac{x}{2}\right)\Gamma\left(\frac{\lambda+1}{2}-\frac{x}{2}\right)}{\Gamma^2\left(\frac{2\lambda+1}{4}-\frac{x}{2}\right)},$$

According to the definition of hypergeometric series, the equation above can be transformed into

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{32^n}\frac{(1)_n}{(\lambda-x)_n}=\frac{\Gamma\left(\frac{\lambda}{2}-\frac{x}{2}\right)\Gamma\left(\frac{\lambda+1}{2}-\frac{x}{2}\right)}{\Gamma^2\left(\frac{2\lambda+1}{4}-\frac{x}{2}\right)}. \tag{6}$$

Setting $x=0$ in the equation(6), the equation (4)can be obtained.

Next, we will prove the equation(5).Take the derivative of both sides of the equation (6) with respect to x, we have

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{1}{32^n}\binom{2n}{n}^2\frac{(1)_n}{(\lambda-x)_n}\right)\bigg|_{x=0}=\frac{d}{dx}\left(\frac{\Gamma\left(\frac{\lambda}{2}-\frac{x}{2}\right)\Gamma\left(\frac{\lambda+1}{2}-\frac{x}{2}\right)}{\Gamma^2\left(\frac{2\lambda+1}{4}-\frac{x}{2}\right)}\right)\bigg|_{x=0}.$$

From Lemma 2.2, we attain

$$\sum_{n=0}^{\infty}\frac{1}{32^n}\binom{2n}{n}^2\frac{(1)_n}{(\lambda)_n}H_n(\lambda)=\frac{\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(\frac{\lambda+1}{2}\right)}{2\Gamma^2\left(\frac{\lambda}{2}+\frac{1}{4}\right)}\left(2\psi\left(\frac{\lambda}{2}+\frac{1}{4}\right)-\psi\left(\frac{\lambda}{2}\right)-\psi\left(\frac{\lambda+1}{2}\right)\right).$$

The equation (5) is follows.

III. Main results

In this section, we obtain two identities by taking some particular values in the above lemmas.

Theorem 3.1 $\sum_{n=0}^{\infty}\frac{1}{32^n}\binom{2n}{n}^2\frac{1}{(n+1)^2}=\frac{32\sqrt{\pi}}{\Gamma^2\left(\frac{1}{4}\right)}+\frac{2\Gamma^2\left(\frac{1}{4}\right)}{\pi\sqrt{\pi}}-8.$ (7)

ProofIn view of the definition of hypergeometric series, one has

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{\left(\frac{1}{2}\right)^n}{(n+1)!}=\frac{2(c-1)}{(a-1)(b-1)}\left({}_2F_1\left(\begin{matrix}a-1,b-1\\c-1\end{matrix};\frac{1}{2}\right)-1\right). \tag{8}$$

Setting $a=\frac{3}{2},b=\frac{3}{2},c=2$ in formula above, it follows that

$$\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n}{(2)_n}\frac{\left(\frac{1}{2}\right)^n}{(n+1)!}=8\left({}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\1\end{matrix};\frac{1}{2}\right)-1\right),$$

i.e.,

$$\sum_{n=0}^{\infty}\frac{1}{32^n}\binom{2n}{n}^2\frac{(2n+1)^2}{(n+1)^2}=8\left(\frac{\sqrt{\pi}}{\Gamma^2\left(\frac{3}{4}\right)}-1\right).$$

Expanding the formula above, we candeduce

$$4\sum_{n=0}^{\infty}\frac{1}{32^n}\binom{2n}{n}^2-4\sum_{n=0}^{\infty}\frac{1}{32^n}\binom{2n}{n}^2\frac{1}{n+1}+\sum_{n=0}^{\infty}\frac{1}{32^n}\binom{2n}{n}^2\frac{1}{(n+1)^2}=8\left(\frac{\sqrt{\pi}}{\Gamma^2\left(\frac{3}{4}\right)}-1\right). \tag{9}$$

Setting $\lambda=1,2$ respectively in equation(4), it yields that

$$\sum_{n=0}^{\infty}\frac{1}{32^n}\binom{2n}{n}^2=\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi\sqrt{\pi}},\qquad \sum_{n=0}^{\infty}\frac{1}{32^n}\binom{2n}{n}^2\frac{1}{n+1}=\frac{8\sqrt{\pi}}{\Gamma^2\left(\frac{1}{4}\right)}.$$

From formulas above and equation(9), Theorem 3.1 follows.

Theorem 3.2 $\sum_{n=0}^{\infty}\frac{1}{32^n}\binom{2n}{n}^2\frac{H_{n+1}}{(n+1)^2}=\frac{\Gamma^2\left(\frac{1}{4}\right)}{\sqrt{\pi}}\left(1-\frac{4\ln 2+8}{\pi}\right)+\frac{\sqrt{\pi}}{\Gamma^2\left(\frac{1}{4}\right)}\left(96-64\ln 2-16\pi\right).$ (10)

ProofSetting $a=\frac{3}{2},b=\frac{3}{2},c=2-x$ in equation(8), one has

$$\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n}{(2-x)_n}\frac{\left(\frac{1}{2}\right)^n}{(n+1)!}=8(1-x)\left({}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\1-x\end{matrix};\frac{1}{2}\right)-1\right)=8(1-x)\frac{\Gamma\left(\frac{\lambda}{2}-\frac{x}{2}\right)\Gamma\left(\frac{\lambda+1}{2}-\frac{x}{2}\right)}{\Gamma^2\left(\frac{2\lambda+1}{4}-\frac{x}{2}\right)}-8(1-x). \tag{11}$$

According to Lemma 2.2, we can derive

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n}{(2-x)_n}\frac{\left(\frac{1}{2}\right)^n}{(n+1)!}\right)\Bigg|_{x=0}=\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n\left(\frac{1}{2}\right)^n}{(n+1)!}\frac{d}{dx}\frac{1}{(2-x)_n}\Bigg|_{x=0}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{32^n}\frac{(2n+1)^2}{(n+1)^2}H_n(2)$$

$$=4\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{32^n}H_n-4\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{32^n}\frac{H_n(2)}{n+1}+\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{32^n}\frac{H_{n+1}}{(n+1)^2}-\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{32^n}\frac{(2n+1)^2}{(n+1)^2},$$

$$\frac{d}{dx}\left(8(1-x)\frac{\Gamma\left(\frac{\lambda}{2}-\frac{x}{2}\right)\Gamma\left(\frac{\lambda+1}{2}-\frac{x}{2}\right)}{\Gamma^2\left(\frac{2\lambda+1}{4}-\frac{x}{2}\right)}-8(1-x)\right)\Bigg|_{x=0}=\frac{4\sqrt{\pi}}{\Gamma^2\left(\frac{3}{4}\right)}\left(\pi-4\ln 2-2\right)+8,$$

i.e.,

$$4\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{32^n}H_n-4\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{32^n}\frac{H_n(2)}{n+1}+\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{32^n}\frac{H_{n+1}}{(n+1)^2}-\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{32^n}\frac{(2n+1)^2}{(n+1)^2}=\frac{4\sqrt{\pi}}{\Gamma^2\left(\frac{3}{4}\right)}\left(\pi-4\ln 2-2\right)+8. \tag{12}$$

Setting $\lambda=1,2$ respectively in equation(5), it yields that

$$\sum_{n=0}^{\infty}\frac{1}{32^n}\binom{2n}{n}^2 H_n=\frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}}\left(1-\frac{4\ln 2}{\pi}\right),\qquad \sum_{n=0}^{\infty}\frac{1}{32^n}\binom{2n}{n}^2\frac{H_n(2)}{n+1}=\frac{4\sqrt{\pi}}{\Gamma^2\left(\frac{1}{4}\right)}\left(6-4\ln 2-\pi\right).$$

Making use of formulas above and equation (12), Theorem 3.2 follows.

IV. Conclusion

In this paper, two identities involving central binomial coefficients and harmonic numbers are obtained by differential method.It is obvious that more identities can be obtained by applying the method. If the formula in Theorem 3.2 is further explored,we can derive the following formula

$$\sum_{n=0}^{\infty}\frac{1}{32^n}\binom{2n}{n}^2\frac{H_{n+1}}{(n+1)^2}=\sum_{n=0}^{\infty}\frac{1}{32^n}\binom{2n}{n}^2\frac{H_n}{(n+1)^2}+\sum_{n=0}^{\infty}\frac{1}{32^n}\binom{2n}{n}^2\frac{1}{(n+1)^3}.$$

If we can find the value of any one of the term of RHS of the formula above, we may be able to obtain the method of computing the value of $\sum_{n=0}^{\infty}\frac{1}{32^n}\binom{2n}{n}^2\frac{1}{(n+1)^k}$, which needs further exploration.

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