IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 18, Issue 4 Ser. III (Jul. – Aug. 2022), PP 42-46
www.iosrjournals.org

Every Cycle C_n with Parallel Chords of Pendant Edge Extension is Even Graceful

A. Elumalai
Department of Mathematics
RajalakshmiInstituteofTechnologyChennai-600124
Tamilnadu

A graph G is called a pendant edge extension graph of a graph H if G is obtained from H by adjoining a new pendant edge with each vertex of H and denote pendant edge extension graph of a graph H by $H \odot K_1$.

In this paper, I have proved

1. Every cycle with parallel chords is even graceful for all $n \geq 6$.
2. Every cycle with parallel chords of pendant edge extension is even graceful for all $n \geq 6$.

Keywords: Graph labeling, Graceful graph, Even and odd graceful graph, Cycle with parallel chords.

AMS Subject Classification: 05C78.

Date of Submission: 28-07-2022 Date of Acceptance: 09-08-2022

I. Introduction

A function f is called a graceful labeling of a graph G with m edges if f is aninjection from the vertex set of G to the set $\{ 0, 1, 2, \ldots m \}$ such that, when each edge xy is assigned the label $| f(x) - f(y) |$ the resulting edge labels are distinct.

A function f is called an even graceful labeling of a graph G with m edges if f is an injection from the vertices of G to the set $\{ 0, 1, 2, \ldots 2m \}$ such that when each edge uv is assigned the label $| f(u) - f(v) |$, the resulting edge labels are distinct even numbers that ranges from 2 to 2m.

A graph G with m edges to be odd graceful if there is an injection from f from $V(G)$ to $\{ 0, 1, 2, \ldots 2m - 1 \}$ such that when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are $\{ 1, 3, 5, \ldots, 2m - 1 \}$.

A line of work on graceful graphs has concentrated on graphs related to the cycles stemming from Rosa's result that a cycle C_n is graceful iff $n = 0$ or 3 (mod 4). A chord of a cycle is an edge joining two non-adjacent vertices of the cycle. Bodendiek, Schumacher and Wegner conjectured in that every cycle with a chord is graceful.

The validity of this conjecture has been proved by Delorme, Mahev et al in [2].

A natural extension of the structure of a cycle with a chord is that of a cycle with a P_k-chord. A cycle with a P_k-chord ($k>2$) is a graph obtained by joining a pair of non-adjacent vertices of a cycle of order n ($n > 4$) by a path of order k.

Koh and Yap [6] have shown that cycles with P_3-chords are graceful and conjectured that all cycle with P_k-chords are graceful. This was proved for $k \geq 4$ by Punnim and Prabhapote [7]. For an excellent survey on graceful labeling see [5].

A graph G is called cycle with parallel chords if G is obtained from a cyclec_n : $v_0v_1 \ldots v_{n-1}v_0$ $(n \geq 6)$ by adding the chords $v_1v_{n-1}, v_2v_{n-2}, \ldots,$ $v_\alpha v_\beta$, $\alpha = \left\lfloor\frac{n}{2}\right\rfloor - 1$and $\beta = \left\lfloor\frac{n}{2}\right\rfloor + 2$, if n is odd or $\beta = \left\lfloor\frac{n}{2}\right\rfloor + 1$, if n is even [3].

A cycle with pendant edges extension is a graph obtained by attaching edges if non adjacent chords of a cycle of order n [1, 8].

In this direction I have proved the following results, every cycle with parallel chords is even graceful for all $n \geq 6$ and every cycle with parallel chords of pendant edges is even graceful for all $n \geq 6$.

II. Even Gracefulness of a Cycle with Parallel Chords and Parallel Chords of Pendant Edge Extension of Graphs

In this section I have proved that certain pendent edge extension of cycle related graphs.

Theorem 2.1. *For $n \geq 6$, every cycle with parallel chords is an even graceful.*

Proof. Let G be a cycle C_n with parallel chords for $n \geq 6$.

Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of a cycle C_n of G. Observe that by definition, G has n vertices and $M = \frac{3n-p}{2}$ edges, if $p = 3$, if n is odd or $p = 2$, if n is even.

I give labels to the vertices of cycle C_n in the following four cases:

Case 1: When n is even (i.e., $C_n = C_{4k}$; $k = 1, 2, \ldots$)
Define

$$f(v_0) = 0$$
$$f(v_{n-(2i-1)}) = M - 6(i-1), \quad \text{for } 1 \leq i \leq \frac{n-4}{4}$$
$$f(v_{n-2i}) = 6i, \quad \text{for } 1 \leq i \leq \frac{n-4}{4}$$
$$f(v_{2i-1}) = 6i - 4, \quad \text{for } 1 \leq i \leq \frac{n}{4}$$
$$f(v_{2i}) = M - 6i + 4, \quad \text{for } 1 \leq i \leq \frac{n-2}{4}$$
$$f(v_{\frac{n}{2}}) = 6k - 2, \quad \text{for } k \geq 2$$
$$f(v_{\frac{n}{2}+1}) = 6k + 2, \quad \text{for } k \geq 2$$

Case 2: When n is even (i.e., $C_n = C_{4k+2}$; $k = 1, 2, \ldots$)

Define

$$f(v_0) = 0$$
$$f(v_{n-(2i-1)}) = M - 6(i-1), \quad \text{for } 1 \leq i \leq \frac{n-2}{4}$$
$$f(v_{n-2i}) = 6i, \quad \text{for } 1 \leq i \leq \frac{n-6}{4}$$
$$f(v_{2i-1}) = 6i - 4, \quad \text{for } 1 \leq i \leq \frac{n-2}{4}$$
$$f(v_{2i}) = M - 6i + 4, \quad \text{for } 1 \leq i \leq \frac{n-2}{4}$$
$$f(v_{\frac{n}{2}}) = 6k - 2, \quad \text{for } k \geq 2$$
$$f(v_{\frac{n}{2}+1}) = 6k + 2, \quad \text{for } k \geq 2$$

Case 3: When n is odd (i.e., $C_n = C_{4k+1}$; $k = 1, 2, \ldots$)
Define

$$f(v_0) = 0$$

$$f(v_{n-(2i-1)}) = M - 6(i-1), \quad \text{for } 1 \le i \le \tfrac{n-1}{4}$$

$$f(v_{n-2i}) = 6i, \quad \text{for } 1 \le i \le \tfrac{n-1}{4}$$

$$f(v_{2i-1}) = 6i - 4, \quad \text{for } 1 \le i \le \tfrac{n-1}{4}$$

$$f(v_{2i}) = M - 6i + 4, \quad \text{for } 1 \le i \le \tfrac{n-1}{4}$$

Case 4: When n is odd (i.e., $C_n = C_{4k+3}$; $k = 1, 2, \ldots$)
Define

$$f(v_0) = 0$$

$$f(v_{n-(2i-1)}) = M - 6(i-1), \quad \text{for } 1 \le i \le \tfrac{n+1}{4}$$

$$f(v_{n-2i}) = 6i, \quad \text{for } 1 \le i \le \tfrac{n-3}{4}$$

$$f(v_{2i-1}) = 6i - 4, \quad \text{for } 1 \le i \le \tfrac{n+1}{4}$$

$$f(v_{2i}) = M - 6i + 4, \quad \text{for } 1 \le i \le \tfrac{n-3}{4}$$

It is clear that f is injective and the edge values are distinct and range from 1 to M . Thus f is even graceful labeling. Hence the graph G is even graceful. □

Theorem 2.2. *For $n \ge 6$, every cycle with parallel chords of pendant edge extension is an even graceful.*

Proof. Let G denote a cycle C_n with parallel chords of pendant edge extension with $n \ge 6$. By definition of G, G is obtained from the cycle C_n of order n : $v_0, v_1, \ldots, v_{n-1}v_0$ ($n \ge 6$) by attaching the pendent edge extension of non-adjacent chords of cycle with parallel chords of vertices of order n that is the vertices are u_1, u_2 & u_3 respectively.

Observe that G has $n + p$ vertices where $p = 2$, if n is even or $p = 3$, if n is odd and $M = \frac{3n-p}{2}$ edges, where $p = 2$, if n is even or $p = 3$, if n is odd.

I give labels to the vertices $v_0v_1 \ldots v_nv_{n-1}$ and u_1, u_2 & u_3 in the following four cases

Case 1: When n is even (i.e., $C_n = C_{4k}$; $k = 1, 2, \ldots$)
Define

$$f(v_0) = 0$$

$$f(v_{n-(2i-1)}) = M - 6(i-1), \quad \text{for } 1 \le i \le \tfrac{n-4}{4}$$

$$f(v_{n-2i}) = 6i, \quad \text{for } 1 \le i \le \tfrac{n-4}{4}$$

$$f(v_{2i-1}) = 6i - 4, \quad \text{for } 1 \le i \le \tfrac{n}{4}$$

$$f(v_{2i}) = M - 6i + 4, \quad \text{for } 1 \le i \le \tfrac{n-2}{4}$$

$$f(v_{\frac{n}{2}}) = 6k + 4, \quad \text{for } k \ge 2$$

and pendant vertices

$f(u_1) = 10, f(u_2) = 6k + 10,$ for $k \geq 2$

Case 2: When n is even (i.e., $C_n = C_{4k+2}$; $k = 1, 2, \ldots$)
Define

$$f(v_0) = 0$$

$f(v_{n-(2i-1)}) = M - 6(i-1),$ for $1 \leq i \leq \frac{n-4}{4}$

$f(v_{n-2i}) = 6i,$ for $1 \leq i \leq \frac{n-4}{4}$

$f(v_{2i-1}) = 6i - 4,$ for $1 \leq i \leq \frac{n}{4}$

$f(v_{2i}) = M - 6i + 4,$ for $1 \leq i \leq \frac{n-2}{4}$

$f(v_{\frac{n}{2}}) = 6k + 8,$ for $k \geq 1$

and pendant vertices

$f(u_1) = 10, f(u_2) = 6k + 2,$ for $k \geq 2$

Case 3: When n is odd (i.e., $C_n = C_{4k+1}$; $k = 1, 2, \ldots$)
Define

$$f(v_0) = 0$$

$f(v_{n-(2i-1)}) = M - 6(i-1),$ for $1 \leq i \leq \frac{n-1}{4}$

$f(v_{n-2i}) = 6i,$ for $1 \leq i \leq \frac{n-1}{4}$

$f(v_{2i-1}) = 6i - 4,$ for $1 \leq i \leq \frac{n-1}{4}$

$f(v_{2i}) = M - 6i + 4,$ for $1 \leq i \leq \frac{n-1}{4}$

and pendant vertices

$f(u_1) = 4, f(u_2) = 6k + 6,$ for $k \geq 1, f(u_3) = 6k + 2,$ for $k \geq 1$

Case 4: When n is odd (i.e., $C_n = C_{4k+3}$; $k = 1, 2, \ldots$)
Define

$$f(v_0) = 0$$

$f(v_{n-(2i-1)}) = M - 6(i-1),$ for $1 \leq i \leq \frac{n-1}{4}$

$f(v_{n-2i}) = 6i,$ for $1 \leq i \leq \frac{n-1}{4}$

$f(v_{2i-1}) = 6i - 4,$ for $1 \leq i \leq \frac{n-1}{4}$

and pendant vertices
$f(u_1) = 4, f(u_2) = 6k + 8,$ for $k \geq 1, f(u_3) = 6k + 4,$ for $k \geq 1$

It is clear that f is injective and the edge values are distinct and range from 1 to M . Thus f is even graceful labeling. Hence the graph G is even graceful. □

References

[1] J. Ayel and O. Favaron, Helms are graceful, Progress in Graph Theory (Waterloo, Ont, 1982), Academic Press, Toronto, Ont. (1984), 89–92.

[2] C. Delorme, K.M. Koh, M. Maheo, Teo H. Thuillier, Cycles with a chord are graceful, J. Graph Theory, 4 (1980), 409–415.

[3] A. Elumalai and G. Sethuraman, Gracefulness of a cycle with parallel P_k-chords, Australasian J. Combin., 32 (2005), 205–211.

[4] A. Elumalai and G. Sethuraman, Gracefulness of a cycle with parallel chords and parallel P_k-chords of different lengths, Ars Comb., 104 (2012), 143–148.

[5] J.A. Gallian, A dynamic survey of graph labeling, The Electronic Journal of Combinatorics, (2018), # DS6.

[6] K.M. Koh and K.Y. Yap, Graceful numbering of cycles with a P_3-chords, Bull. Inst. Math. Acad. Sinica, 12 (1985), 41–48.

[7] N. Punnim and N. Pabhapote, On graceful graphs: Cycles with a P_k-chord, $k \geq 4$, Ars Combin, 23A (1987), 225–228.

[8] G. Sethuraman and A. Elumalai, On graceful graphs: pendant edge extensions of a family of complete bipartite and complete tripartite graphs, Indian J. Pure Appl. Math., 32(9) (2001), 1283–1296.

Illustrative examples

1.

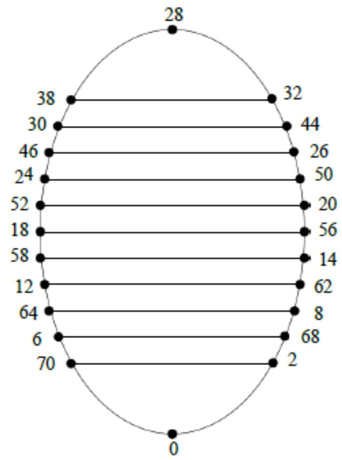

Figure 1: Even graceful labeled C_{24} with parallel chords

2.

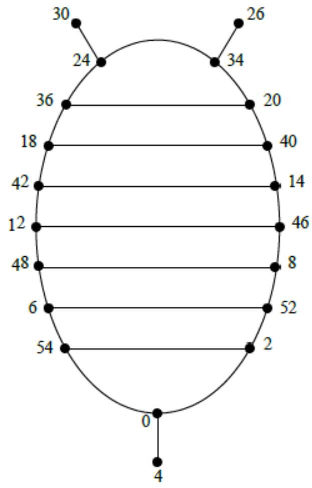

Figure 2: Even graceful labeled C_{17} with parallel chords of pendant edge extension