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Fractional Fourier transform in extended Sobolev type spaces

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Abstract In this paper, we briefly introduce here extended Sobolev type space ${}^*\mathbb{H}_\alpha^{m,p}$. Some properties of fractional Fourier transform such as completeness, product, duality and convolution are investigated in ${}^*\mathbb{H}_\alpha^{m,p}$.

Keywords Fractional Fourier transform, Generalized schwartz space, Sobolev space.

Mathematics Subject Classification (2000) 35S05, 46E10, 46E35, 46F05, 47G30.

1 Introduction

The idea of the fractional powers of the Fourier operator has been "discovered" several times in the literature. Initially, the idea appeared in the mathematical literature between First world War, 1914 to Second World War, 1945 (e.g., [1, 2]). A large number of publications relating to this idea appeared after Second World War, 1945. The fractional Fourier operator re-gains a momentum in 1980s with publications by Namias (e.g.[3]). Following Namias's contribution, a large number of papers appeared in the mathematical literature during 1990s to 2012s tying the concept of the fractional Fourier operators to mathematical analysis, Distribution theory, theoretical research and many other fields, Time-frequency analysis as described in [4].

The fractional Fourier Transform (FRFT) is an elegant generalization of the ordinary Fourier transform [1–3, 5]. The fractional Fourier transform with a parameter α, has many applications in several areas including Communications, Optics, Quantum Physics and Singal processing. For more details of the fractional Fourier transform, see [6, 7]. The α-th order fractional Fourier transform represents the α-th power of the Fourier transform, when $\alpha = 90^\circ$, we obtain the Fourier transform, while for $\alpha = 0^\circ$, we obtain the signal itself. Any intermediate value of α ($0^\circ < \alpha < 90^\circ$) produces a signal representation that can be considered as a rotated time–frequency representation of the signal [4]. The fractional Fourier transform with a parameter α, $\varphi(x) \in L_1(\mathbb{R})$ is defined as [6–10]

$$(\mathscr{F}_\alpha\varphi)(\xi) = \widehat{\varphi}_\alpha(\xi) = \int_{\mathbb{R}} K_\alpha(x,\xi)\varphi(x)dx \tag{1.1}$$

with $K_\alpha(x,\xi)$ representing the kernel function is defined by

$$K_\alpha(x,\xi) = \begin{cases} C_\alpha e^{\frac{i(x^2+\xi^2)\cot\alpha}{2} - ix\xi\csc\alpha}, & \alpha \neq n\pi, n\epsilon\mathbb{Z} \\ \frac{1}{\sqrt{2\pi}}e^{-ix\xi}, & \alpha = \frac{\pi}{2} \\ \delta(x-\xi), & \alpha = 2n\pi \\ \delta(x+\xi), & \alpha = (2n+1)\pi \end{cases}$$

where $C_\alpha = (2\pi i\sin\alpha)^{-\frac{1}{2}}e^{\frac{i\alpha}{2}} = \sqrt{\frac{1-i\cot\alpha}{2\pi}}$ and $\delta(x)$ representing the Dirac delta function. Throughout the manuscript, we use $\mathscr{F}_\alpha$ or $\widehat{\varphi}_\alpha$ to denote the operator associated with the fractional Fourier transform.

The corresponding inversion formula of $(\mathscr{F}_\alpha\varphi)(\xi)$ is given by

$$\varphi(x) = \int_{\mathbb{R}} \overline{K_\alpha(x,\xi)}(\mathscr{F}_\alpha\varphi)(\xi)d\xi \tag{1.2}$$

$$\overline{K_\alpha(x,\xi)} = C'_\alpha e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2}+ix\xi\csc\alpha}$$

and

$$C'_\alpha = \overline{C_\alpha} = \overline{(2\pi i\sin\alpha)^{-\frac{1}{2}}e^{-\frac{i\alpha}{2}}} = \sqrt{\frac{1+i\cot\alpha}{2\pi}} = C_{-\alpha}.$$

Hence,

$$\overline{K_\alpha(x,\xi)} = K_{-\alpha}(x,\xi).$$

The inverse of a FRFT with the parameter α is the FrFT with the parameter $-\alpha$.

The current manuscript is was primarily inspired/motivated by the works of Ram Shankar Pathak [11]

The brief of the paper is given as follows: Section 1 is the introduction. In Section 2, some preliminaries results are discussed. In Section 3, extended Sobolev type space ${}^*\mathbb{H}^l_\alpha$ is defined and a theorem is suggested that characterizes the denseness property of this space. In Section 4, properties asssociated with the continuity of convolution are investigated. Finally, duality of the space is investigated in Section 5.

2 Preliminaries

In this section, we consider generalized Schwartz space, generalized fractional Fourier transform and define a linear space ${}^*\mathscr{D}_\alpha$ with two examples and extended Sobolev type space ${}^*\mathbb{H}^{m,p}_\alpha$. We also discuss continuity, linearity, completeness and denseness.

Definition 1 (Generalized Schwartz space) The space ${}^*\mathscr{G}_\alpha$ is defined as follows: φ is a member of ${}^*\mathscr{G}_\alpha$ iff it is a complex valued $\mathbb{C}^\infty$-function on $\mathbb{R}$ and for every choice of β and γ of non-negative integers, it satisfies that

$${}^*\chi^\alpha_{\beta,\gamma}(\varphi) = \sup_{x\in\mathbb{R}} \left|x^\beta(\Delta^*_x)^\gamma\varphi(x)\right| < \infty,$$

where

$$\Delta^*_x = -\left(\frac{d}{dx} + ix\cot\alpha\right), \quad \alpha \neq n\pi,\ n\in\mathbb{Z}.$$

Theorem 1 *The fractional Fourier transform $\mathscr{F}_\alpha : {}^*\mathscr{G}_\alpha \to {}^*\mathscr{G}_\alpha$is a continuous onto isomorphism. Its inverse $\mathscr{F}_\alpha^{-1} : {}^*\mathscr{G}_\alpha \to {}^*\mathscr{G}_\alpha$ is also a continuous onto isomorphism.*

Proof Let $\varphi(x) \in {}^*\mathscr{G}_\alpha \subset L_1(\mathbb{R})$, then its fractional Fourier transform

$$(\mathscr{F}_\alpha\varphi)(\xi) = \widehat{\varphi}_\alpha(\xi) = \int_{\mathbb{R}} K_\alpha(x,\xi)\varphi(x)dx \quad \text{exists.}$$

Now [12], we have

$$\begin{aligned}(\Delta^*_\xi)^r\widehat{\varphi}_\alpha(\xi) &= \mathscr{F}_\alpha[(-ix\csc\alpha)^r\varphi(x)](\xi)\\ &= ((\Delta^*_\xi)^r(\mathscr{F}_\alpha\varphi)(x))(\xi), \quad \forall\, \mathrm{r}\in\mathbb{N}.\end{aligned}$$

Since

$$\varphi \in {}^*\mathscr{G}_\alpha, \quad (-ix\csc\alpha)^r\varphi(x) \in {}^*\mathscr{G}_\alpha \ \Rightarrow \mathscr{F}_\alpha[(-ix\csc\alpha)^r\varphi(x)](\xi) \in {}^*\mathscr{G}_\alpha.$$

Hence,

$$
\begin{aligned}
{}^{*}\chi^{\alpha}_{\beta,\gamma}(\widehat{\varphi}_{\alpha}) &= \sup_{\xi\in\mathbb{R}} \left|\xi^{\beta}(\Delta^{*}_{\xi})^{\gamma}\widehat{\varphi}_{\alpha}(\xi)\right| \\
&= \sup_{\xi\in\mathbb{R}} \left|\xi^{\beta}\mathscr{F}_{\alpha}[(-ix\csc\alpha)^{\gamma}\varphi(x)](\xi)\right| < \infty,
\end{aligned} \tag{2.3}
$$

where β and γ are non-negative integers. This proves that $\widehat{\varphi}_{\alpha} \in \mathscr{G}_{\alpha}(\mathbb{R})$.
Also, from (1.1) and (1.2) we see that for all $\varphi \in \mathscr{G}_{\alpha}(\mathbb{R})$,

$$\mathscr{F}_{\alpha}^{-1}(\mathscr{F}_{\alpha}\varphi) = \varphi = \mathscr{F}_{\alpha}(\mathscr{F}_{\alpha}^{-1}\varphi).$$

It follows that $\mathscr{F}_{\alpha} : {}^{*}\mathscr{G}_{\alpha}(\mathbb{R}) \to {}^{*}\mathscr{G}_{\alpha}(\mathbb{R})$ is an one-one and onto function. To show that it is continuous. We assume that the sequence $\{\varphi_j\}_{j\in\mathbb{N}}$ converges in ${}^{*}\mathscr{G}_{\alpha}(\mathbb{R})$ to zero, then from (1.3) it follows that ${}^{*}\chi^{\alpha}_{\beta,\gamma}(\mathscr{F}_{\alpha}\varphi_j) \to 0$ as $j \to \infty$. This shows the continuity of fractional Fourier transform. Similary we can show that the inverse fractional Fourier transform $\mathscr{F}_{\alpha}^{-1} : {}^{*}\mathscr{G}_{\alpha}(\mathbb{R}) \to {}^{*}\mathscr{G}_{\alpha}(\mathbb{R})$ is also a continuous ishomorphism onto map.
This completes the proof.

Definition 2 The generalized fractional Fourier transform $\mathscr{F}_{\alpha}f$ of $f \in {}^{*}\mathscr{G}'_{\alpha}(\mathbb{R})$ is defined by

$$\langle \mathscr{F}_{\alpha}f \,,\, \varphi\rangle = \langle f \,,\, \mathscr{F}_{\alpha}\varphi\rangle, \tag{2.4}$$

where $\varphi \in {}^{*}\mathscr{G}_{\alpha}(\mathbb{R})$.

Theorem 2 *The generalized fractional Fourier transform $\mathscr{F}_{\alpha}$ is a continuous linear map of ${}^{*}\mathscr{G}'_{\alpha}$ onto itself.*

Proof Proof is similar of Theorem 3.3 [12].

Definition 3 If φ is a function defined on an open subset Ω of $\mathbb{R}$, the closure of the set $\{x \in \Omega : \varphi(x) \neq 0\}$ is called the support of the function φ and denoted by supp φ.
The set of all complex valued function φ defined on Ω and $(\Delta^{*}_{x})^{n}\varphi(x)$ exists for all $n \in \mathbb{N}$ and $\forall x \in \Omega$ as well as having compact support, $\alpha \neq n\pi$, is denoted by ${}^{*}\mathscr{D}_{\alpha}$. It is a linear space.

Example 1 *A function $\varphi : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is defined as follows:*

$$\varphi(x) = \frac{sin\pi x}{1+x^2} + \log_e|x - \frac{3}{2}|.$$

The set $\overline{\{x \in \mathbb{R} : \varphi(x) \neq 0\}} = \mathbb{R}$, which is not a compact set.
Therefore, $\varphi \notin {}^{*}\mathscr{D}_{\alpha}$.

Example 2 *A function φ is defined on $\mathbb{R}$ as follows:*

$$\varphi(x) = \begin{cases} e^{-\left(\frac{2}{x^2}+\frac{3}{(x-a)^2}\right)}, & 0 < x < a \\ 0 & elsewhere \end{cases}$$

Now,
$\Delta^{*}_{x}\varphi(x) = -\left(\frac{d}{dx} + ix\cot\alpha\right)e^{-\left(\frac{2}{x^2}+\frac{3}{(x-a)^2}\right)} = \left\{\frac{4}{x^3} + \frac{6}{(x-a)^3} - ix\cot\alpha\right\}\varphi(x)$ *exists $\forall x \in \mathbb{R}$. By mathematical induction, $(\Delta^{*}_{x})^{n}\varphi(x)$ exists for all $n \in \mathbb{N}$ and $\forall x \in \mathbb{R}$.*
Now, the set $\overline{\{x \in \mathbb{R} : \varphi(x) \neq 0\}} = \overline{(0,a)} = [0,a]$, which is a compact set.
Hence $\varphi \in {}^{*}\mathscr{D}_{\alpha}(\mathbb{R})$.

Theorem 3 ${}^{*}\mathscr{D}_{\alpha}(\mathbb{R})$ *is a dense in $L^{p}(\mathbb{R})$, $\quad 1 \leq p < \infty$.*

Proof Similar proof of Theorem 1.2.4 [14].

Definition 4 A mapping $f : {}^*\mathscr{D}_\alpha(\mathbb{R}) \to \mathbb{C}$ is called functional. A functional on ${}^*\mathscr{D}_\alpha(\mathbb{R})$ is said to be linear if $\langle f\,,\,\beta\phi_1+\gamma\phi_2\rangle = \langle f\,,\,\beta\phi_1\rangle + \langle f\,,\,\gamma\phi_2\rangle$ for any two complex numbers β and γ and $\phi_1,\,\phi_2 \in {}^*\mathscr{D}_\alpha(\mathbb{R})$.
The linear functional f is said to be continuous if for any sequence of functions $\{\phi_n\}$ that converges in ${}^*\mathscr{D}_\alpha(\mathbb{R})$ to zero, the sequence of numbers $\{\langle f\,,\,\phi_n\rangle\}$ also converges to zero as $n\to\infty$.
The set of all continuous linear functionals on ${}^*\mathscr{D}_\alpha(\mathbb{R})$ is denoted by ${}^*\mathscr{D}'_\alpha(\mathbb{R})$ or ${}^*\mathscr{D}'_\alpha$.

Definition 5 (The generalized Sobolev type space ${}^*\mathscr{H}^{m,p}_\alpha(\Omega)$) : We denote by ${}^*\mathscr{H}^{m,p}_\alpha(\Omega)$, is the space of all distributions $f \in {}^*\mathscr{D}'_\alpha(\mathbb{R})$ such that $(\Delta_x^*)^m f \in L^p(\Omega)$. We equip ${}^*\mathscr{H}^{m,p}_\alpha(\Omega)$) with the norm

$$ {}^\alpha\|f\|_{m,p} = \left(\int_\Omega |(\Delta_x^*)^m f(x)|^p dx\right)^{\frac{1}{p}}, \tag{2.5}$$

$$ {}^\alpha\|f\|_{m,\infty} = \|(\Delta_x^*)^m f\|_\infty, \quad \forall\, m\in\mathbb{N}, \quad 1\le p<\infty, \tag{2.6}$$

where Ω is an open subset of $\mathbb{R}$ and Δ_x^* as above.
We note that ${}^*\mathbb{H}^{0,p}_\alpha(\Omega) = L^p(\Omega)$ and ${}^*\mathbb{H}^{m,p}_\alpha(\Omega) \subset {}^*\mathbb{H}^{m',p}_\alpha(\Omega)$ if $m > m'$.

Theorem 4 *Δ_x^* is a continuous, linear operator in ${}^*\mathscr{D}'_\alpha(\Omega)$ in following sense:*
Linearity: $(\Delta_x^*)^m(af+bg) = a(\Delta_x^*)^m f + b(\Delta_x^*)^m g$, *for all* $f,\,g\in{}^*\mathscr{D}'_\alpha(\Omega)$ *and* $a,\,b\in\mathbb{C}$.
Continuity: If $f_n\to f$ *in* ${}^*\mathscr{D}'_\alpha(\Omega)$ *then* $(\Delta_x^*)^m f_n \to (\Delta_x^*)^m f$ *in* ${}^*\mathscr{D}'_\alpha(\Omega)$ *as* $n\to\infty$.

Proof The linearity is trivial. To prove continuity, let $\varphi\in{}^*\mathscr{D}'_\alpha(\Omega)$ and $\Delta_x^* = -\left(\frac{d}{dx}+ix\cot\alpha\right)$ then, $(\Delta_x^*)^m\varphi\in{}^*\mathscr{D}'_\alpha(\Omega)$.
Therefore, $\langle(\Delta_x^*)^m f_n\,,\,\varphi\rangle = \langle f_n\,,\,(\Delta_x^*)^m\varphi\rangle \to \langle f\,,\,(\Delta_x^*)^m\varphi\rangle = \langle(\Delta_x^*)^m f\,,\,\varphi\rangle$ as $n\to\infty$.
Consequently, $f_n\to f$ in ${}^*\mathscr{D}'_\alpha(\Omega)$ then $(\Delta_x^*)^m f_n\to(\Delta_x^*)^m f$ in ${}^*\mathscr{D}'_\alpha(\Omega)$ as $n\to\infty$.

Theorem 5 ${}^*\mathbb{H}^{m,p}_\alpha(\Omega)$ *ia a Banach space.*

Proof We need to show that ${}^*\mathbb{H}^{m,p}_\alpha(\Omega)$ is complete with respect to the above norms (1.6) and (1.7). Let $\{f_j\}_{j\in\mathbb{N}}$ be a Cauchy sequence in ${}^*\mathbb{H}^{m,p}_\alpha(\Omega)$. For every $m\in\mathbb{N}$, $\{(\Delta_x^*)^m f_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega)$. We know $L^p(\Omega)$ is complete. There exists g_m in $L^p(\Omega)$ such that $(\Delta_x^*)^m f_j\to g_m$ in $L^p(\Omega)$ as $j\to\infty$. In particular, $f_j\to g_0$ in $L^p(\Omega)$, hence $f_j\to g_0$ in ${}^*\mathbb{D}'_\alpha(\Omega)$. On other hand for every $m\in\mathbb{N}$, $(\Delta_x^*)^m$ is a continuous operator from ${}^*\mathbb{D}'_\alpha(\Omega)$ into ${}^*\mathbb{D}'_\alpha(\Omega)$ by Theorem 1.4. Hence $(\Delta_x^*)^m f_j\to(\Delta_x^*)^m g_0$ as $j\to\infty$.
By the uniqueness of the limit, we get $g = (\Delta_x^*)^m g_0$.
Therefore, $g_0\in{}^*\mathbb{H}^{m,p}_\alpha(\Omega)$ and $f_j\to g_0$ as $j\to\infty$ in ${}^*\mathbb{H}^{m,p}_\alpha(\Omega)$.
Hence ${}^*\mathbb{H}^{m,p}_\alpha(\Omega)$ is a Banach space.

For p=2, we denote ${}^*\mathbb{H}^{m,p}_\alpha(\Omega)$ by ${}^*\mathbb{H}^{m}_\alpha(\Omega)$ instead of ${}^*\mathbb{H}^{m,2}_\alpha(\Omega)$. In ${}^*\mathbb{H}^{m,2}_\alpha(\Omega)$, the norm

$$\|f\|_m = \|f\|_{m,2} = \left(\int_\Omega |(\Delta_x^*)^m f(x)|^2 dx\right)^{\frac{1}{2}}, \tag{2.7}$$

is induced by the scalar product

$$\langle f\,,\,g\rangle_m = \int_\Omega (\Delta_x^*)^m f(x)\overline{(\Delta_x^*)^m g(x)}dx. \tag{2.8}$$

Proposition 1 ${}^*\mathbb{H}^{m}_\alpha(\Omega)$ *is a Hilbert space with respect to the inner product* (1.9).

3 The Sobolev type Space ${}^*\mathbb{H}^l_\alpha$.

Sobolev space is named after the Russian mathematicians "Sergei Sobolev (1908-1989)", although they were known before the rise of the Russian mathematician to academic stardom. Although, more than 72 years have passed since the birth of Sobolev spaces, they still remain an active field of research. Nowadays, Sobolev spaces are the subject of countless papers, articles and monographs.
In this section, we shall define extended Sobolev type space ${}^*\mathbb{H}^l_\alpha$ and develop the theorem that characterizes the denseness property of this space.

By definition, $f\in{}^*\mathbb{H}^l_\alpha$ iff $(\Delta_x^*)^{2l} f(x)\in L^2(\mathbb{R})$. But by fractional Fourier transform, this is equivalent to $(-i\xi\csc\alpha)^{2l}\,\widehat{f}_\alpha(\xi)\in L^2(\mathbb{R}),\quad l\in\mathbb{N}$.

This motivates us to define the Sobolev type space by imposing suitable condition on the fractional Fourier transform of f instead of the condition on its Δ_x^*.

Definition 6 Let $l \in \mathbb{R}$, we denote by ${}^*\mathbb{H}^l_\alpha(\Omega)$, the space of all $f \in {}^*\mathscr{G}_\alpha$, $\left(1+|\xi \csc\alpha|^2\right)^{2l} \widehat{f}_\alpha(\xi) \in L^2(\mathbb{R})$. Obviously, for any real number l, ${}^*\mathbb{H}^l_\alpha(\Omega)$ is linear space.

We equip ${}^*\mathbb{H}^l_\alpha(\Omega)$ with the inner product

$$\langle f\,,\,g\rangle_l = \langle f\,,\,g\rangle_{{}^*\mathbb{H}^l_\alpha} = \int_{\mathbb{R}} \left(1+|\xi \csc\alpha|^2\right)^{2l} \widehat{f}_\alpha(\xi)\overline{\widehat{g}_\alpha(\xi)}d\xi, \tag{3.9}$$

which induces the norm

$${}^\alpha\|f\|_{{}^*\mathbb{H}^l_\alpha} = \left(\int_{\mathbb{R}} (1+|\xi \csc\alpha|^2)^{2l} \left|\widehat{f}_\alpha(\xi)\right|^2 d\xi\right)^{\frac{1}{2}}. \tag{3.10}$$

Example 3 *If $\mathscr{P}(\Delta_x^*)$ is a linear differential operator with constant co-efficient of order m and $u \in {}^*\mathbb{H}^l_\alpha(\mathbb{R})$, then $\mathscr{P}(\Delta_x^*)u \in {}^*\mathbb{H}^{l-m}_\alpha(\mathbb{R})$ and the map $\mathscr{P}(\Delta_x^*) : {}^*\mathbb{H}^l_\alpha(\mathbb{R}) \to {}^*\mathbb{H}^{l-m}_\alpha(\mathbb{R})$ is continuous.*

Proof Let

$$\mathscr{P}(\Delta_x^*) = \sum_{r=0}^{2m} a_r(\Delta_x^*)^r; \quad a_r \in \mathbb{C}.$$

Then for $u \in \mathscr{G}'_\alpha$ we have

$$\begin{aligned}
\|\mathscr{P}(\Delta_x^*)u\|_{{}^*\mathbb{H}^{l-m}_\alpha(\mathbb{R})} &= \left(\int_{\mathbb{R}} \left(1+|\xi \csc\alpha|^2\right)^{2l-2m} |\mathscr{F}_\alpha[\mathscr{P}(\Delta_x^*)u](\xi)|^2 d\xi\right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbb{R}} \left(1+|\xi \csc\alpha|^2\right)^{2l-2m} |\mathscr{P}(-i\xi \csc\alpha)\widehat{u}_\alpha(\xi)|^2 d\xi\right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbb{R}} \left(1+|\xi \csc\alpha|^2\right)^{2l-2m} \left|\sum_{r=0}^{m} a_r(-i\xi \csc\alpha)^r \widehat{u}_\alpha(\xi)\right|^2 d\xi\right)^{\frac{1}{2}} \\
&\leq \sum_{r=0}^{m} |a_r| \left(\int_{\mathbb{R}} \left(1+|\xi \csc\alpha|^2\right)^{2l-2m} \left(1+|\xi \csc\alpha|^2\right)^{2m} |\widehat{u}_\alpha(\xi)|^2 d\xi\right)^{\frac{1}{2}} \\
&= \sum_{r=0}^{m} |a_r|\, \|u\|_{{}^*\mathbb{H}^l_\alpha(\mathbb{R})},
\end{aligned}$$

which proves that $\mathscr{P}(\Delta_x^*)u \in {}^*\mathbb{H}^{l-m}_\alpha(\mathbb{R})$. To show that continuity assume that $\{u_j\}_{j\in\mathbb{N}}$ is a sequence in ${}^*\mathbb{H}^l_\alpha(\mathbb{R})$ which converges to zero in ${}^*\mathbb{H}^l_\alpha(\mathbb{R})$. Then, from above inequality it follows that

$$\|\mathscr{P}(\Delta_x^*)u\|_{{}^*\mathbb{H}^{l-m}_\alpha(\mathbb{R})} \leq \sum_{r=0}^{m} |a_r|\, \|u\|_{{}^*\mathbb{H}^l_\alpha(\mathbb{R})} \to 0 \text{ as } j \to \infty,$$

which implies the continuity of the operator $\mathscr{P}(\Delta_x^*)$.

Theorem 6 *For an integer $m \geq 0$, we have*

$${}^*\mathbb{H}^m_\alpha(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : (\Delta_x^*)^{2m} f \in L^2(\mathbb{R})\},$$

where Δ_x^* as above.

Proof Let $f \in {}^*\mathbb{H}^m_\alpha(\mathbb{R})$, then $f \in {}^*\mathscr{G}'_\alpha(\mathbb{R})$. Hence its fractional Fourier transform exists and we have

$$\left[\mathscr{F}_\alpha\left((\Delta_x^*)^{2m} f(x)\right)\right](\xi) = (-i\xi \csc\alpha)^{2m} \widehat{f}_\alpha(\xi).$$

Since $f \in {}^*\mathbb{H}^m_\alpha(\mathbb{R})$, we have

$$\int_{\mathbb{R}} (1+|\xi \csc\alpha|^2)^{2m} \left|\widehat{f}_\alpha(\xi)\right|^2 d\xi < \infty.$$

Now using the Parseval's identity for fractional Fourier transform,

$$\begin{aligned}
\int_{\mathbb{R}} \left|(\Delta_x^*)^{2m} f(x)\right|^2 dx &= \int_{\mathbb{R}} \left|\left[\mathscr{F}_\alpha\left((\Delta_x^*)^{2m} f(x)\right)\right](\xi)\right|^2 d\xi \\
&= \int_{\mathbb{R}} \left|(-i\xi \csc\alpha)^{2m} \widehat{f}_\alpha(\xi)\right|^2 d\xi \\
&\leq \int_{\mathbb{R}} \left|(1+|\xi \csc\alpha|^2)^{2m} \widehat{f}_\alpha(\xi)\right|^2 d\xi < \infty.
\end{aligned}$$

Next, we assume that $(\Delta_x^*)^{2m} f \in L^2(\mathbb{R}), \quad \forall\, m \in \mathbb{N}$, we have

$$\begin{aligned}\int_{\mathbb{R}} \left(1+|\xi \csc\alpha|^2\right)^{2m} \left|\widehat{f}_\alpha(\xi)\right|^2 d\xi &= \int_{\mathbb{R}} \sum_{k=0}^{2m} \binom{2m}{k} \left(|\xi \csc\alpha|^2\right)^k \left|\widehat{f}_\alpha(\xi)\right|^2 d\xi \\ &= \sum_{k=0}^{2m} \binom{2m}{k} \int_{\mathbb{R}} \left|\left[\mathscr{F}_\alpha\left((\Delta_x^*)^k f\right)\right](\xi)\right|^2 d\xi \\ &< \infty.\end{aligned}$$

Theorem 7 *The generalized Schwartz space ${}^*\mathscr{G}_\alpha$ is dense in ${}^*\mathbb{H}^l_\alpha$.*

Proof We know that

$${}^*\mathscr{G}_\alpha(\mathbb{R}) \subset {}^*\mathbb{H}^l_\alpha(\mathbb{R}), \quad \forall\, l \in \mathbb{R}.$$

Let $f \in {}^*\mathbb{H}^l_\alpha(\mathbb{R})$, then

$$\left(1+|\xi \csc\alpha|^2\right)^l \widehat{f}_\alpha(\xi) \in L^2(\mathbb{R}).$$

Since ${}^*\mathscr{D}_\alpha(\mathbb{R})$ is a set of all complex valued infinitely differentiable functions φ defined on $\mathbb{R}$ and having compact support. ${}^*\mathscr{D}_\alpha(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. There exists a sequence $\{\varphi_j\}_{j\in\mathbb{N}} \subset {}^*\mathscr{D}_\alpha(\mathbb{R})$ such that

$$\varphi_j \to \left(1+|\xi \csc\alpha|^2\right)^l \widehat{f}_\alpha(\xi) \quad \text{in} \quad L^2(\mathbb{R}) \quad \text{as} \quad j \to \infty.$$

Since $\left(1+|\xi \csc\alpha|^2\right)^{-l} \varphi_j(\xi) \in {}^*\mathscr{D}_\alpha(\mathbb{R})$, the functions $\psi_j(x) = \mathscr{F}_\alpha^{-1}\left(\left(1+|\xi \csc\alpha|^2\right)^{-l} \varphi_j(\xi)\right)$ are in ${}^*\mathscr{G}_\alpha \subset {}^*\mathscr{G}'_\alpha$. It follows that $(\mathscr{F}_\alpha \psi_j)(\xi) = \left(1+|\xi \csc\alpha|^2\right)^{-l} \varphi_j(\xi)$ and then

$$\begin{aligned}\|f-\psi_j\|^2_{{}^*\mathbb{H}^l_\alpha} &= \int_{\mathbb{R}} \left(1+|\xi \csc\alpha|^2\right)^{2l} |\mathscr{F}_\alpha(f-\psi_j)(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}} \left(1+|\xi \csc\alpha|^2\right)^{2l} |(\mathscr{F}_\alpha f)(\xi) - (\mathscr{F}_\alpha \psi_j)(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}} \left(1+|\xi \csc\alpha|^2\right)^{2l} \left|\widehat{f}_\alpha(\xi) - \left(1+|\xi \csc\alpha|^2\right)^{-l} \varphi_j(\xi)\right|^2 d\xi \\ &= \int_{\mathbb{R}} \left|\left(1+|\xi \csc\alpha|^2\right)^l \widehat{f}_\alpha(\xi) - \varphi_j(\xi)\right|^2 d\xi \\ &\to 0 \quad \text{as} \quad j \to \infty.\end{aligned}$$

This implies that $\|f-\psi_j\|_{{}^*\mathbb{H}^l_\alpha} \to 0 \quad \text{as} \quad j \to \infty$. Hence ${}^*\mathscr{G}_\alpha(\mathbb{R})$ is dense in ${}^*\mathbb{H}^l_\alpha(\mathbb{R})$.

4 Convolution for the fractional Fourier transform

To make the paper self content, we recall the definition of convolution for the fractional Fourier transform that can be found in [9].

Definition 7 For any function f(x), let us define the functions $\check{f}(x)$ and $\check{\check{f}}(x)$ by $\check{f}(x) = f(x)e^{ix^2\frac{\cot\alpha}{2}}$ and $\check{\check{f}}(x) = f(x)e^{-ix^2\frac{\cot\alpha}{2}}$. For any two functions f and g, we define the convolution operator $\star$ by

$$h(x) = (f \star g)(x) = C_\alpha e^{-ix^2\frac{\cot\alpha}{2}} \left(\check{f} * \check{g}\right)(x),$$

where $*$ is the convolution operation for the Fourier transform as defined by

$$(f * g)(x) = \int_{\mathbb{R}} f(y) g(x-y) dy.$$

Moreover

$$(\mathscr{F}_\alpha h)(\xi) = [\mathscr{F}_\alpha(f \star g)](\xi) = (\mathscr{F}_\alpha f)(\xi)(\mathscr{F}_\alpha g)(\xi) e^{-i\xi^2\frac{\cot\alpha}{2}}$$

Theorem 8 *If $\varphi \in {}^*\mathbb{H}^l_\alpha(\mathbb{R})$ and $\psi \in {}^*\mathscr{G}_\alpha(\mathbb{R})$, then the convolution $\varphi \star \psi \in {}^*\mathbb{H}^l_\alpha(\mathbb{R})$, and the map $\varphi \to \varphi \star \psi$ is continuous from ${}^*\mathbb{H}^l_\alpha(\mathbb{R})$ into ${}^*\mathbb{H}^l_\alpha(\mathbb{R})$, $\quad \forall\, l \in \mathbb{R}$.*

Proof If $\varphi \in {}^*\mathbb{H}^l_\alpha(\mathbb{R})$ and $\psi \in {}^*\mathscr{G}_\alpha(\mathbb{R})$, then from above definition, we have

$$[\varphi \star \psi](x) = C_\alpha e^{-ix^2\frac{\cot\alpha}{2}} \int_{\mathbb{R}} \varphi(y) e^{iy^2\frac{\cot\alpha}{2}} \psi(x-y) e^{i(x-y)^2\frac{\cot\alpha}{2}} dy.$$

It is obvious that $\varphi \star \psi \in C^\infty(\mathbb{R})$. Since $\left(1+|\xi \csc\alpha|^2\right)^l \widehat{\varphi}_\alpha(\xi) \in L^2(\mathbb{R})$ and $\widehat{\psi}_\alpha(\xi) \in {}^*\mathscr{G}_\alpha(\mathbb{R})$, the product $\left(1+|\xi \csc\alpha|^2\right)^l \widehat{\varphi}_\alpha(\xi)\widehat{\psi}_\alpha(\xi) \in L^2(\mathbb{R})$, hence $\varphi \star \psi \in {}^*\mathbb{H}^l_\alpha(\mathbb{R})$. Moreover

$$\begin{aligned}
\|\varphi \star \psi\|_{{}^*\mathbb{H}^l_\alpha} &= \left(\int_{\mathbb{R}} \left(1+|\xi \csc\alpha|^2\right)^{2l} |[\mathscr{F}_\alpha(\varphi \star \psi)](\xi)|^2 d\xi\right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbb{R}} \left(1+|\xi \csc\alpha|^2\right)^{2l} |[\mathscr{F}_\alpha(\varphi)](\xi)|^2 |[\mathscr{F}_\alpha(\psi)](\xi)|^2 d\xi\right)^{\frac{1}{2}} \\
&\le \sup_{\xi\in\mathbb{R}} |\widehat{\psi}(\xi)|\, \|\varphi\|_{{}^*\mathbb{H}^l_\alpha} \\
&< \infty.
\end{aligned}$$

Consequently, $\varphi \star \psi \in {}^*\mathbb{H}^l_\alpha(\mathbb{R})$.

Theorem 9 *If $\check{f}(x) = f(x)e^{ix^2\frac{\cot\alpha}{2}} \in {}^*\mathbb{H}^l_\alpha(\mathbb{R})$ and $\check{g}(x) = g(x)e^{ix^2\frac{\cot\alpha}{2}} \in {}^*\mathscr{G}_\alpha(\mathbb{R})$, then the product $\check{f}\check{g} \in {}^*\mathbb{H}^l_\alpha(\mathbb{R})$, and the map $f \to \check{f}\check{g}$ is continuous from ${}^*\mathbb{H}^l_\alpha(\mathbb{R})$ into ${}^*\mathbb{H}^l_\alpha(\mathbb{R})$ $\;\forall\; l \in \mathbb{R}$.*

Proof Let $\check{f} \in {}^*\mathbb{H}^s_\alpha(\mathbb{R})$, $\quad \check{g} \in {}^*\mathscr{G}_\alpha(\mathbb{R})$ such that

$$\check{f}(x) = e^{ix^2\frac{\cot\alpha}{2}} f(x)$$

and

$$\check{g}(x) = e^{ix^2\frac{\cot\alpha}{2}} g(x),$$

where $f \in {}^*\mathbb{H}^s_\alpha(\mathbb{R})$ and $g \in {}^*\mathscr{G}_\alpha(\mathbb{R})$. Now as per [9], we have

$$\begin{aligned}
\left[\mathscr{F}_\alpha\left(\check{f}(x)\check{g}(x)\right)\right](\xi) &= \left[\mathscr{F}_\alpha\left(f(x)g(x)e^{x^2 i\cot\alpha}\right)\right](\xi) \\
&= C_{-\alpha} e^{\frac{i}{2}\xi^2\cot\alpha} \int_{\mathbb{R}} e^{-\frac{i}{2}\eta^2\cot\alpha} \widehat{f}_\alpha(\eta) e^{-\frac{i}{2}(\xi-\eta)^2\cot\alpha} \widehat{g}_\alpha(\xi-\eta) d\eta.
\end{aligned}$$

Now, using [13], we have

$$\begin{aligned}
\left|\left(1+|\xi \csc\alpha|^2\right)^l \left[\mathscr{F}_\alpha\left(\check{f}(x)\check{g}(x)\right)\right](\xi)\right| &\le C_{-\alpha} \int_{\mathbb{R}} \left(1+|\xi \csc\alpha|^2\right)^l |\widehat{f}_\alpha(\eta)||\widehat{g}_\alpha(\xi-\eta)| d\eta \\
&= C_{-\alpha} \int_{\mathbb{R}} \frac{\left(1+|\xi \csc\alpha|^2\right)^l}{\left(1+|\eta \csc\alpha|^2\right)^l} \left(1+|\eta \csc\alpha|^2\right)^l | \\
&\times \widehat{f}_\alpha(\eta)||\widehat{g}_\alpha(\xi-\eta)| d\eta \\
&\le C_{-\alpha} 2^{|l|} \int_{\mathbb{R}} \left(1+|(\xi-\eta)\csc\alpha|^2\right)^l \left(1+|\eta \csc\alpha|^2\right)^l \\
&\times |\widehat{f}_\alpha(\eta)||\widehat{g}_\alpha(\xi-\eta)| d\eta.
\end{aligned}$$

Let $G_\alpha(\xi) = \left(1+|\xi \csc\alpha|^2\right)^l |\widehat{g}_\alpha(\xi)|$, $F_\alpha(\xi) = \left(1+|\xi \csc\alpha|^2\right)^l |\widehat{f}_\alpha(\xi)|$.
Then we have

$$\left|\left(1+|\xi \csc\alpha|^2\right)^l \left[\mathscr{F}_\alpha\left(\check{f}(x)\check{g}(x)\right)\right](\xi)\right| \le C_{-\alpha} 2^{|l|} \left(G_\alpha \star F_\alpha\right)(\xi).$$

Now, we note that

$$\begin{aligned}|(G_\alpha \star F_\alpha)(\xi)| &= \left|\int_{\mathbb{R}} G^{\frac{1}{2}}(\xi-\eta)F_\alpha(\eta)G^{\frac{1}{2}}(\xi-\eta)d\eta\right| \\ &\le \left(\int_{\mathbb{R}} |G(\xi-\eta)||F_\alpha(\eta)|^2 d\eta\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |G(\xi-\eta)|^2 d\eta\right)^{\frac{1}{2}}.\end{aligned}$$

Hence

$$\begin{aligned}\int_{\mathbb{R}} \left(1+|\xi \csc\alpha|^2\right)^{2l} \left|\left[\mathscr{F}_\alpha\left(\check{f}(x)\check{g}(x)\right)\right](\xi)\right|^2 d\xi &\le C_1 \int_{\mathbb{R}} |(G_\alpha \star F_\alpha)(\xi)|^2 d\xi \\ &\le C_1 \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |G_\alpha(\xi-\eta)||F_\alpha(\eta)|^2 d\eta\right) d\xi \\ &\quad \times \|G_\alpha\|_{L^1(\mathbb{R})} \\ &= C_1 \|G_\alpha\|_{L^1(\mathbb{R})} \int_{\mathbb{R}} |F_\alpha(\eta)| d\eta,\end{aligned}$$

where $C_1 = |C_{-\alpha}|^2 2^{|l|}$.
Therefore

$$\left\|\check{f}\check{g}\right\|_{{}^*\mathbb{H}^l_\alpha} \le C_1 \|g\|_{L^1(\mathbb{R})} \|f\|_{{}^*\mathbb{H}^l_\alpha}(\mathbb{R}).$$

It implies that $\check{f}\check{g} \in {}^*\mathbb{H}^l_\alpha(\mathbb{R})$.
The proof of the theorem is complete.

5 The space ${}^*\mathbb{H}^{-l}_\alpha(\mathbb{R})$

In this section, we will investigate the duality of the space ${}^*\mathbb{H}^l_\alpha(\mathbb{R})$ with some its properties.

A Hilbert space ${}^*\mathbb{H}^l_\alpha(\mathbb{R})$ has a dual space with respect to the inner product space $\langle f\,,\,\varphi\rangle_{{}^*\mathbb{H}^l_\alpha}$. A function $f \in {}^*\mathscr{G}_\alpha(\mathbb{R})$ defines a continuous linear functional on ${}^*\mathscr{G}_\alpha(\mathbb{R})$ by

$$\langle f\,,\,\varphi\rangle = \int_{\mathbb{R}} f(x)\varphi(x)dx = \int_{\mathbb{R}} \widehat{f}_\alpha(\xi)\widehat{\varphi}_\alpha(\xi)d\xi$$

by Parseval's relation. Since

$$\widehat{f}_\alpha(\xi)\widehat{\varphi}_\alpha(\xi) = (1+|\xi \csc\alpha|)^{-l}\widehat{f}_\alpha(\xi)(1+|\xi \csc\alpha|)^{l}\widehat{\varphi}_\alpha(\xi),$$

using Schwartz's inequality, we obtain

$$|\langle f\,,\,\varphi\rangle| \le \|f\|_{{}^*\mathbb{H}^{-l}_\alpha} \|\varphi\|_{{}^*\mathbb{H}^l_\alpha}.$$

Theorem 10 *The space ${}^*\mathbb{H}^{-l}_\alpha(\mathbb{R})$ is the dual space of ${}^*\mathbb{H}^l_\alpha$ for all $l \in \mathbb{R}$.*

Proof Since ${}^*\mathscr{G}_\alpha$ is dense in ${}^*\mathbb{H}^l_\alpha$ and ${}^*\mathscr{G}_\alpha \subset {}^*\mathbb{H}^l_\alpha \subset {}^*\mathscr{G}'_\alpha$. It implies that the dual space $\left({}^*\mathbb{H}^l_\alpha\right)'$ of ${}^*\mathbb{H}^l_\alpha$ is a subspace of ${}^*\mathscr{G}'_\alpha$. Now let $f \in {}^*\mathbb{H}^{-l}_\alpha(\mathbb{R})$ and $\varphi \in {}^*\mathbb{H}^l_\alpha(\mathbb{R})$, then

$$
\begin{aligned}
|\langle f\,,\,\varphi\rangle| &= \left|\int_{\mathbb{R}} \widehat{f}_\alpha(\xi)\widehat{\varphi}_\alpha(\xi)d\xi\right| \\
&= \left|\int_{\mathbb{R}} (1+|\xi\csc\alpha|^2)^{-l}\widehat{f}_\alpha(\xi)(1+|\xi\csc\alpha|)^{l}\widehat{\varphi}_\alpha(\xi)d\xi\right| \\
&\leq \left(\int_{\mathbb{R}} (1+|\xi\csc\alpha|^2)^{-2l}\left|\widehat{f}_\alpha(\xi)\right|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} (1+|\xi\csc\alpha|^2)^{2l}|\widehat{\varphi}_\alpha(\xi)|^2\right)^{\frac{1}{2}} \\
&= \|f\|_{{}^*\mathbb{H}^{-l}_\alpha}\|\varphi\|_{{}^*\mathbb{H}^{l}_\alpha}.
\end{aligned}
$$

It implies that f is a continuous linear functional on ${}^*\mathbb{H}^l_\alpha$. Hence $f \in \left({}^*\mathbb{H}^l_\alpha\right)'$.
Thus

$$ {}^*\mathbb{H}^{-l}_\alpha \subset \left({}^*\mathbb{H}^l_\alpha\right)'. \tag{5.11} $$

Now, we want to show that

$$ \left({}^*\mathbb{H}^l_\alpha\right)' \subset {}^*\mathbb{H}^{-l}_\alpha. $$

Let $f \in \left({}^*\mathbb{H}^l_\alpha\right)'$ be arbitrary.
By Riesz Representation Theorem, we know that if f is a bounded linear functional on a Hilbert space ${}^*\mathbb{H}^l_\alpha$, there exists a function $\varphi \in {}^*\mathbb{H}^l_\alpha$ such that

$$
\begin{aligned}
f(\varphi) = \langle f\,,\,\varphi\rangle_{{}^*\mathbb{H}^l_\alpha} &= \int_{\mathbb{R}} (1+|\xi\csc\alpha|^2)^{2l}\widehat{\varphi}_\alpha(\xi)\overline{\widehat{f}_\alpha(\xi)}d\xi, \\
&= \int_{\mathbb{R}} \varphi(x)\overline{k(x)}dx,
\end{aligned}
$$

where

$$ k(x) = \mathscr{F}^{-1}_\alpha\left[(1+|\xi\csc\alpha|^2)^{2l}\widehat{f}_\alpha(\xi)\right]. $$

Since the function

$$ (1+|\xi\csc\alpha|^2)^{-l}\widehat{k}_\alpha(\xi) = (1+|\xi\csc\alpha|^2)^{l}\widehat{f}_\alpha(\xi) \in L^2(\mathbb{R}). $$

Hence $\overline{k} \in {}^*\mathbb{H}^l_\alpha$ and $f(\varphi) = \langle \overline{k}\,,\,\varphi\rangle \quad \varphi \in {}^*\mathbb{H}^l_\alpha$.
It implies that

$$ \left({}^*\mathbb{H}^l_\alpha\right)' \subset {}^*\mathbb{H}^{-l}_\alpha. \tag{5.12} $$

From (4.12) and (4.13), we obtain

$$ {}^*\mathbb{H}^{-l}_\alpha = \left({}^*\mathbb{H}^l_\alpha\right)'. $$

The dual space of space ${}^*\mathbb{H}^l_\alpha$ is ${}^*\mathbb{H}^{-l}_\alpha, \quad \forall\, l \in \mathbb{R}$.

Theorem 11 *Let* $f \in \mathbb{H}^{-l}_\alpha, \quad l \in \mathbb{N}$. *Then*

$$ f(x) = \sum_{r=1}^{n} (\Delta^*_x)^r g_r(x); \quad g_r \in L^2(\mathbb{R}), \quad r = 0,1,2,3..............,n. $$

Proof Since $f \in \mathbb{H}_\alpha^{-l}$ iff $(1+|\xi \csc\alpha|^2)^{-l}\widehat{f}_\alpha(\xi) \in L^2(\mathbb{R})$.
Now let

$$(1+|\xi \csc\alpha|^2)^{-l}\widehat{f}_\alpha(\xi) = \widehat{g}'_\alpha(\xi),$$

then we have

$$\widehat{f}_\alpha(\xi) = (1+|\xi \csc\alpha|^2)^{l}\widehat{g}'_\alpha(\xi).$$

$$\begin{aligned}\widehat{f}_\alpha(\xi) &= \sum_{r=0}^{l}\binom{l}{r}|\xi \csc\alpha|^{2r}\widehat{g}'_\alpha(\xi)\\ &= \sum_{r=0}^{l}\binom{l}{r}\frac{|\xi \csc\alpha|^{2r}}{(-i\xi \csc\alpha)^r}(-i\xi \csc\alpha)^r\widehat{g}'_\alpha(\xi)\\ &= \sum_{r=0}^{l}\binom{l}{r}(-i\xi \csc\alpha)^r(\mathscr{F}_\alpha g_r)(\xi)\end{aligned}$$

where $\frac{|\xi \csc\alpha|^{2r}}{(-i\xi \csc\alpha)^r}\widehat{g}'_\alpha(\xi) = (\mathscr{F}_\alpha g_r)(\xi)$
Hence, by inverse fractional Fourier transform,

$$f(x) = \sum_{r=0}^{l}\binom{l}{r}(\Delta_x^*)^r g_r(x); \quad g_r \in L^2(\mathbb{R}), \quad r = 0,1,2,3.............,l.$$

This completes the proof of the theorem.

Gilbert G Walter and Xiaoping Shen [14] introduced the subspace B_0 of $H^{-1}(\mathbb{R})$ which is very important in Wavelet Analysis.

$$B_0 = \{f \in H^{-1} : \operatorname{supp} f \subset \mathbb{Z}\}$$

. This motivates us to define the subspace ${}^*\mathbb{B}_\alpha$ of ${}^*\mathbb{H}_\alpha^{-1}(\mathbb{R})$ using the theory of distributional fractional Fourier transform as follows

$${}^*\mathbb{B}_\alpha = \{h \in {}^*\mathbb{H}_\alpha^{-1}(\mathbb{R}) : \operatorname{supp} f \subset \mathbb{Z}\}. \tag{5.13}$$

The following characterization of ${}^*\mathbb{B}_\alpha$ is due to Pathak [11].

Theorem 12 *A necessary and sufficient condition for*

$$h(x) = \sum_{n=-\infty}^{\infty} d_n\delta(x-n), \quad \{d_n\} \in l^2, \tag{5.14}$$

the series $\sum d_n\delta(x-n)$ to be convergent in ${}^*\mathscr{G}_\alpha(\mathbb{R})$ is that $h \in {}^*\mathbb{B}_\alpha$.

Proof **Necessary Condition.** Here we suppose that h is given by (4.15) and the series $\sum d_n\delta(x-n)$ being convergent in ${}^*\mathscr{G}_\alpha(\mathbb{R})$.We want to show that $h \in {}^*\mathbb{B}_\alpha$.
Then it has support in $\mathbb{Z}$ and its fractional Fourier transform is given by

$$\widehat{h}_\alpha(\xi) = \sum_{n=-\infty}^{\infty} d_n K_\alpha(\xi,n).$$

Since $\{d_n\} \in l^2$,

$$\begin{aligned}\int_{-\infty}^{\infty}\left(1+|\xi \csc\alpha|^2\right)^{2l}\left|\widehat{h}_\alpha(\xi)\right|d\xi &= \int_{-\infty}^{\infty}\left(1+|\xi \csc\alpha|^2\right)^{2l}\left|\sum_{n=-\infty}^{\infty} d_n K_\alpha(\xi,n)\right|d\xi\\ &\le \int_{-\infty}^{\infty}\left(1+|\xi \csc\alpha|^2\right)^{2l}\sum_{n=-\infty}^{\infty}|d_n||K_\alpha(\xi,n)|d\xi\\ &\le |C_\alpha|\int_{-\infty}^{\infty}\left(1+|\xi \csc\alpha|^2\right)^{2l}\left(\sum_{n=-\infty}^{\infty}|d_n|\right)d\xi < \infty\end{aligned}$$

for all $l < -\frac{1}{4}$ and $|C_\alpha| > 0$.
Hence $h \in {}^*\mathbb{B}_\alpha$.

Sufficient Condition. Here we suppose that $h \in {}^*\mathbb{B}_\alpha$. Then it is a tempered distribution in ${}^*\mathscr{G}'_\alpha(\mathbb{R})$. with point support.We have to prove that the series $\sum d_n \delta(x-n)$ is convergent in ${}^*\mathscr{G}'_\alpha(\mathbb{R})$. For any $\psi \in {}^*\mathscr{G}_\alpha(\mathbb{R})$ and from (4.15) we have

$$\left| \langle \sum_{n=-\infty}^{\infty} d_n \delta(x-n)\,,\,\psi \rangle \right| = \left| \sum_{n=-\infty}^{\infty} d_n \psi(n) \right| \leq \left(\sum_{n=-\infty}^{\infty} |d_n|^2 \right)^{\frac{1}{2}} \left(\sum_{n=-\infty}^{\infty} |\psi(n)|^2 \right)^{\frac{1}{2}},$$

Since $\psi \in {}^*\mathscr{G}_\alpha(\mathbb{R})$, $\psi(0)$ is bounded and $|\psi(n)| \leq D|n|^{-1}$, $D > 0$, for all $n \neq 0$. Hence

$$\left(\sum_{n=-\infty}^{\infty} |\psi(n)|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{n=-\infty}^{\infty} |Dn^{-1}|^2 \right)^{\frac{1}{2}} \leq D \left(\sum_{n=-\infty}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} < \infty.$$

Thus, if $d_n \in l^2$, the series $\sum d_n \delta(x-n)$ is convergent in ${}^*\mathscr{G}'_\alpha(\mathbb{R})$.
Hence the theorem.

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