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Spectral and Numerical Gaps of Some Operators

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Abstract *In this paper, we investigate the relationship between some classes of operators and their spectral and numerical gaps. We characterize these gaps for equivalent operators.*

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1. Introduction

Let $\mathcal{H}$ denote a Hilbert space and $B(\mathcal{H})$ denote the Banach algebra of bounded linear operators. If $T \in B(\mathcal{H})$, then T^* denotes the adjoint of T, while $\mathrm{Ker}(T), \mathrm{Ran}(T), \overline{\mathcal{M}}$ and $\mathcal{M}^{\perp}$ stands for the kernel of T, range of T, closure of $\mathcal{M}$ and orthogonal complement of a closed subspace $\mathcal{M}$ of $\mathcal{H}$, respectively. We denote by $\sigma(T)$, $\| T \|$ and $W(T)$, the spectrum, norm and numerical range of T, respectively.

Two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are said to be similar (denoted $A \sim B$) if there exists an invertible operator $N \in B(\mathcal{H}, \mathcal{K})$ such that $NA = BN$ or equivalently $A = N^{-1}BN$, and are unitarily equivalent (denoted by $A \cong B$)if there exists a unitary operator $U \in B_+(\mathcal{H}, \mathcal{K})$ (Banach algebra of all invertible operators in $B(\mathcal{H})$) such that $UA = BU$(i.e. $A = U^*BU$, equivalently, $A = U^{-1}BU$). Two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are said to be metrically equivalent (denoted by $A \sim_m B$) if $\| Ax \| = \| Bx \|$, (equivalently, $|\langle Ax, Ax\rangle|^{\frac{1}{2}} = |\langle Bx, Bx\rangle|^{\frac{1}{2}}$ for all $x \in \mathcal{H}$). The concept of metric equivalence of operators was initially introduced in [17] in 2013. Clearly similarity, unitary equivalence and metric equivalence are equivalence relations on $B(\mathcal{H})$.
T and S are nearly-equivalent if T^*T and S^*S are similar and are unitarily-quasi-equivalent if there is a unitary operator U such that $T^*T = US^*SU^*$. An operator T is said to be nearly normal if $T^*T = ATT^*A^{-1}$, where A is an invertible operator.

2. Main Results

Let A be an operator on a separable Hilbert space $\mathcal{H}$. The spectrum of A is defined as $\sigma(A) = \{\lambda \in \mathbb{C} : \lambda I - A$ is not invertible $\}$. Note that $\sigma(A) \subset \{\lambda \in \mathbb{C}: |\lambda| \leq \| A \|\}$. The numerical range of A is defined to be the set $W(A) = \{\langle Ax, x\rangle: \| x \| = 1\}$ and the numerical radius of A is defined as $w(A) = \sup\{|\lambda|: \lambda \in W(A)\}$. (See [9], [15]). An important use of $W(T)$ is to bound the spectrum $\sigma(T)$ of an operator T.(cf. [9], [6]).

The spectral radius of an operator A is defined as $r(A) = \sup\{|\lambda|: \lambda \in \sigma(A)\}$ (see [10], § 88). Clearly, for any $T, S \in B(\mathcal{H}), r(ST) = r(TS), \sigma(ST)$ and $\sigma(TS)$ differ at most by 0, which is irrelevant for the definition of the spectral radius.

It is well known (cf. [9], [15], [6]) that $w(.)$is a norm on $B(\mathcal{H})$ and that for any $T \in B(\mathcal{H})$:
(i). $w(T) \geq 0$ and $w(T) = 0$ if and only if $T = 0$.
(ii). $w(\lambda T) = |\lambda| w(T)$, for any $\lambda \in \mathbb{C}$.
(iii). $w(T + S) \leq w(T) + w(S)$, for any $T, S \in B(\mathcal{H})$.

Recall (cf. [15]) that an operator $T \in B(\mathcal{H})$ is
self-adjoint if $T^* = T$; normal if $T^*T = TT^*$; quasinormal if $TT^*T = T^*TT$; hyponormal if $T^*T \geq TT^*$; paranormal if $\| Tx \|^2 \leq \|T^2x\|$, for every unit vector $x \in \mathcal{H}$; normaloid if $r(T) = \| T \|$ or equivalently, if $\|T^n\| = \| T \|^n$, for all positive integers n and spectraloid if $r(T) = w(T)$. Equivalently, if $w(T^n) = (w(T))^n$. An operator T on a Hilbert space is a rank-one operator if $\dim(\operatorname{Ran}(T)) = 1$.

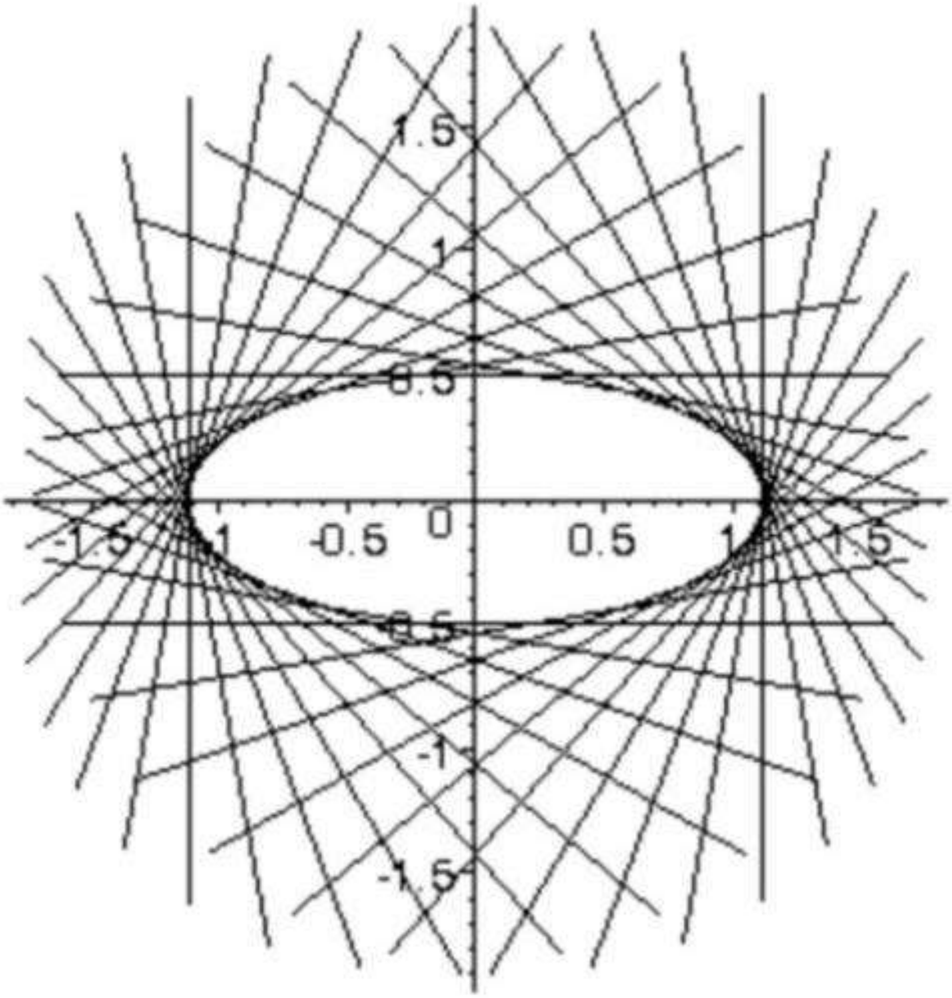
The spectral gap of $A \in B(\mathcal{H})$ is defined as $G_p(A) = \| A \| - r(A)$ and the numerical gap of $A \in B(\mathcal{H})$ is defined as $wG_p(A) = \| A \| - w(A)$. (cf. [11]). The notion of a numerical gap was initially introduced by S.S. Dragomir (see [5]) but the name was coined in 2021 by Z. I. Ismailov and P.I. Al([11]) while the concept of the spectral gap of an operator was introduced by M. Demuth in 2015(see [4]). These two concepts have been investigated by several authors (cf. [11], [3])

Remark. The spectral gap is the difference between the moduli of the two largest eigenvalues of an operator. The spectral and numerical gaps show how the spectral picture of an operator changes under perturbation. They also show how the numerical range $W(T)$ bounds the spectrum $\sigma(T)$.
Even though $W(T)$ is often used to bound the spectrum $\sigma(T)$ of an operator $T, \sigma(T)$ could be much smaller. Consider $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $W(T) = \left\{\lambda \in \mathbb{C}: |\lambda| \leq \frac{1}{2}\right\}$, which is the closed disk with center 0 and radius $\frac{1}{2}$. However, $\sigma(T) = \{0\}$. By contrast, for self-adjoint and generally spectraloid operators T, the spectrum $\sigma(T)$ is sharply bounded by $W(T)$ (cf. [7], §2.5).

Example. For $A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$, the following figure obtained by a MAPLE procedure (see [2]) shows the spectral picture of A, where

$$\| A \| \approx 1.309, \sigma(A) = \{-1,1\}, r(A) = 1, w(A) \approx 1.115.$$

and the $W(A)$ is the ellipse with center (0,0), with foci at -1 and 1 and with minor axis length 1 and major axis length approximately 2.23.



Theorem 2. 1 ([15], Proposition 6.25) For any $A \in B(\mathcal{H})$,$0 \leq r(A) \leq w(A) \leq \| A \| \leq 2w(A)$.

For any operator T, the operator $A = T^*T - TT^*$ is a self-adjoint operator and $T^*T - TT^* = 0$, whenever T is normal. Given $T, S \in B(\mathcal{H})$, we let $A = T^*T - TT^*$ and $B = S^*S - SS^*$. Clearly, A and B are normal operators.

2.1 Spectral and Numerical Gaps for Equivalent Operators

In this subsection, we investigate the spectral and numerical gaps of in some equivalence relations.

Remark. We characterize $A = T^*T - TT^*$ and $B = S^*S - SS^*$ when T, S are equivalent in some sort.

Theorem 2.2 Suppose T and S are unitarily equivalent. Then $A = T^*T - TT^*$ and $B = S^*S - SS^*$ are unitarily equivalent.

Proof. Suppose $T = U^*SU$, for some unitary operator U. Then a simple computation shows that

$$T^*T - TT^* = U^*(S^*S - SS^*)U = A.$$

So, $S^*S - SS^* = UAU^* = B$. This proves the claim.

Remark. For any $A, B \in B(\mathcal{H}), G_p(A) - G_p(B) = (\| A \| - \| B \|) - (r(A) + r(B))$ and $wG_p(A) - wG_p(B) = (\| A \| - \| B \|) - (w(A) + w(B))$.

Corollary 2.3 Suppose A and B are unitarily equivalent. Then $G_p(A) = G_p(B)$ and $wG_p(A) = wG_p(B)$.

Proof. The proof follows immediately from the definition of the respective gaps and the fact that unitary equivalence preserves spectral radii, numerical radii and also the norm of operators.

Remark. Note that in general, similarity need not preserve norm of operators. However, for similar normal operators, it does. Similarly, metric equivalence preserves norm (cf. [17], Theorem 2.14) but it does not preserve the spectrum ([17], Proposition 2.16). But clearly, although metrically equivalent operators need not have equal spectra, they must have equal spectral radius.

Theorem 2. 4 ([17], Theorem 2.14) Suppose T and S are metrically equivalent operators. Then $\| S \|=\| T \|$.

Theorem 2.5 Suppose A and B are metrically equivalent operators. Then for every $\lambda \in \sigma(A)$ there exists a$\beta \in \sigma(B)$ (and vice versa) such that $|\lambda| = |\beta|$.

Theorem 2.5 says that metric equivalence preserves the spectral radii of operators, although it need not preserve the spectra of operators.

Corollary 2.6 Suppose T and S are metrically equivalent operators. Then $r(T) = r(S)$.

Proof. The proof follows from Theorem 2.5.

Remark. It is worth noting that the fact that operators have equal spectral radii does not in general imply that they are metrically equivalent. The operators represented by the matrices $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ are such that $r(A) = r(B) = 1$, but are not metrically equivalent. Note also that in this example, $W(A) = [0,1] \neq [-1,0] = W(B)$ but $w(A) = w(B) = 1$.
Note also that A and B are not similar but A^*A and B^*B are similar. That is, A and B are nearly equivalent.

Theorem 2.7 Suppose A and B are similar normal operators. Then $G_p(A) = G_p(B)$ and $wG_p(A) = wG_p(B)$.

Proof. The proof follows immediately from the fact that similar normal operators are unitarily equivalent (cf. [17], Proposition 2.13) and the proof of Corollary 2.3.

Theorem 2.8 Suppose A and B are metrically equivalent operators. Then $G_p(A) = G_p(B)$.

Proof. The proof follows from an application of Theorem 2.4 and Corollary 2.6.

Theorem 2. 9 ([17], Theorem 2.18) Suppose A and B are metrically equivalent normal operators on a Hilbert space $\mathcal{H}$. Suppose $A = U|A|$ and $B = V|B|$ are the polar decompositions of A and B, respectively. Then $|A| = |B|$.

Remark. We note that Theorem 2.9 also holds if we replace normality with invertibility of A and B.

Corollary 2.10 Suppose A and B are metrically equivalent invertible operators on a Hilbert space $\mathcal{H}$. Suppose $A = U|A|$ and $B = V|B|$ are the polar decompositions of A and B, respectively. Then $|A| = |B|$.
Proof. By hypothesis, $A^*A = B^*B$ and this implies that $|A|U^*U|A| = |B|V^*V|B|$. Since A and B are invertible both U and V are unitary operators (see [14], Remark 0.10 and [15], Corollary 5.91) and hence we have $|A|^2 = |B|^2$. Both $|A|^2$ and $|B|^2$ are positive operators and so they have unique positive square roots. Taking square roots proves the claim.

Remark. It was proved in ([17], Theorem 2.15) that metrically equivalent operators A and B need not have equal numerical range. However, it was shown by the same author that $W(|A|) = W(|B|)$. This is equivalent to saying that both A and B have the same numerical radius. For instance, let A be the unilateral shift on $\ell^2(\mathbb{N})$ and $B = I$, the identity operator on $\ell^2(\mathbb{N})$. Clearly A and B are metrically equivalent. But $W(A) = \{\lambda \in \mathbb{C}: \lambda \leq 1\} \neq \{1\} = W(B)$. However, $w(A) = w(B)$.

Theorem 2.11 Suppose A and B are metrically equivalent invertible operators. Then $G_p(|A|) = G_p(|B|)$ and $wG_p(|A|) = wG_p(|B|)$.

Proof. Follows from the application of Corollary 2.6 and Corollary 2.10.

Theorem 2.12 Suppose A and B are metrically equivalent invertible operators. Then $w(A) = w(B)$.

Corollary 2.13 Suppose A and B are metrically equivalent invertible operators. Then $G_p(A) = G_p(B)$ and $wG_p(A) = wG_p(B)$.

Proof. The proof of the first part of the claim follows from Theorem 2.8 and the proof of the second part of the claim follows from an application of Theorem 2.11 and Theorem 2.12.

2.2 **Spectral and Numerical Gaps for Some Classes of Operators**

In this subsection, we investigate the spectral and numerical gaps operators belonging to some operator classes.

Clearly $G_p(A) \geq 0$ and $wG_p(A) \geq 0$ for any $A \in B(\mathcal{H})$. For normaloid operators, $r(A) = \parallel A \parallel$ and so $G_p(A) = 0$ whenever A is a normaloid operator.

Theorem 2.14 If A is normal, then $wG_p(A) = 0$.

Proof. The proof follows from the fact that for a normal operator $A, w(A) = \parallel A \parallel$.

Remark. Kubrusly [15] has proved that in a Hilbert space, if $r(T) = \parallel T \parallel$, then $r(T) = w(T)$ and $w(T) = \parallel T \parallel$
Theorem 2.15 If A is normaloid, then $G_p(A) = wG_p(A) = 0$.

Proof. The proof follows from the fact that for a normaloid operator $A, r(A) = \parallel A \parallel$, which in turn implies that $w(A) = \parallel A \parallel$ (see [15]).

Recall that an operator $T \in B(\mathcal{H})$ is called an isometry if $T^*T = I$ and a co-isometry if $TT^* = I$.

Theorem 2.16 If A is an isometry or a co-isometry, then $G_p(A) = 0$.

Proof. The proof follows from the definition and the fact that $\parallel A \parallel = 1$ and $r(A) = 1$ for any isometry or a co-isometry A.

We note that the class of normaloid operators on a Hilbert space $\mathcal{H}$ coincides with the class of all operators on $\mathcal{H}$ for which $\parallel T \parallel = \sup\{|\langle Tx, x\rangle| : \parallel x \parallel = 1, x \in \mathcal{H}\}$. This includes the normal operators, isometries, quasinormal operators, hypornormal and paranormal operators (cf. [19]).

Theorem 2.17 If A is a non-zero quasinilpotent operator, then $G_p(A) > 0$ and $wG_p(A) > 0$.

Remark. As a consequence of Theorem 2.17, it has been shown in ([6],[9]) that if $A \in B(\mathcal{H})$ is non-zero and $A^2 = 0$, then $w(A) = \frac{\|A\|}{2}$. Therefore, $r(A) = 0, \parallel A \parallel > 0$ and so $G_p(A) = \parallel A \parallel - 0 = \parallel A \parallel > 0$ and $wG_p(A) = \parallel A \parallel - \frac{\|A\|}{2} = \frac{\|A\|}{2} > 0$.

Theorem 2. 18 For any operator A

(a). $r(A) \leq w(A)$.
(b). $wG_p(A) \leq G_p(A)$.

Theorem 2.19 If A is spectraloid, then $G_p(A) = wG_p(A) = 0$.

Proof. Follows from the fact that for a spectraloid operator $A, r(A) = w(A) = \parallel A \parallel$.

Theorem 2.20 If A is unitarily equivalent to a normal operator, then A is normal.

Proof. Suppose $A = U^*BU$, where U is a unitary operator and B is normal. Then
$A^*A = U^*B^*BU = U^*BB^*U = AA^*$.

Corollary 2.21 If A is unitarily equivalent to a normal operator, then $G_p(A) = wG_p(A) = 0$.

Proof. The proof follows from an application of Theorem 2.14, Theorem 2.20 and the fact that every normal operator is normaloid.

We note that if A is a normal operator, then $\overline{W(A)} = \text{conv}(\sigma(A))$, the convex hull of $\sigma(A)$.

Theorem 2.22 Let $A, B \in B(\mathcal{H})$ are normal operators such that $\sigma(A) = \sigma(B)$, then $\overline{W(A)} = \overline{W(B)}$.

Proof. Since $A, B \in B(\mathcal{H})$ are normal and $\sigma(A) = \sigma(B)$, we have that

$$\overline{W(A)} = \text{conv}(\sigma(A)) = \text{conv}(\sigma(B)) = \overline{W(B)}.$$

Theorem 2.22 says that for normal operators $A, B \in B(\mathcal{H})$ with the same spectrum, their numerical ranges can differ only by their boundaries $\partial W(A)$ and $\partial W(B)$ (see also [16]).

Corollary 2.23 Let $A, B \in B(\mathcal{H})$ are normal operators such that $\sigma(A) = \sigma(B)$, then $w(A) = w(B)$.

Proof. The proof follows from Theorem 2.22, the convexity of the numerical range and the fact that for any $T \in B(\mathcal{H})$, the set $\overline{W(T)}$ is closed and contains all its boundary points $\lambda \in \partial W(T)$ such that $|\lambda| = w(T)$.

3. Approximation of Spectral and Numerical Gaps of sums and products of Operators

For any $A \in B(\mathcal{H})$, small changes (or perturbations) may lead to big changes in the spectral picture. For instance, in finite dimensional settings, one eigenvalue of an operator matrix can be shifted arbitrarily by a rank-one perturbation, without disturbing the other eigenvalues. However, for a self-adjoint operator A, small changes in A will generally not lead to large changes in the spectrum and the numerical range. This is due to the fact that the spectrum of a self-adjoint operator is bounded sharply by the numerical range. In general, for spectraloid operators, since $r(T) = w(T)$, the spectrum of a spectraloid operator is bounded sharply by its numerical range.

Recall that for any $T \in B(\mathcal{H}), r(T) = r(T^*), \| T \| = \|T^*\|, w(T) = w(T^*)$.

Proposition 3. 1 ([18], Proposition 1) If $T, S \in B(\mathcal{H})$ and $ST = TS$, then $r(T + S) \leq r(T) + r(S)$ and $r(TS) \leq r(T)r(S)$.

The following result is an application of Proposition 3.1.

Theorem 3.2 If $T \in B(\mathcal{H})$ is normal, then $G_p(T + T^*) = 0$.

Proof. Normality of T ensures T and T^* commute. Applying Proposition 3.1 we have $r(T + T^*) \leq r(T) + r(T^*) = 2r(T)$ and $\|T + T^*\| \leq \| T \| + \|T^*\| = 2 \| T \|$. Therefore $G_p(T + T^*) = \|T + T^*\| - r(T + T^*) \leq 2(\| T \| - r(T)) = 0$.

Remark. Note also that if T is normal, then $G_p(TT^*) = \|TT^*\| - r(TT^*) \leq \| T \|^2 - (r(T))^2 = 0$.
Note that the assumption that T is normal in Theorem 3.2 cannot be dropped since it ensures the commutativity of T and T^*. This result is trivial since the operators $T + T^*$ and TT^* are self-adjoint and hence normaloid, which means that $r(T + T^*) = \|T + T^*\|$ and $r(TT^*) = \|TT^*\|$.
We note also that the assumption that S and T commute in Theorem 3.1 cannot be dropped. In fact, if S and T do not commute, it is difficult to say how $\sigma(S + T)$ is related to $\sigma(S)$ and $\sigma(T)$. Consequently, it would be difficult to determine how $r(S + T)$ is related to $r(S)$ and $r(T)$. In particular, $\sigma(S + T)$ need not be contained in $\sigma(S) + \sigma(T)$. To see, this let $S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $\sigma(S + T) = \{-1,1\}$ and $\sigma(S) = \sigma(T) = \{0\}$. This shows also that $r(S + T)$ need not be less than $r(T) + r(S)$.

The following result estimates the spectral gap of the operator pencil $T + I$, after T is given a perturbation I. This will show how sensitive a system is upon perturbation or disturbance.

Theorem 3.3 Let $T \in B(\mathcal{H})$. Then $G_p(T + I) \leq G_p(T)$.

Proof. Note that $[I, T] = 0$ and so we can apply Proposition 3.1 to get $\| T + I \| \leq \| T \| + \| I \| = 1 + \| T \|$ and $r(T + I) \leq r(T) + r(I) = 1 + r(T)$ Therefore, $G_p(T + I) = \| T + I \| - r(T + I) \leq 1 + \| T \| - 1 - r(T) = \| T \| - r(T) = G_p(T)$.

Proposition 3.4 Let $T, S \in B(\mathcal{H})$ and $ST = TS$. If $\varepsilon \in \mathbb{C}$, then $G_p(T + \varepsilon S) \leq G_p(T) + |\varepsilon| G_p(S)$ and $G_p(\varepsilon ST) \leq |\varepsilon|(\| S \| \| T \| - r(S)r(T))$.

Proof. Since $[S, T] = 0$ it follows that $[\varepsilon S, T] = 0$ for any $\varepsilon \in \mathbb{C}$. A simple computation shows that $G_p(T + \varepsilon S) = \| T + \varepsilon S \| - r(T + \varepsilon S) \leq (\| T \| + |\varepsilon| \| S \|) - (r(T) + |\varepsilon|)r(S) = (\| T \| - r(T)) + |\varepsilon|(\| S \| - r(S)) = G_p(T) + |\varepsilon| G_p(S)$. That is, $G_p(T + \varepsilon S) \leq G_p(T) + |\varepsilon| G_p(S)$, which proves the first claim. The second claim is proved similarly.

Note that the assumption that the change εS is sufficiently small, will result in a small perturbation of the spectral gap. Note that in the second claim if $|\varepsilon| \longrightarrow 0$, we have $G_p(ST) \longrightarrow 0$. This is because the operator εST will tend to 0, the zero operator, which is normaloid. Theorem 3.5 (Equivalent norm) For any $T \in B(\mathcal{H})$, we have$w(T) \leq \| T \| \leq 2w(T)$.

Theorem 3. 6 ([6], Theorem 5) Let $T \in B(\mathcal{H})$. If $w(T) = \| T \|$, then $r(T) = \| T \|$.

It is noted in (cf. [15]) that in a Hilbert space $r(T) = \| T \|$ implies that $r(T) = w(T)$. Moreover, $r(T) = \| T \|$ also implies $w(T) = \| T \|$. Thus, $w(T) = \| T \|$ is a property of every normaloid operator on a Hilbert space.

Proposition 3. 7 Let $x, y \in \mathbb{R}$. Then $\max\{x, y\} = \frac{1}{2}(x + y + |x - y|)$.

Proof. The case when $x = y$ is trivial. Suppose that $x < y$ or $x > y$. Then $\max\{x, y\} + \min\{x, y\} = x + y$ and $\max\{x, y\} - \min\{x, y\} = |x - y|$. Adding/subtracting these equations, we have that $\max\ \{x, y\} = \frac{1}{2}(x + y + |x - y|)$ and $\min\{x, y\} = \frac{1}{2}(x + y - |x - y|)$.

Lemma 3. 8 ([1], Exercise I.3.1(ii)) Let $A \in B(\mathcal{H})$ and suppose A has a direct sum decomposition $A = A_1 \oplus A_2$ with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$. Then
(i). $\| A \| = \max\{\|A_1\|, \|A_2\|\}$.
(ii). $r(A) = \max\{r(A_1), r(A_2)\}$.
(iii). $w(A) = \max\{w(A_1), w(A_2)\}$.

Theorem 3.9 Let $A \in B(\mathcal{H})$ and suppose A has a direct sum decomposition $A = A_1 \oplus A_2$ with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$. Then $G_p(A) = \frac{1}{2}\big(G_p(A_1) + G_p(A_2) + |\|A_1\| - \|A_2\|| + |r(A_1) - r(A_2)|\big)$and $wG_p(A) = \frac{1}{2}\big(wG_p(A_1) + wG_p(A_2) +$ $A1 - A2 + wA1 - wA2$.

Proof. Using Proposition 3.7 and Lemma 3.8(i) and (ii), we have

$$
\begin{aligned}
&G_p(A) = \| A \| - r(A) \\
&= \max\{\|A_1\|, \|A_2\|\} - \max\{r(A_1), r(A_2)\} \\
&= \frac{1}{2}(\|A_1\| + \|A_2\| + |\|A_1\| - \|A_2\||) - \frac{1}{2}(r(A_1) + r(A_2) + |r(A_1) - r(A_2)|) \\
&= \frac{1}{2}(\|A_1\| - r(A_1) + \|A_2\| - r(A_2) + |\|A_1\| - \|A_2\|| - |r(A_1) - r(A_2)|) \\
&= \frac{1}{2}\big(G_p(A_1) + G_p(A_2) + |\|A_1\| - \|A_2\|| - |r(A_1) - r(A_2)|\big)
\end{aligned}
$$

The proof of the second claim is similar and follows easily from the definition of $wG_p(A)$, Proposition 3.7 and Lemma 3.8(i) and (iii).

4. Gap Equivalences of Operators

We say that two operators A and B are spectrally gap-equivalent if $G_p(A) = G_p(B)$, and we denote it by $A \overset{sge}{\sim} B$ We say that two operators A and B are numerically gap-equivalent if $wG_p(A) = wG_p(B)$, and we denote it by $A \overset{nge}{\sim} B$.

Theorem 4.1 Spectral gap equivalence and numerical gap equivalence are equivalence relations on B(H).

Theorem 4.2 Metric equivalence implies spectral gap equivalence on $B(\mathcal{H})$.

Proof. The proof follows from Theorem 2.8.

Theorem 4.3 If $A, B \in B(\mathcal{H})$ are invertible or normal and metrically equivalent then they are numerically gap-equivalent.

Proof. The proof follows from the application of Theorem 2.11, Theorem 2.12 and Corollary 2.13.

Remark. We note that for normaloid operators, numerical gap equivalence and spectral gap equivalence coincide. However, there exist non-normaloid operators with this property.

From Corollary 2.3, it is clear that unitary equivalence preserves spectral gap and numerical gap between operators. This means that unitary equivalence implies spectral gap equivalence and also numerical gap equivalence of operators. However, similarity of operators does not in general imply spectral gap equivalence and also numerical gap equivalence of operators, unless the operators are normal. This proof of this claim follows from Theorem 2.7.

5. Discussion

The numerical radius provides a norm equivalent to the operator norm in complex Hilbert spaces. Together with the convexity of the numerical range, these properties can be used in eigenvalue approximationby constructing an initial guess for iterative methods, in particular the convergence of the steepest descent method for solving the system $Ax = b$ and also those methods that estimate eigenvalues instead of using Gershgorin circles- in areas such as fluid dynamics and stability analysis of finite difference approximations of solutions to hyperbolic initial value problems(cf. [8]). This finds application in many areas of theoretical and applied mathematics, including in quantum information processing and mathematical modelling. Spectral and numerical gaps of operators find application in how sensitive a system is under perturbation(cf. [9]). They may show how eigenvalues of an operator change under perturbation.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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