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Semi-Slant Pseudo-Riemannian Submersions From Indefinite Almost Para-Contact Manifolds

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***Abstract**. In this paper, we introduce semi-slant pseudo-Riemannian submersions from indefinite almost para-contact manifolds onto pseudo-Riemannian manifolds. We investigate necessary and sufficient conditions for foliations determined by horizontal and vertical distributions to be totally geodesic. We also obtain a necessary and sufficient condition for submersions to be totally geodesic and provide a non- trivial example. Moreover, we discuss the harmonicity of such submersions.*

***Key words and phrases**. Pseudo-Riemannian manifold, Riemannian submersion, almost para-contact pseudo-metric manifold, para-complex para-contact pseudo-Riemannian submersion.*

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I. Introduction

The theory of Riemannian submersions was introduced by O' Neill [13] and Gray [8]. Several geometers studied Riemannian submersions with suitable surjective maps ([7], [12], [13], [16]). It is known that Riemannian submersions are related with physics and have their applications in the Kaluza-Klein theories ([3], [10]), Yang-Mills theory ([2], [24]), supergravity and superstring theories ([10], [11]), the theory of robotics ([1]).

In 1984, Chinea studied Riemannian submersions between almost contact manifolds and investigated some geometric properties and interrelations of structures between such manifolds [4], [5]. In 2010, Sahin introduced anti-invariant and semi-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian mani- folds [17], [18]. He also gave the notion of a slant submersion from an almost Her- mitian manifold onto a Riemannian manifold as a generalization of almost Hermitian submersions and anti-invariant submersions [19]. Further, in 2013, K. S. Park intro- duced semi-slant submersions from an almost Hermitian manifold onto a Riemannian manifold and obtained interesting results [14], [15].

In the present paper, our aim is to study semi-slant pseudo-Riemannian submersions from indefinite almost para-contact manifolds onto pseudo-Riemannian manifolds.

The composition of the paper is as follows. In section 2, we collect some basic definitions, formulas and results on indefinite almost para-contact manifolds and pseudo-Riemannian submersions. In section 3, we give an example of semi-slant pseudo-Riemannian submersions from indefinite almost para-contact manifolds onto pseudo-Riemannian manifolds. We investigate necessary and sufficient conditions for foliations determined by horizontal and vertical distributions to be totally geodesic. We also obtain a necessary and sufficient condition for submersions to be totally geodesic and check the harmonicity of such submersions.

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II. Preliminaries:

2.1 Almost Para-Contact Manifolds. Let M be a $(2n+1)$-dimensional Riemann- ian manifold and ϕ be a (1,1) type tensor field, ξ a vector field, called characteristic vector field and η a 1-form on M. Then, (ϕ, ξ, η) is called an almost para-contact structure on M if

(2.1) $$\phi^2 X = X - \eta(X)\xi, \text{ for any } X \in \Gamma(TM), \ \eta(\xi) = 1$$

and the tensor field ϕ induces an almost para-complex structure on the distribution $D = \ker(\eta)$ ([20], [25]).

M is said to be an almost para-contact manifold, if it is equipped with an almost para-contact structure. Again, M with an almost para contact structure (ϕ, ξ, η) is called an indefinite almost para-contact manifold if it is endowed with a pseudo-Riemannian metric g such that

(2.2) $$g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \ \forall X, Y \in \Gamma(TM),$$

where $\varepsilon = 1$ or -1, according as the characteristic vector field ξ is spacelike or timelike. It follows that

(2.3) $$g(\xi, \xi) = \varepsilon, g(\xi, X)$$

(2.4) $$= \varepsilon\eta(X),$$

(2.5) $$g(X, \phi Y) = g(\phi X, Y),$$

for all $X, Y \in \Gamma(TM)$.

In particular, if index $g = 1$, then the manifold (M, ϕ, ξ, η, g) is called a Lorentzian almost para-contact manifold. If the metric g is positive definite, then the manifold (M, ϕ, ξ, η, g) is the usual almost para-contact metric manifold ([21]).

The fundamental 2-form Φ on M is defined by

(2.6) $$\Phi(X, Y) = g(X, \phi Y),$$

for all $X, Y \in \Gamma(TM)$.

Let M^{2n+1} be an almost para-contact manifold with the structure (ϕ, ξ, η). An almost para-complex structure J on $M^{2n+1} \times \mathbb{R}$ is defined by

(2.7) $$J\left(X, f\frac{d}{dt}\right) = (\phi X + f\xi, \ \eta(X)\frac{d}{dt})$$

where X is tangent to M^{2n+1}, t is the coordinate on $\mathbb{R}$ and f is a smooth function on M^{2n+1}

An almost para-contact structure (ϕ, ξ, η) is said to be normal, if the Nijenhuis tensor N_J of almost para-complex structure

J defined as where X is tangent to M^{2n+1}, t is the coordinate on $\mathbb{R}$ and f is a smooth function on M^{2n+1}.

An almost para-contact structure (ϕ, ξ, η) is said to be normal, if the Nijenhuis tensor N_J of almost para-complex structure J defined as

(2.8) $$N_J(X,Y) = [J,J](X,Y) = [JX,JY] + J^2[X,Y] - J[JX,Y] - J[X,JY],$$

for any vector fields $X, Y \in \Gamma(TM)$, vanishes.

If X and Y are vector fields on M^{2n+1}, then we have ([20], [25])

(2.9) $$N_J = ((X,0),(Y,0)) = (N_\phi(X,Y) - 2d\eta(X,Y)\xi\{(\mathcal{L}_\phi X\eta)Y - (\mathcal{L}_\phi X\eta)\}\frac{d}{dt}$$

(2.10) $$N_J = \left((X,0),\left(0,\frac{d}{dt}\right)\right) = -\left((\mathcal{L}_\phi X\eta)Y - (\mathcal{L}_\phi X\eta)\frac{d}{dt}\right)$$

where $\mathcal{L}_X$ is the Lie derivative respect to a vector field X and N_ϕ, $N^{(1)}$, $N^{(2)}$, $N^{(3)}$, $N^{(4)}$ are defined as

(2.11) $$N_\phi(X,Y) = [\phi,\phi](X,Y) = [\phi X,\phi Y] + \phi^2[X,Y] - \phi[\phi X,Y] - \phi[X,\phi Y],$$

(2.12) $$N^{(1)}(X,Y) = N_\phi(X,Y) - 2d\eta(X,Y)\xi,$$

(2.13) $$N^{(2)}(X,Y) = (\mathcal{L}_{\phi X\eta})Y - (\mathcal{L}_{\phi Y\eta})X,$$

(2.14) $$N^{(3)}(X) = (L_{\xi\phi})X$$

(2.15) $$N^{(4)}(X) = (L\xi\eta)X.$$

For an almost para-contact structure (ϕ, ξ, η), vanishing of $N^{(1)}$ implies the vanishing of $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$. Moreover, $N^{(2)}$ vanishes if and only if ξ is a killing vector field.

An indefinite almost para-contact manifold $(M^{2n+1}, \varphi, \xi, \eta, g, \varepsilon)$ is called

(i) normal, if $N_\phi - 2d\eta \otimes \xi = 0$,

(ii) para-contact, if $\Phi = d\eta$,

(iii) K-para-contact, if M is para-contact and ξ is killing,

(iv) para-cosymplectic, if $\nabla\Phi = 0$, which implies $\nabla\eta = 0$, where ∇ is the Levi- Civita connection on M,

(v) almost para-cosymplectic, if $d\eta = 0$ and $d\Phi = 0$,

(vi) weakly para-cosymplectic, if M is almost para-cosymplectic and

$$[R(X,Y),\phi] = R(X,Y)\phi - \phi R(X,Y) = 0,$$

where R is Riemannian curvature tensor on M,

(vii) para-Sasakian, if $\Phi = d\eta$ and M is normal,

(viii) quasi-para-Sasakian, if $d\Phi = 0$ and M is normal.

2.2 Pseudo -Riemannian Submersions: Let $(\bar{M}^m, \bar{g})$ and (M^n, g) be two connected pseudo-Riemannian manifolds of indices $\bar{s} (0 \leq \bar{s} \leq m)$ and $s\ (0 \leq s \leq n)$ respectively, with $s > \bar{s}$.

A pseudo-Riemannian submersion is a smooth map $f : \overline{M}^m \longrightarrow \overline{M}^n$ which is onto and satisfies the following conditions ([7], [8], [13], [16]):

(i) the derivative map $f_{*p}: T_p\bar{M} \longrightarrow T_{f(p)}M$ is surjective at each point $p \in \bar{M}$

(ii) fibres $f^{-1}(q)$ of f over $q \in M$ are pseudo-Riemannian submanifolds of $\bar{M}$;

(iii) f_* preserves the length of horizontal vectors.

A vector field on $\overline{M}$ is called vertical if it is always tangent to fibers and it is called horizontal if it is always orthogonal to fibres. horizontal if it is always orthogonal to fibres. We denote by V the vertical distribution and by H the horizontal distribution. Also, we denote vertical and horizontal projections of a vector field E on $\bar{M}$ by vE and by hE respectively. A horizontal vector field $\bar{X}$ on $\bar{M}$ is said to be basic if $\bar{X}$ is f-related to a vector field X on M i.e. $f_*\bar{X} = Xof$.

Thus, every vector field X on M has a unique horizontal lift $\bar{X}$ on $\bar{M}$.

We recall the following lemma for later use:

Lemma 2.1. ([7], [12]) If $f: \bar{M} \longrightarrow M$ is a pseudo-Riemannian submersion and $\bar{X}, \bar{Y}$ are basic vector fields on $\bar{M}$ that are f-related to the vector fields X, Y on M respectively, then we have the following properties:

(i) $\bar{g}(\bar{X}, \bar{Y}) = g(X, Y) \circ f$,

(ii) $h[\bar{X}, \bar{Y}]$ is a vector field and $h[\bar{X}, \bar{Y}] = [X, Y] \circ f$,

(iii) $h(\bar{\nabla}_X Y)$ is a basic vector field f-related to $\nabla_X Y$, where $\bar{\nabla}$ and ∇ are the Levi-Civita connections on $\bar{M}$ and M respectively,

(iv) $[E, U] \in \mathcal{V}$, for any vector field $U \in \mathcal{V}$ and for any vector field $E \in \Gamma(T\bar{M})$.

A pseudo-Riemannian submersion $f: \bar{M} \longrightarrow M$ determines tensor fields T and A of type $(1,2)$ on $\bar{M}$ defined by formulas ([7], [12], [13])

(2.16) $T(E, F) = T_E F = h(\bar{\nabla}_{hE} hF) + h(\bar{\nabla}_{hE} hF)$,

(2.17) $\mathcal{A}(E, F) = \mathcal{A}_E F = v(\bar{\nabla}_{hE} hF) + h(\bar{\nabla}_{hE} vF)$, for any $E, F \in \Gamma(T\bar{M})$.

Let $\bar{X}, \bar{Y}$ be horizontal vector fields and U, V be vertical vector fields on $\bar{M}$. Then, we have

(2.18) $$\mathcal{T}_U\bar{X} = v(\bar{\nabla}_U\bar{X}), \mathcal{T}_U V = h(\bar{\nabla}_U V),$$

(2.19) $$\bar{\nabla}_U\bar{X} = \mathcal{T}_U\bar{X} + h(\bar{\nabla}_U\bar{X}),$$

(2.20) $$\mathcal{T}_U\bar{F} = 0, \ \mathcal{T}_{\bar{E}}F = \mathcal{T}_U\bar{F},$$

(2.21) $$\bar{\nabla}_U \mathrm{V} = \mathcal{T}_U\mathrm{V} + u(\bar{\nabla}_U\mathrm{V}),$$

(2.22) $$\mathcal{A}_{\bar{X}}\bar{Y} = v(\bar{\nabla}_{\bar{X}}\bar{Y}), \mathcal{A}_{\bar{X}}U = h(\bar{\nabla}_{\bar{X}}U),$$

(2.23) $$\bar{\nabla}_{\bar{X}}U = \mathcal{A}_{\bar{X}}U + v(\bar{\nabla}_{\bar{X}}U),$$

(2.24) $$\mathcal{A}_U F = 0, \mathcal{A}_E F = \mathcal{A}_{hE}F,$$

(2.25) $$\bar{\nabla}_{\bar{X}}\bar{Y} = \mathcal{A}_{\bar{X}}\bar{Y} + h(\nabla_{\bar{X}}\bar{Y}),$$

(2.26) $$h(\bar{\nabla}_U\bar{X}) = h(\bar{\nabla}_{\bar{X}}U) = \mathcal{A}_{\bar{X}}U,$$

(2.27) $$\mathcal{A}_{\bar{X}}\bar{Y} = \frac{1}{2}v[\bar{X}, \bar{Y}],$$

(2.28) $$\mathcal{A}_{\bar{X}}\bar{Y} = -\mathcal{A}_{\bar{Y}}\bar{X},$$

(2.29) $$\mathcal{T}_U V = \mathcal{T}_V U,$$

$\forall E, F \in \Gamma(T\bar{M})$.

It can be easily shown that a Riemannian submersion $f: \bar{M} \longrightarrow M$ has totally geodesic fibres if and only if T vanishes identically. By lemma (2.1), the horizontal distribution H is integrable if and only if $A = 0$. Also, in view of equations (2.28) and (2.29), $\mathcal{A}$ is alternating on the horizontal distribution and T is symmetric on the vertical distribution.

Now, we recall the notion of harmonic maps between pseudo-Riemannian manifolds. Let $(\bar{M}, \bar{g})$ and (M, g) be pseudo-Riemannain manifolds and let $f: \bar{M} \longrightarrow M$ be a smooth map. Then the second fundamental form of the map f is given by

(2.30) $$(\bar{\nabla}f_*)(X, Y) = \left(\nabla^f_X f_* Y\right) \circ f - f_*(\bar{\nabla}_X Y),$$

for all $X, Y \in \Gamma(T\bar{M})$, where ∇^f denotes the pullback connection of ∇ with respect to f and the tension field τ of f is defined by

(2.31) $$\tau(f) = \mathrm{trace}(\bar{\nabla}f_*) = \sum^{m} (\bar{\nabla}f_*)(e_i, e_i),$$

where $\{e_1, e_2, \cdots e_m\}$ is an orthonormal frame on $\bar{M}$.

It is known that f is harmonic if and only if $\tau(f) = 0$[6].

In this paper, we study pseudo Riemannian submersions $f: \bar{M} \longrightarrow M$ such that fibres $f^{-1}(q)$ over $q \in M$ be pseudo-Riemannian submanifolds admitting non-lightlike vector fields.

III. Semi-Slant PSEUDO-Riemannian submersions

Definition 3.1. Let $(\bar{M}^{2m_1+1}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be an indefinite almost para-contact manifold and (M^{m_2}, g) be a pseudo-Riemannian manifold with $m_1 > m_2$. A pseudoRiemannian submersion $f: \bar{M} \longrightarrow M$ is called a semi-slant pseudo-Riemannian submersion if the structure vector field $\bar{\xi}$ is horizontal and there exists a distribution $\bar{\mathcal{D}} \subseteq \ker f_*$ such that
(i) $\ker f_* = \bar{\mathcal{D}} \oplus \bar{\mathcal{D}}^{\perp}$;
(ii) $\bar{\phi}(\bar{\mathcal{D}}) = \bar{\mathcal{D}}$; and
(iii) for any non-zero vector field $\bar{X}_p \in \bar{\mathcal{D}}_p^{\perp}$, the angle θ between $\bar{\phi}\bar{X}_p$ and the space $\bar{\mathcal{D}}_p^{\perp}$ isConstant
This angle θ is called semi-slant angle of the submersion.
If dimension $\bar{\mathcal{D}} = 0$, then the map f is a slant pseudo-Riemannian submersion and
if $\theta = \frac{\pi}{2}$, then it is a semi-invariant pseudo-Riemannian submersion.
For any vector field $U \in \mathcal{V}$, we put

(3.1) $$U = PU + QU,$$
where $PU \in \bar{\mathcal{D}}$ and $QU \in \bar{\mathcal{D}}^{\perp}$.
Also, for any vector field $U \in \bar{\mathcal{D}}^{\perp}$, we set

(3.2)$\bar{\phi}U = \psi U + \omega U,$
where ψU and ωU are horizontal and vertical components of $\bar{\phi}U$ respectively.
For any vector field $\bar{X} \in \mathcal{H}$, we put
(3.2)$\bar{\phi}\bar{X} = t\bar{X} + n\bar{X},$
where $t\bar{X}$ and $n\bar{X}$ are horizontal and vertical components of $\bar{\phi}\bar{X}$ respectively.

Example 3.2. Let $\{(\mathbb{R}_6^{13}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g}); (x_1, \cdots, x_6, y_1, \cdots, y_6, z)^t\}$ be an indefinite almost para-contact manifold with
$$\bar{\phi}\left(\frac{\partial}{\partial x_i}\right)_{i=1,2,\cdots,6} = \frac{\partial}{\partial y_i}, \quad \bar{\phi}\left(\frac{\partial}{\partial y_i}\right)_{i=1,2,\cdots,6} = \frac{\partial}{\partial x_i}, \quad \bar{\phi}\left(\frac{\partial}{\partial z}\right) = 0,$$
$\bar{\xi} = 2\frac{\partial}{\partial z}, \bar{\eta} = 2dz$, signature of $\bar{g} = (-,-,-,+,+,+,-,-,-,+,+,+,+)$ and let $(\mathbb{R}_2^6, g)$ be a pseudo-Riemannian manifold.
Define a submersion $f: \{\mathbb{R}_6^{13}; (x_1, \cdots, x_6, y_1, \cdots, y_6, z)^t\} \longrightarrow \{\mathbb{R}_2^6; (u_1, u_2, \cdots, u_6)^t\}$

by

$$f((x_1, \cdots, x_6, y_1, \cdots, y_6, z)^t) \mapsto (x_2 \sin\alpha + y_2 \cos\alpha, x_3 \cos\alpha + y_3 \sin\alpha, x_4 \sin\alpha - y_4 \cos\alpha, -x_5 \cos\alpha + y_5 \sin\alpha, x_6 \sin\alpha - y_6 \cos\alpha . z)^t,$$

where $\alpha \in \mathbb{R}$.

The vertical distribution $\mathcal{V}$ is span of

$$\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \cos\alpha \frac{\partial}{\partial x_2} - \sin\alpha \frac{\partial}{\partial y_2}, -\sin\alpha \frac{\partial}{\partial x_3} + \cos\alpha \frac{\partial}{\partial y_3}, \cos\alpha \frac{\partial}{\partial x_4} + \sin\alpha \frac{\partial}{\partial y_4} \right.$$
$$\left. \sin\alpha \frac{\partial}{\partial x_5} + \cos\alpha \frac{\partial}{\partial y_5}, \cos\alpha \frac{\partial}{\partial x_6} + \sin\alpha \frac{\partial}{\partial y_6}\right\}$$

We have $\bar{\mathcal{D}} = \mathrm{Span}\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}\right\} \subset \mathcal{V}$ and

$$\bar{\mathcal{D}}^{\perp} = \mathrm{Span}\left\{\cos\alpha \frac{\partial}{\partial x_2} - \sin\alpha \frac{\partial}{\partial y_2}, \quad -\sin\alpha \frac{\partial}{\partial x_3} + \cos\alpha \frac{\partial}{\partial y_3}, \cos\alpha \frac{\partial}{\partial x_4} + \sin\alpha \frac{\partial}{\partial y_4} \right.$$
$$\left. \sin\alpha \frac{\partial}{\partial x_5} + \cos\alpha \frac{\partial}{\partial y_5}, \cos\alpha \frac{\partial}{\partial x_6} + \sin\alpha \frac{\partial}{\partial y_6}\right\}.$$

It can be seen that $\bar{\mathcal{D}}$ is invariant with respect to $\bar{\phi}$ and semi-slant angle of submersion f is $\theta = \frac{\pi}{2} - 2\alpha$.
Proposition 3.1. Let $f: \bar{M} \longrightarrow M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost para-

contact manifold $(\bar{M},\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ onto a pseudoRiemannian manifold (M,g). If U,V are vertical and $\bar{X},\bar{Y}$ are horizontal vector fields on $\bar{M}$, then we have

$$\begin{aligned} \bar{g}(\omega QU,V) &= \bar{g}(U,\omega QV), \\ \bar{g}(t\bar{X},\bar{Y}) &= \bar{g}(\bar{X},t\bar{Y}), \quad (3.5) \\ \bar{g}(\psi QU,\bar{X}) &= \bar{g}(U,n\bar{X}). \end{aligned}$$

Proof. Using equations (2.5), (3.1) and (3.2), for any vector fields $U,V\in\mathcal{V}$, we have

$$\bar{g}(\phi PU+\psi QU+\omega QU,V)=\bar{g}(U,\phi PV+\psi QV+\omega QV)$$

which implies

$$\bar{g}(\omega QU,V)=\bar{g}(U,\omega QV).$$

Similarly, for any vector fields $U,V\in\mathcal{V}$ and $\bar{X},\bar{Y}\in\mathcal{H}$, we have equations (3.5) and (3.6).

Theorem 3.3. Let $(\bar{M},\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ be an indefinite almost para-contact manifold and (M,g) a pseudo-Riemannian manifold. Then, a pseudo-Riemannian submersion $f:\bar{M}\longrightarrow M$ is a semi-slant pseudo-Riemannian submersion if and only if there exists $\lambda\in[0,1]$ such that

(3.7) $$(Q\omega)^2=\lambda\bar{\phi}^2.$$

Moreover, if θ is semi-slant angle of the pseudo-Riemannian submersion, then $\lambda=\cos^2\theta$.

Proof. Let $V\in\bar{\mathcal{D}}^\perp$. Then, we have

(3.8) $$\cos\theta=\frac{\bar{g}(\bar{\phi}V,Q\omega V)}{|\bar{\phi}V||Q\omega V|}.$$

Again, we have

(3.9) $$\cos\theta=\frac{|Q\omega V|}{|\bar{\phi}V|}.$$

From equations (2.5), (3.2) and (3.8), we have

(3.10) $$\cos\theta=\frac{\bar{g}(V,(Q\omega)^2V)}{|\bar{\phi}V||Q\omega V|}.$$

In view of equations (3.9) and (3.10), we have

(3.11) $$\cos^2\theta=\frac{\bar{g}(V,(Q\omega)^2V)}{g(V,\bar{\phi}^2V)}$$

Equation (3.11) implies that $\cos^2\theta=$ constant if and only if $(Q\omega)^2$ and $\bar{\phi}^2$ are conformally parallel.
Hence, we get $(Q\omega)^2=\lambda\bar{\phi}^2$, for some $\lambda\in[0,\infty)$.
Again, by using equations (3.7) and (3.11), we have $\lambda=\cos^2\theta$. Consequently, we get $\lambda\in[0,1]$.

Proposition 3.2. Let $f:\bar{M}\longrightarrow M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost para-contact manifold $(\bar{M},\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ onto a pseudoRiemannian manifold (M,g). If θ is semi-slant angle of the pseudo-Riemannian submersion, then for $U,V\in\bar{\mathcal{D}}^\perp$,

(3.12) $$\bar{g}(Q\omega U,Q\omega V)=(\bar{g}(U,V)-\varepsilon\bar{\eta}(U)\bar{\eta}(V))\cos^2\theta,$$

(3.13) $$\bar{g}(\psi U\psi V)=\bar{g}(\bar{\phi}U,\bar{\phi}V)\sin^2\theta-\bar{g}(P\omega U,P\omega V).$$

Proof. Let $U,V\in\bar{\mathcal{D}}^\perp$. On replacing V by $Q\omega V$ in equation (3.4), we get

$$\bar{g}(Q\omega U,Q\omega V)=\bar{g}(U,(Q\omega)^2V).$$

In view of equations (2.2) and (3.7), above equation implies equation (3.12). Similarly, we have equation (3.13).

Theorem 3.4. Let $f:\bar{M}\longrightarrow M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost para-contact manifold $(\bar{M},\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ onto a pseudo-Riemannia: manifold (M,g). Then, $(\psi\bar{\mathcal{D}}^\perp)^\perp$ is invariant with respect to $\bar{\phi}$.

Proof. Let $U \in (\psi\overline{\mathcal{D}}^{\perp})^{\perp} \subset \mathcal{H}$. For any $V \in \overline{\mathcal{D}}^{\perp}$, we have

$$\begin{aligned}\bar{g}(\bar{\phi}U, Q\omega V) &= \bar{g}(U, \bar{\phi}(Q\omega V)) \\ &= \bar{g}(U, \psi Q\omega V + \omega Q\omega V) \\ &= 0\end{aligned}$$

which implies $\bar{\phi}U \in (\psi\overline{\mathcal{D}}^{\perp})^{\perp}$.
Again, for any $V \in \overline{\mathcal{D}}^{\perp}$, we get

$$\begin{aligned}\bar{g}(\bar{\phi}U, V) &= \bar{g}(U, \bar{\phi}V) \\ &= \bar{g}(U, \psi V + P\omega V + Q\omega V) \\ &= 0\end{aligned}$$

which implies $\bar{\phi}U \in (\psi\bar{D})^{\perp}$.
Hence, $\bar{\phi}(\psi D^{\perp})^{\perp} \subseteq (\psi D^{\perp})^{\perp}$.

Proposition 3.3. Let $\left\{\frac{\partial}{\partial x_1}, \cdots \frac{\partial}{\partial x_k}\right\}$ be a basis of D^1 and $\left\{\frac{\partial}{\partial y^1}, \ldots \frac{\partial}{\partial y^1}\right\}$ a basis of

$$\bar{D}l = 2m_1 + 1 - m_2 - k.$$

Then$\{\ \psi\left(\frac{\partial}{\partial x_2}\right) csc\,\theta,\ \psi\left(\frac{\partial}{\partial x_2}\right) csc\,\theta, \psi\left(\frac{\partial}{\partial x_3}\right) csc\,\theta, \ldots\ \psi\left(\frac{\partial}{\partial x_k}\right) csc\,\theta\}$ is an orthonormal basis of $\psi\bar{D}^1$.
Proof. We have

$$\bar{g}\left(\psi\left(\frac{\partial}{\partial x_i}\right) csc\,\theta,\ \psi\left(\frac{\partial}{\partial x_j}\right) csc\,\theta\right) = \bar{g}\left(\psi\left(\frac{\partial}{\partial x_i}\right),\ \psi\left(\frac{\partial}{\partial x_j}\right)\right) \csc^2\theta$$

$$= \bar{g}\left(\emptyset\left(\frac{\partial}{\partial x_i}\right) \csc\theta,\ \emptyset\left(\frac{\partial}{\partial x_j}\right)\right) \sin^2\theta \csc^2\theta - \bar{g}\left(P_\omega\left(\frac{\partial}{\partial x_i}\right), P_\omega\left(\frac{\partial}{\partial x_j}\right)\right)\csc^2\theta$$

$$= \delta ij$$

Proposition 3.4. If $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \ldots \frac{\partial}{\partial x_k}$ are any orthonormal vector fields in $\overline{D^{\perp}}$,then

$\{\frac{\partial}{\partial x_1}$, $Q\omega\left(\frac{\partial}{\partial x_2}\right)\sec\theta$, $\frac{\partial}{\partial x_2}$, $Q\omega\left(\frac{\partial}{\partial x_3}\right)\sec\theta$, $\ldots \frac{\partial}{\partial x_k}$, $Q\omega\left(\frac{\partial}{\partial x_k}\right)\sec\theta\}$is anorthonormal basis of $\overline{D^{\perp}}$.

Proof. We have

$$\bar{g}\left(Q\omega\left(\frac{\partial}{\partial x_i}\right) sec\,\theta,\ Q\omega\left(\frac{\partial}{\partial x_j}\right) sec\,\theta\right) = \bar{g}\left(Q\omega\left(\frac{\partial}{\partial x_i}\right),\ Q\omega\left(\frac{\partial}{\partial x_j}\right)\right) \sec^2\theta$$

$$= \bar{g}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) - \varepsilon^{\eta}\left(\frac{\partial}{\partial x_i}\right)\eta\left(\frac{\partial}{\partial x_j}\right)$$

$$= \delta ij.$$

Lemma 3.5. Let f: $\bar{M} \longrightarrow M$be a semi-slant pseudo-Riemannian submersion froman indefinite almost para-contact manifold ($\bar{M}$, $\bar{\emptyset}$, $\bar{\xi}$, $\bar{\eta}$, $\bar{g}$) onto a pseudo-Riemannian manifold (M, g). Then, for any $U, V \in \mathcal{V}$ and $\bar{X}, \bar{Y} \in H$, we have

$$\begin{aligned}&h(\bar{\nabla}_{PU}(\bar{\emptyset}PV)) + v(\bar{\nabla}_{PU}(\bar{\emptyset}PV)) + h(\bar{\nabla}_{QU}(\bar{\emptyset}PV)) + v(\bar{\nabla}_{QU}(\bar{\emptyset}PV)) \\ &+h(\bar{\nabla}_{PU}(\psi QV)) + \mathcal{T}_{PU}(\psi QV) + \mathcal{T}_{PU}(\omega QV) + v(\bar{\nabla}_{PU}(\omega QV)) \\ (3.14)\quad &+ h(\bar{\nabla}_{QU}(\psi QV)) + \mathcal{T}_{QU}(\psi QV) + h(\bar{\nabla}_{QU}(\omega PV)) + \mathcal{T}_{QU}(\omega QV) \\ &= (\bar{\nabla}_U\bar{\emptyset})V + t(\mathcal{T}uV) + n(\mathcal{T}uV) + \bar{\emptyset}(P(u\bar{\nabla}_U V)) + \psi(Q(u\bar{\nabla}_U V)) + \omega(Q(u\bar{\nabla}_U V)),\end{aligned}$$

$$\begin{aligned}&h\bar{\nabla}_{\bar{X}}(t\bar{Y}) + \mathcal{A}_{\bar{X}}(t\bar{Y}) + \mathcal{A}_{\bar{X}}(n\bar{Y}) + v(\bar{\nabla}_{\bar{X}}(n\bar{Y})) \\ (3.15)\qquad = \ &(\bar{\nabla}_{\bar{X}}\bar{\phi})\bar{Y} + t(h\bar{\nabla}_{\bar{X}}\bar{Y}) + n(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \bar{\phi}\big(P(\mathcal{A}_{\bar{X}}\bar{Y})\big) \\ &+\psi\big(Q(\mathcal{A}_{\bar{X}}\bar{Y})\big) + \omega\big(Q(\mathcal{A}_{\bar{X}}\bar{Y})\big)\end{aligned}$$

$$\begin{aligned}&\mathcal{A}_{\bar{X}}(\bar{\phi}PU) + v(\bar{\nabla}_{\bar{X}}(\bar{\phi}PU)) + h(\bar{\nabla}_{\bar{X}}(\psi QU)) \\ (3.16)\qquad &+\mathcal{A}_{\bar{X}}(\psi QU) + \mathcal{A}_{\bar{X}}(\omega QU) + v(\nabla_{\bar{X}}(\omega QU))\end{aligned}$$

$$= \left(\bar{\nabla}_{\bar{X}}^{\bar{\phi}}\right) U + t(\mathcal{A}_{\bar{X}} U) + n(\mathcal{A}_{\bar{X}} U) + \bar{\phi}\big(P(v\bar{\nabla}_{\bar{X}} U)\big)$$
$$+\psi\big(Q(v\bar{\nabla}_{\bar{X}} U)\big) + \omega\big(Q(v\bar{\nabla}_{\bar{X}} U)\big),$$

$$h\big(\bar{\nabla}_U(t\bar{X})\big) + \mathcal{T}_U(t\bar{X}) + \mathcal{T}_U(n\bar{X}) + v\big(\bar{\nabla}_U(n\bar{X})\big)$$
$$= (\nabla_U \phi)X + t(h\nabla_U X) + n(h\nabla_U X) + \phi\big(P(\mathcal{T}_U X)\big) \tag{3.17}$$
$$+\psi\big(Q(\mathcal{T}_U \bar{X})\big) + \omega\big(Q(\mathcal{T}_U \bar{X})\big).$$

Proof. For U, V ∈ $\mathcal{V}$, we have
$\bar{\nabla}_U(\bar{\phi}V) = \bar{\nabla}_{P\bar{U}+Q\bar{U}}(\bar{\phi}(PV + QV))$,
which gives

$$(\bar{\nabla}_U \bar{\phi})V + t(\mathcal{T}_U V) + n(\mathcal{T}_U V) + \bar{\phi}\big(P(v\bar{\nabla}_U V)\big) + \psi\big(Q(v\bar{\nabla}_U V)\big) + \omega\big(Q(v\bar{\nabla}_U V)\big)$$
$$= h(\bar{\nabla}_{PU}(\bar{\phi}PV)) + v(\bar{\nabla}_{PU}(\bar{\phi}PV)) + h\big(\bar{\nabla}_{QU}(\bar{\phi}PV)\big) + v\big(\bar{\nabla}_{QU}(\bar{\phi}PV)\big)$$
$$+h\big(\bar{\nabla}_{PU}(\psi QV)\big) + \mathcal{T}_{PU}(\omega QV) + \mathcal{T}_{PU}(\psi QV) + v(\bar{\nabla}_{PU}(\omega QV))$$
$$+h\left(\bar{\nabla}_{QU}(\psi QV)\right) + h\left(\bar{\nabla}_{QV}(\omega QV)\right) + \mathcal{T}_{QU}(\psi QV) + \mathcal{T}_{QU}(\omega QV)$$

Similarly, we have other equations
By using similar steps as in lemma (3.5), we have

Lemma 3.6. Let f: $\bar{M} \longrightarrow M$ be a semi-slant pseudo-Riemannian submersion froman indefinite almost para-contact manifold ($\bar{M}$, $\bar{\emptyset}$, $\bar{\xi}$, $\bar{\eta}$, $\bar{g}$) onto a pseudo-Riemannian manifold (M, g). Then, for any $U, V \in \mathcal{V}$ and $\bar{X}, \bar{Y} \in H$, we have

$$\mathcal{T}_{PU}(\bar{\phi}PV) + h\big(\bar{\nabla}_{QU}(\bar{\phi}PV)\big) + h(\bar{\nabla}_{PU}(\psi QV))$$
$$+\mathcal{T}_{PU}(\omega QV) + h\big(\bar{\nabla}_{QU}(\psi QV)\big) + h\big(\bar{\nabla}_{QV}(\omega QV)\tag{3.18}$$
$$= t(\mathcal{T}_U V) + \psi\big(Q(v\bar{\nabla}_U V)\big);$$

$$u(\bar{\nabla}_{PU}(\bar{\phi}PV)) + v\big(\bar{\nabla}_{QU}(\bar{\phi}PV)\big) + \mathcal{T}_{PU}(\psi QV)$$
$$+v(\bar{\nabla}_{PU}(\omega QV)) + \mathcal{T}_{QU}(\psi QV) + \mathcal{T}_{QU}(\omega QV) \tag{3.19}$$
$$= n(\mathcal{T}_U V) + \bar{\phi}P(v\bar{\nabla}_U V) + \omega\big(Q(v\bar{\nabla}_U V)\big).$$

$$h(\bar{\nabla}_{\bar{X}}(t\bar{Y})) + \mathcal{A}_{\bar{X}}(n\bar{Y}) = t(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \psi\big(Q(\mathcal{A}_{\bar{X}}\bar{Y})\big); \tag{3.20}$$

$$\mathcal{A}_{\bar{X}}(t\bar{Y}) + v(\bar{\nabla}_{\bar{X}}(n\bar{Y})) = n(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \omega\big(Q(\mathcal{A}_{\bar{X}}\bar{Y})\big) + \bar{\phi}P(\mathcal{A}_{\bar{X}}\bar{Y}). \tag{3.21}$$

$$\mathcal{A}_{\bar{X}}(\bar{\phi}PU) + h\big(\bar{\nabla}_{\bar{X}}(\psi QU)\big) + \mathcal{A}_{\bar{X}}(\omega QU) = t(\mathcal{A}_{\bar{X}}U) + \psi\big(Q(v\bar{\nabla}_{\bar{X}}U)\big); \tag{3.22}$$

$$v\left(\bar{\nabla}_{\bar{X}}(\bar{\phi}PU)\right) + \mathcal{A}_{\bar{X}}(\psi QU) + v\big(\bar{\nabla}_{\bar{X}}(\omega QU)\big) = n(\mathcal{A}_{\bar{X}}U) + \bar{\phi}P(v\bar{\nabla}_{\bar{X}}U) + \omega Q(v\bar{\nabla}_{\bar{X}}U) \tag{3.23}$$

$$h(\bar{\nabla}_U(t\bar{X})) + \mathcal{T}_U(n\bar{X}) = t(h\bar{\nabla}_U\bar{X}) + \psi(Q\mathcal{T}_U\bar{X}); \tag{3.24}$$

$$\mathcal{T}_U(t\bar{X}) + v(\bar{\nabla}_U(n\bar{X})) = n(h\bar{\nabla}_U\bar{X}) + \bar{\phi}(P\mathcal{T}_U\bar{X}) + \omega Q(\mathcal{T}_U\bar{X}). \tag{3.25}$$

Lemma 3.7. Let f: $\bar{M} \longrightarrow M$ be a semi-slant pseudo-Riemannian submersion froman indefinite almost para-contact manifold ($\bar{M}$, $\bar{\emptyset}$, $\bar{\xi}$, $\bar{\eta}$, $\bar{g}$) onto a pseudo-Riemannian manifold (M, g). Then, for any $U, V \in v$ and $\bar{X}, \bar{Y} \in H$, we have

$$h(\bar{\nabla}_{PU}(\bar{\phi}PV)) + v(\bar{\nabla}_{PU}(\bar{\phi}PV)) + h\big(\bar{\nabla}_{QU}(\bar{\phi}PV)\big) + v\big(\bar{\nabla}_{QU}(\bar{\phi}PV)\big)$$
$$+h\big(\bar{\nabla}_{PU}(\psi QV)\big) + \mathcal{T}_{PU}(\psi QV) + \mathcal{T}_{PU}(\omega QV) + v\big(\bar{\nabla}_{PU}(\omega QV)\big)$$
$$+h\left(\bar{\nabla}_{QU}(\psi QV)\right) + \mathcal{T}_{QU}(\psi QV) + h\left(\bar{\nabla}_{QU}(\omega QV)\right) + \mathcal{T}_{QU}(\omega QV),$$
$$= \bar{g}(U, V)\bar{\xi} + t(\mathcal{T}_U V) + n(h\bar{\nabla}_U V) + \bar{\phi}P\big(v(\bar{\nabla}_U V)\big) \tag{3.26}$$
$$+\psi Q(v\bar{\nabla}_U V) + \omega Q(v\bar{\nabla}_U V)$$

$$h(\bar{\nabla}_{\bar{X}}(t\bar{Y})) + \mathcal{A}_{\bar{X}}(t\bar{Y}) + \mathcal{A}_{\bar{X}}(n\bar{Y}) + v(\bar{\nabla}_{\bar{X}}(n\bar{Y}))$$
$$= \bar{g}(\bar{X}, \bar{Y})\bar{\xi} - \varepsilon\bar{\eta}(\bar{Y})\bar{X} + t(h\bar{\nabla}_{\bar{X}}\bar{Y}) + n(h\bar{\nabla}_{\bar{X}}\bar{Y}) \tag{3.27}$$
$$+\bar{\phi}P(\mathcal{A}_{\bar{X}}\bar{Y}) + \psi Q(\mathcal{A}_{\bar{X}}\bar{Y}) + \omega Q(\mathcal{A}_{\bar{X}}\bar{Y}),$$

$$\mathcal{A}_{\bar{X}}(\bar{\phi}PU) + v(\bar{\nabla}_{\bar{X}}(\bar{\phi}PU)) + h(\bar{\nabla}_{\bar{X}}(\psi QU))$$
$$+\mathcal{A}_{\bar{X}}(\psi QU) + \mathcal{A}_{\bar{X}}(\omega QU) + v(\nabla_{\bar{X}}(\omega QU))$$
$$= \bar{g}(\bar{X}, U)\bar{\xi} + t(\mathcal{A}_{\bar{X}}U) + n(\mathcal{A}_{\bar{X}}U) \tag{3.28}$$
$$+\bar{\phi}P(v\bar{\nabla}_{\bar{X}}U) + \psi Q(v\bar{\nabla}_{\bar{X}}U) + \Omega Q(v\bar{\nabla}_{\bar{X}}U),$$

$$h(\overline{\nabla}_U(t\bar{X})) + \mathcal{T}_U(t\bar{X}) + \mathcal{T}_U(n\bar{X}) + v(\overline{\nabla}_U(n\bar{X}))$$
(3.29) $$= \bar{g}(U,\bar{X})\bar{\xi} - \varepsilon\bar{\eta}(\bar{X})U + t(h\overline{\nabla}_U\bar{X}) + n(h\overline{\nabla}_U\bar{X})$$
$$+\bar{\phi}P(\mathcal{T}_U\bar{X}) + \psi Q(\mathcal{T}_U\bar{X}) + \omega Q(v\overline{\nabla}_U\bar{X}).$$

Theorem 3.8. Let f: $\bar{M} \longrightarrow M$be a semi-slant pseudo-Riemannian submersion froman indefinite almost para-contact manifold $(\bar{M}, \bar{\emptyset}, \bar{\xi}, \bar{\eta}, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, the fibres of f are totally geodesic if and only if

(3.30) $$\overline{\nabla}_U(\bar{\phi}V) = \bar{\phi}(v\overline{\nabla}_U V)$$

for any U,V ∈ v.

Proof. Let U,V ∈ v. By using equation (3.1), we have

$$\overline{\nabla}_U(\bar{\phi}V) = (\overline{\nabla}_U\bar{\phi})V + \bar{\phi}\big(\overline{\nabla}_{PU}PV + \overline{\nabla}_{PU}QV + \overline{\nabla}_{QU}PV + \overline{\nabla}_{QU}QV\big).$$

In view of equation (2.18) and (3.1), the above equation implies

$$\overline{\nabla}_U(\bar{\phi}V) = (\overline{\nabla}_U\bar{\phi})V + \bar{\phi}(\mathcal{T}_U V) + \bar{\phi}(v\overline{\nabla}_U V).$$

Since the manifold $(\bar{M}, \bar{\emptyset}, \bar{\xi}, \bar{\eta}, \bar{g})$ is para-cosymplectic, we have $(\overline{\nabla}_U\bar{\phi}) = 0$. So from above equation, we have

(3.31) $$\overline{\nabla}_U(\bar{\phi}V) = \bar{\phi}(\mathcal{T}_U V) + \bar{\phi}(v\overline{\nabla}_U V)$$

Now, fibres are totally geodesic if and only if TuV $= 0$, which implies equation (3.30)

Theorem 3.9. Let f: $\bar{M} \longrightarrow M$ be a semi-slant pseudo-Riemannian submersion froman indefinite almost para-contact manifold $(\bar{M}, \bar{\emptyset}, \bar{\xi}, \bar{\eta}, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, the horizontal distribution $\mathcal{H}$ defines a totally geodesic foliation if and only if

$$\bar{g}(t(h\nabla_{\bar{X}}Y) + \psi Q(v\nabla_{\bar{X}}Y), \psi QU)$$
(3.32) $$+\bar{g}(n(h\overline{\nabla}_{\bar{X}}\bar{Y}) + \bar{\phi}P(v\overline{\nabla}_X Y) + \omega Q(v\overline{\nabla}_{\bar{X}}\bar{Y}), \bar{\phi}PU)$$
$$+\bar{g}(n(h\overline{\nabla}_{\bar{X}}\bar{Y}) + \bar{\phi}P(v\overline{\nabla}_X Y) + \omega Q(v\overline{\nabla}_{\bar{X}}\bar{Y}), \omega QU) = 0,$$

for any $\bar{X}, \bar{Y} \in \mathcal{H}$. and $U, V \in \mathcal{V}$.

Proof. Let $\bar{X}, \bar{Y} \in \mathcal{H}$. For any $U, V \in \mathcal{V}$, equation (2.2) implies

$$\bar{g}(\overline{\nabla}_{\bar{X}}\bar{Y}, U) = \bar{g}(\bar{\phi}(\overline{\nabla}_{\bar{X}}\bar{Y}), \bar{\phi}U).$$

By splitting horizontal and vertical components and the using equation (3.1), (3.2), (3.3), we have

$$\begin{aligned}
\bar{g}(\overline{\nabla}_{\bar{X}}\bar{Y}, U) &= \bar{g}(t(h\overline{\nabla}_{\bar{X}}\bar{Y}'), \bar{\phi}PU) + \bar{g}(t(h\overline{\nabla}_{\bar{X}}\bar{Y}), \psi QU) + \bar{g}(t(h\overline{\nabla}_{\bar{X}}\bar{Y}), \omega QU)\\
&\quad+\bar{g}(n(h\overline{\nabla}_{\bar{X}}\bar{Y}), \bar{\phi}PU) + \bar{g}(n(h\overline{\nabla}_{\bar{X}}\bar{Y}), \psi QU) + \bar{g}(n(h\overline{\nabla}_{\bar{X}}\bar{Y}), \omega QU)\\
&\quad+\bar{g}(\bar{\phi}P(v\overline{\nabla}_{\bar{X}}\bar{Y}), \bar{\phi}PU) + \bar{g}(\bar{\phi}P(v\overline{\nabla}_{\bar{X}}\bar{Y}), \psi QU) + \bar{g}(\bar{\phi}P(v\overline{\nabla}_{\bar{X}}\bar{Y}), \omega QU)\\
&\quad+\bar{g}(\psi Q(v\overline{\nabla}_{\bar{X}}\bar{Y}), \bar{\phi}PU) + \bar{g}(\psi Q(v\overline{\nabla}_{\bar{X}}\bar{Y}), \psi QU) + \bar{g}(\psi Q(v\overline{\nabla}_{\bar{X}}\bar{Y}), \omega QU)\\
&\quad+\bar{g}(\omega Q(v\overline{\nabla}_{\bar{X}}\bar{Y}), \bar{\phi}PU) + \bar{g}(\omega Q(v\overline{\nabla}_{\bar{X}}\bar{Y}), \psi QU) + \bar{g}(\omega Q(v\overline{\nabla}_{\bar{X}}\bar{Y}), \omega QU)\\
&= \bar{g}(t(h\overline{\nabla}_{\bar{X}}\bar{Y}), \psi QU) + \bar{g}(n(h\overline{\nabla}_{\bar{X}}\bar{Y}), \bar{\phi}PU) + \bar{g}(n(h\overline{\nabla}_{\bar{X}}\bar{Y}), \omega QU)\\
&\quad+\bar{g}(\bar{\phi}P(v\overline{\nabla}_{\bar{X}}\bar{Y}), \bar{\phi}PU) + \bar{g}(\bar{\phi}P(v\overline{\nabla}_{\bar{X}}\bar{Y}), \omega QU) + \bar{g}(\psi Q(v\overline{\nabla}_{\bar{X}}\bar{Y}), \psi QU)\\
&\quad+\bar{g}(\omega Q(v\overline{\nabla}_{\bar{X}}\bar{Y}), \bar{\phi}PU) + \bar{g}(\omega Q(v\overline{\nabla}_{\bar{X}}\bar{Y}), \omega QU)\\
&= \bar{g}(t(h\overline{\nabla}_{\bar{X}}\bar{Y}) + \psi Q(v\overline{\nabla}_{\bar{X}}\bar{Y}), \psi QU)\\
&\quad+\bar{g}(n(h\overline{\nabla}_{\bar{X}}\bar{Y}) + \bar{\phi}P(v\overline{\nabla}_{\bar{X}}\bar{Y}) + \omega Q(v\overline{\nabla}_{\bar{X}}\bar{Y}), \bar{\phi}PU)\\
&\quad+\bar{g}(n(h\overline{\nabla}_{\bar{X}}\bar{Y}) + \bar{\phi}P(v\overline{\nabla}_{\bar{X}}\bar{Y}) + \omega Q(v\overline{\nabla}_{\bar{X}}\bar{Y}), \omega QU),
\end{aligned}$$

which implies $\overline{\nabla}_{\bar{X}}\bar{Y} \in \mathcal{H}$ if and only if right side of a above equation vanishes. Hence $\mathcal{H}$ defines a totally geodesic foliation if and only if equation (3.32) is satisfied.

Corollary 3.1.Let $f: \bar{M} \longrightarrow M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost para-contact manifold $(\bar{M}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ onto a pseudoRiemannian manifold (M, g). Then following statements are equivalent:

(a) The horizontal distribution $\mathcal{H}$ defines a totally geodesic foliation,

(b) $\bar{g}(t(h\overline{\nabla}_{\bar{X}}\bar{Y}) + \psi Q(v\overline{\nabla}_{\bar{X}}\bar{Y}), \psi QU)$
$+\bar{g}(n(h\overline{\nabla}_{\bar{X}}\bar{Y}) + \bar{\phi}P(v\overline{\nabla}_X Y) + \omega Q(v\overline{\nabla}_{\bar{X}}\bar{Y}), \bar{\phi}PU)$
$+\bar{g}(n(h\overline{\nabla}_{\bar{X}}\bar{Y}) + \bar{\phi}P(v\overline{\nabla}_X Y) + \omega Q(v\overline{\nabla}_{\bar{X}}\bar{Y}), \omega QU) = 0,$

(c) $\bar{g}(\overline{\nabla}_{\bar{X}}\bar{Y}, t\psi QU \pm \psi Q\omega QU) = 0,$

(d) $\bar{g}(nt(h\overline{\nabla}_{\bar{X}}\bar{Y}) + \bar{\phi}Pn(h\overline{\nabla}_{\bar{X}}\bar{Y}) + \omega Qn(h\overline{\nabla}_{\bar{X}}\bar{Y}) + \bar{\phi}^2P(v\overline{\nabla}_{\bar{X}}\bar{Y})$
$+n\psi Q(v\overline{\nabla}_{\bar{X}}\bar{Y}) + \bar{\phi}P\omega Q(v\overline{\nabla}_{\bar{X}}\bar{Y}) + \omega Q\omega Q(v\overline{\nabla}_{\bar{X}}\bar{Y}), \quad U) = 0,$

for all $\bar{X}, \bar{Y} \in \mathcal{H}, U \in \mathcal{V}$.

Proof. In theorem (3.9), we have proved $(a) \Leftrightarrow (b)$. Similarly, we can prove $(b) \Leftrightarrow (c)$, $(c) \Leftrightarrow (d)$ and $(d) \Leftrightarrow (a)$.

s

Theorem 3.10. Let $f:\bar{M} \longrightarrow M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost para-contact manifold $(\bar{M},\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ onto a pseudo – Riemannian manifold (M,g). Then, the horizontal distribution $\mathcal{H}$ defines a totally geodesic foliation if and only if

(3.33) $$\bar{g}(h\bar{\nabla}_{\bar{X}}(t\bar{Y}) + h\bar{\nabla}_{\bar{X}}(n\bar{Y}), \psi QU) + \bar{g}(v\bar{\nabla}_{\bar{X}}(t\bar{Y}), \bar{\phi}PU + \omega QU) + \bar{g}(v\bar{\nabla}_{\bar{X}}(n\bar{Y}), \bar{\phi}PU + \omega QU) = 0,$$

for all $\bar{X},\bar{Y} \in \mathcal{H}$ and $U \in \mathcal{V}$.

Proof. Let $\bar{X},\bar{Y} \in \mathcal{H}, U \in \mathcal{V}$. Then, by using equation (2.2), we have

$\bar{g}(\bar{\nabla}_{\bar{X}}\bar{Y}, U) = \bar{g}(\bar{\nabla}_{\bar{X}}(\bar{\phi}\bar{Y}) - (\bar{\nabla}_{\bar{X}}\bar{\phi})\bar{Y}, \bar{\phi}U)$.

Now, by using equations (3.2) and (3.3) in above equation, we have

$$\bar{g}(\bar{\nabla}_{\bar{X}}\bar{Y}, U) = \bar{g}(h\bar{\nabla}_{\bar{X}}(t\bar{Y}) + h\bar{\nabla}_{\bar{X}}(n\bar{Y}), \psi QU) + \bar{g}(v\bar{\nabla}_{\bar{X}}(t\bar{Y}), \bar{\phi}PU + \omega QU) + \bar{g}(v\bar{\nabla}_{\bar{X}}(n\bar{Y}), \bar{\phi}PU + \omega QU).$$

$\mathcal{H}$ is totally geodesic if and only if $\bar{\nabla}_{\bar{X}}\bar{Y} \in \mathcal{H}$, which implies $\bar{g}(\bar{\nabla}_{\bar{X}}\bar{Y}, U) = 0$. Hence, the proof follows from equation (3.34).

Theorem 3.11. Let $f:\bar{M} \longrightarrow M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost para-contact manifold $(\bar{M},\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ onto a pseudoRiemannian manifold (M,g). Then, the vertical distribution $\mathcal{V}$ defines a totally geodesic foliation if and only if

(3.35) $$\bar{g}(t(h\bar{\nabla}_U V) + \psi Q(v\bar{\nabla}_U V), t\bar{X}) + \bar{g}(n(h\bar{\nabla}_U V) + \bar{\phi}P(v\bar{\nabla}_U V) + \omega Q(v\bar{\nabla}_U V), n\bar{X}) = 0,$$

for all $U, V \in \mathcal{V}, \bar{X} \in \mathcal{H}$.

Proof. Let $U, V \in \mathcal{V}, \bar{X} \in \mathcal{H}$. Using equation (2.2), we have

$\bar{g}(\bar{\nabla}_U V, \bar{X}) = \bar{g}(\bar{\phi}(\bar{\nabla}_U V), \bar{\phi}\bar{X}) + \varepsilon\bar{\eta}(\bar{\nabla}_U V)\bar{\eta}(\bar{X})$.

By using equations (3.2) and (3.3) in above equation, we have

$$\bar{g}(\bar{\nabla}_U V, \bar{X}) = \bar{g}(t(h\bar{\nabla}_U V) + \psi Q(v\bar{\nabla}_U V), t\bar{X}) + \bar{g}(n(h\bar{\nabla}UV) + \bar{\phi}P(v\bar{\nabla}_U V) + \omega Q(v\bar{\nabla}UV), n\bar{X}) + \varepsilon\bar{\eta}(\bar{\nabla}_U V)\bar{\eta}(\bar{X})$$

Now, the proof follows from equation (3.36).

Theorem 3.12. Let $f:\bar{M} \longrightarrow M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost para-cosymplectic manifold $(\bar{M},\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ onto a pseudo-Riemannian manifold (M,g). Then, the vertical distribution $\mathcal{V}$ defines a totally geodesic foliation if and only if

(3.37) $$\bar{g}(h\bar{\nabla}_U(\bar{\phi}PV) + h\bar{\nabla}_U(\psi QV) + h\bar{\nabla}_U(\omega QV), t\bar{X}) + \bar{g}(v\bar{\nabla}_U(\bar{\phi}PV) + v\bar{\nabla}_U(\psi QV) + v\bar{\nabla}_U(\omega QV), n\bar{X}) = 0,$$

for all $U, V \in \mathcal{V}, \bar{X} \in \mathcal{H}$.

Proof. Let $U, V \in \mathcal{V}$ and $\bar{X} \in \mathcal{H}$.

Using equation (2.2), we have

$\bar{g}(\bar{\nabla}_U V, \bar{X}) = \bar{g}(\bar{\phi}(\bar{\nabla}_U V), \bar{\phi}\bar{X}) + \varepsilon\bar{\eta}(\bar{\nabla}_U V)\bar{\eta}(\bar{X})$.

As $\bar{M}$ is almost para-cosymplectic manifold, we have $(\bar{\nabla}_U\bar{\phi})V = 0$ and so by using equations (3.2) and (3.3) in above equation, we get

(3.38) $$\bar{g}(\bar{\nabla}_U V, \bar{X}) = \bar{g}(h\bar{\nabla}_U(\bar{\phi}PV) + h\bar{\nabla}_U(\psi QV) + h\bar{\nabla}_U(\omega QV), t\bar{X}) + \bar{g}(v\bar{\nabla}_U(\bar{\phi}PV) + v\bar{\nabla}_U(\psi QV) + v\bar{\nabla}_U(\omega QV), n\bar{X}) + \varepsilon\bar{\eta}(\bar{\nabla}_U V)\bar{\eta}(\bar{X}).$$

Now, the vertical distribution $\mathcal{V}$ defines a totally geodesic foliation if and only if $\bar{\nabla}_U V \in \mathcal{V}$, for all $U, V \in \mathcal{V}$. This completes the proof.

Now, by using similar steps as in theorem 22 and theorem 24 of [22], we have

Theorem 3.13. Let $f:\bar{M} \longrightarrow M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost para-contact manifold $(\bar{M},\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ onto a pseudoRiemannian manifold (M,g). Then, the submersion f is an affine map on $\mathcal{H}$.

Theorem 3.14. Let $f:\bar{M} \longrightarrow M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost para-contact manifold $(\bar{M},\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ onto a pseudoRiemannian manifold (M,g). Then, the submersion f is an affine map if and only if $h(\bar{\nabla}_E hF) + \mathcal{A}_{hE}vF + \mathcal{T}_{vE}vF$ is f-related to $\nabla_X Y$, for any $E, F \in \Gamma(T\bar{M})$.

Theorem 3.15. Let $f:\bar{M} \longrightarrow M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost para-contact manifold $(\bar{M},\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ onto a pseudo - Riemannian manifold (M,g). Then, the submersion map f is totally geodesic if and only if

(3.39) $\mathcal{T}_U V + \mathcal{A}_{\bar{X}} V + h\bar{\nabla}_U \bar{Y} = 0,$

for any $U, V \in \mathcal{V}$ and $\bar{X}, \bar{Y} \in \mathcal{H}$.

Proof. Let $E = \bar{X} + U, F = \bar{Y} + V \in \Gamma(T\bar{M})$.

In view of equation (2.30) and theorem (3.13), by splitting E, F in horizontal and vertical components, we have

$$\begin{aligned}(\bar{\nabla} f_*)(E,F) &= (\bar{\nabla} f_*)(U,V) + (\bar{\nabla} f_*)(\bar{X},V) + (\bar{\nabla} f_*)(U,\bar{Y}) \\ &= -f_*\big(h(\bar{\nabla}_U V + \bar{\nabla}_{\bar{X}} V + \bar{\nabla}_U \bar{Y})\big),\end{aligned}$$

Which gives

(3.40) $(\bar{\nabla} f_*)(E,F) = -f_*(\mathcal{T}_U V + \mathcal{A}_{\bar{X}} V + h\bar{\nabla}_U \bar{Y}).$

Now, f is totally geodesic if and if $(\bar{\nabla} f_*)(E,F) = 0$, which implies equation (3.39).

Theorem 3.16. Let $f: \bar{M} \longrightarrow M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost para-contact manifold $(\bar{M}^{2m_1+1}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ onto a pseudoRiemannian manifold (M^{m_2}, g). If the fibres $f^{-1}(q)$ of f over $q \in M$ are totally geodesic, then f is a harmonic map.

Proof. The tension field $\tau(f)$ of the map $f: \bar{M} \longrightarrow M$ is defined as

(3.14) $\tau(f) = \text{trace}(\bar{\nabla} f_*).$

Let $\{e_1, e_2, \cdots, e_{2m_1+1-m_2}, e_{2m_1+1-m_2+1} = \bar{e}_1, \bar{e}_2, \cdots \bar{e}_{m_2}\}$ be an orthonormal basis of $\Gamma(TM)$, where $\{e_1, e_2, \cdots, e_{2m_1+1-m_2}\}$ is an orthonormal basis of $\mathcal{V}$ and $\{\bar{e}_1, \bar{e}_2, \cdots \bar{e}_{m_2}\}$ is an orthonormal basis of $\mathcal{H}$. Then, we have

(3.42) $$\tau(f) = \sum_{i=1}^{2m_1+1-m_2} \bar{g}(e_i, e_i)(\bar{\nabla} f_*)(e_i, e_i) + \sum_{j=1}^{m_2} \bar{g}(\bar{e}_j, \bar{e}_j)(\bar{\nabla} f_*)(\bar{e}_j, \bar{e}_j).$$

For any vertical vector fields $U, V \in \mathcal{V}$, using equation (2.18), we have

(3.42) $$\begin{aligned}(\bar{\nabla} f_*)(U,V) &= \left(\nabla_U^f (f_* V)\right) \circ f - f_*(\bar{\nabla}_U V) \\ &= -f_*(h\bar{\nabla}_U V) \\ &= -f_*(\mathcal{T}_U V),\end{aligned}$$

where ∇^f is the pullback connection of ∇ with respect to f. For any horizontal vector fields $\bar{X}, \bar{Y} \in \mathcal{H}$, which are f-related to $X, Y \in \Gamma(TM)$ respectively, lemma 2.1 and theorem 3.13 imply

$$\begin{aligned}(3.44)(\bar{\nabla} f_*)(\bar{X},\bar{Y}) &= \left(\nabla_{\bar{X}}^f (f_* \bar{Y})\right) \circ f - f_*(\bar{\nabla}_{\bar{X}} \bar{Y}) \\ &= \left(\nabla_{f_*\bar{X}} (f_* \bar{Y})\right) \circ f - f_*(h\bar{\nabla}_{\bar{X}} \bar{Y}) \\ &= 0\end{aligned}$$

In view of equations (3.42), (3.43), (3.44) and theorem 3.14, we get

(3.45) $$\tau(f) = -\sum_{i=1}^{4m+3-n} \bar{g}(e_i, e_i) f_*(\mathcal{T}_{e_i} e_i).$$

Now, if the fibres $f^{-1}(q)$ of f over $q \in M$ are totally geodesic, then $\mathcal{T} = 0$. So the proof of the theorem follows from equation (3.45).

References

[1]. Altafini, C.; Redundant robotic chains on Riemannian submersions, IEEE Transactions on Robotics and Automation, 20(2), (2004), 335-340.

[2]. Bourguignon, J. P. and Lawson, H. B.; Stability and isolation phenomena for Yang-mills fields, Commum. Math. Phys. 79, (1981), 189-230.

[3]. Bourguignon, J. P. and Lawson, H. B.; A Mathematicians visit to Kaluza-Klein theory, Rend. Semin. Mat. Torino Fasc. Spec. (1989), 143-163.

[4]. Chinea, D.; Almost contact metric submersion, Rendiconti del Circolo Mate. di Palermo, Series II, 34(1984), 319-330.

[5]. Chinea, D.; Transference of structures on almost complex contact metric submersions, Houston Journal of Mathematics, Vol. 14, No. 1, (1988).

[6]. Eells., Sampson J.H.; Harmonic mapping of Riemannian manifolds, Amer. J. Math., 1964,86. 109-160.

[7]. Falcitelli, M.; Ianus, S. and Pastore, A. M.; Riemannian Submersions and Related Topics, World Scientific, River Edge, NJ, (2004). MR2110043

[8]. Gray, A.; Pseudo- Riemannian almost product manifolds and submersions, J. Math. Mech., 16, (1967), 715-737.

[9]. Ianus, S. and Visinescu, M.; Kaluza-Klein theory with scalar fields and generalized Hopf manifolds, Class. Quantum Gravity 4, (1987), 1317-1325.

[10]. Ianus, S. and Visinescu, M.; Space-time compactification and Riemannian submersions, In: Rassias, G.(ed.) The Mathematical Heritage of C. F. Gauss, (1991), 358-371, World Scientific, River Edge.

[11]. Mustafa, M. T.; Applications of harmonic morphisms to gravity, J. Math. Phys., 41(10), (2000), 6918-6929.

[12]. O'Neill, B.; Semi-Riemannian Geometry with Applications to Relativity, Vol.103 of Pure and Applied Mathematics, Academic Press, New York,(1983).

[13]. O'Neill, B.; The fundamental equations of a submersion, Michigan Math. J., 13,(1966), 459-469.

[14]. Park, K. S.; H-slant submersions, Bull. Korean Math. Soc. 49 No. 2, (2012), 329-338.

[15]. Park, K. S.; V-Semi-slant submersions, arXiv: 1206.1404v1 [Math.DG] 7 June 2012.
[16]. Rio, E. G. and Kupeli, D. N.; Semi-Riemannian maps and their applications, Kluwer Academic Publisher, Dordrecht, (1999).
[17]. Sahin, B.; Anti-invariant Riemannian submersions from almost Hermitian manifolds, Cent. Eur. J. Math., 8(3), (2010), 437-447, DOI: 10.2478/s11533-010-0023-6.
[18]. Sahin, B.; Semi-invariant submersions from almost Hermitian manifolds, Canad. Math. Bull. , 54, No. 3 (2011).
[19]. Sahin, B.; Slant submersions from almost Hermitian manifolds, Bull. Math. Soc. Sci. Math. Roumanie Tome, 54 (102)No. 1, (2011), 93-105.
[20]. Sasaki, S.; On differentiable manifolds with certain structures which are closely related to almost contact structure I, Tohoko Math. J., 12,(1960), 459-476.
[21]. Sato, I.; On a structure similar to the almost contact structure, Tensor. New Series, Vol. 30 , No. 3, (1976), 219-224.
[22]. Shukla, S. S. and Verma, Uma Shankar; Paracomplex Paracontact Pseudo-Riemannian Submersions, Geometry, vol. 2014, Article ID 616487, 2014, 12 pages. doi:10.1155/2014/616487.
[23]. Watson, B.; Almost Hermitian submersions, J. Differential Geometry, 11, No. 1, (1976), 147165.
[24]. Watson, B.; G, G'-Riemannian submersions and nonlinear gauge field equations of general relativity, In: Rassias, T. (ed.) Global Analysis - Analysis on manifolds, dedicated M. Morse. Teubner-Texte Math., , Teubner, Leipzig, 57 (1983), 324-349.
[25]. Zamkovoy, S.; Canonical connections on para-contact manifolds, Ann. Glob. Anal. Geom., $(2009), 36: 37-60$.

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