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Proof of "Axioms" of Propositional Logic

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Abstract
We introduce more basic axioms with which we are able to prove some "axioms" of Propositional Logic. We use the symbols from my other article: "Introduction to Logical Structures". Logical Structures (SrL) are graphs with doubly labelled vertices with edges carrying symbols. The proofs are very mechanical and does not require ingenuity to construct. It is easy to see that in order to transform information, it has to be chopped up. Just look at a kid playing with blocks with letters on them: he has to break up the word into letters to assemble another word. Within SrL we take as our "atoms" propositions with chopped up relations attached to them. We call the results: (incomplete) "structures". We play it safe by only allowing relations among propositions to be choppable. We will see whether this is the correct way of chopping up sentences (it seems to be). This is where our Attractors (Repulsors) and Stoppers come in. Attractors that face away from each other repels and so break a relation between the two propositions. Then a Stopper attaches to the chopped up relation to indicate it can't reconnect. So it is possible to infer sentences from sentences. The rules I stumbled upon, to implement this, seems to be consistent. Modus Ponens is found in most logics.
Keywords: *Structural Logic, Knowledge, Structured Information*
Highlights: The proof statements.
Declaration: I have nothing to declare.

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I. Introduction:

We start with a review of the basics and the nature of Structural Logic (SrL). SrL uses graphs with doubly labelled vertices and labelled (with symbols) edges. The double labelling is accomplished by allowing the vertices to be 2-dimensional enclosures that can have letter, word or symbol content. We prove AND introduction using axioms of the Attractor and Stopper operators. We also prove (the correct version of) AND elimination. Also shown is why dropping Stoppers anywhere in a structure is invalid. We show why rotation of an Attractor through 180 degrees is invalid. We show that there is a left to right bias in SrL. We give axioms from ref. [1] special names. If every statement cannot be stated entirely in symbols then your symbolic language is not adequate. SrL is adequate since you can just put words in concept- or object enclosures. Then we examine how SrL works in a variety of situations.

Definition: An *object* is a name for what specific sense-data points at. A *concept* is an object that is the result of some relation between objects.

Methods: good old reasoning.

Results: various results are in the discussion.

Chapter 1: The Basics.

Chapter 2: Proof of "Axioms" of Propositional Logic.

Appendix A: Operator List.

Appendix B: Relation List.

Appendix C: Enclosure List.

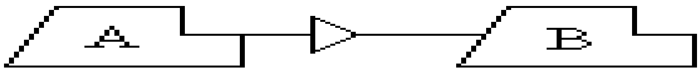
Bibliography.

Chapter 1: The Basics

Discussion:

The book in general tries to explicate (or make explicit) the process of constructing structures from text in as many circumstances as possible. Constructing the structure with enough interlinkages makes possible actual usage of the knowledge. A knowledge structure has at least one operator. The book also tries out (and formalises) proofs in a varied range of other Logics as translated into Structural Logic (SrL). This book relies on my other book ref. [1]. There is another Logic called Structural Logic (SL), but this (SrL) is not it. The main purpose of the book is to make possible the expression of knowledge as it appears in mind, for purpose of comparison so we can check each other's reasoning and learn from one another. To show where it fits in: Knowledge Structures are graphs with edges labelled by symbols and vertices singly or doubly labelled, with at least one operator in the structure. The double labelling is accomplished by allowing the vertex to be some kind of 2D enclosure that can have words or letter "content". We are going to use Structure enclosures. In my view propositions are objects relevant to concepts or object-concept-object (the first object is called "subject" in natural language terminology) structures.

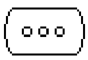
The following structure (graph):



Structure 1.1

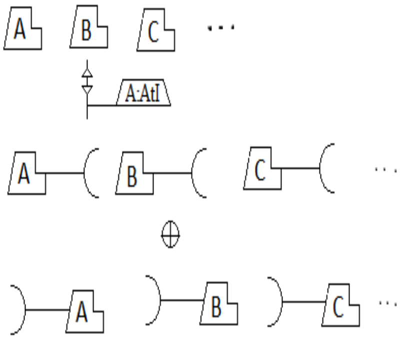
reads: "Structure A therefore structure B". The default meaning is: "Structure A exists therefore Structure B exists". And "exists" entails "is true if all operators are executed". A connective is an operator that is already executed (the edges of the graph). The "therefore" link above is an operator that is already executed. An operator is an edge, with symbol with no underlying meaning but which can transform a structure. Examples of operators are: Introductors, Attractors, Stoppers (symbols to follow). We are going to explore to find the actual argumentation/deduction that happens naturally in mind, and give it symbols. This reasoning would happen in your study, and in ordinary life the logic is mainly: A connects with B causing C. The "connecting" could be physical or informational. Someone said Mathematics uses Classical Logic (CL) and good taste, and this is what we stake our lives on sometimes when we utilize some engineered object. However Engineering Mathematics does not include the more general ideas used in advanced Mathematics (Group Theory, Category Theory).

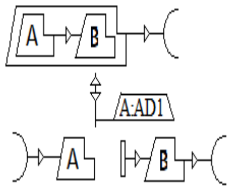
The scope of a structure is expressed as a non-empty vertex drawn as follows:

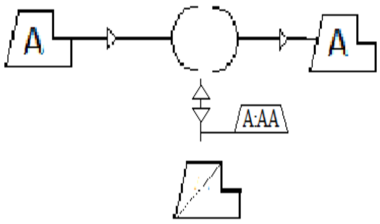


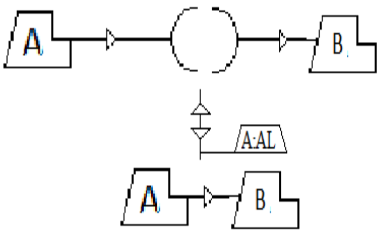
where the dots say the enclosure is not empty. In this book whenever a structure does not have a scope enclosure we assume the scope is **SrL** itself. A scope enclosure is assumed relevant to every enclosure in a structure.

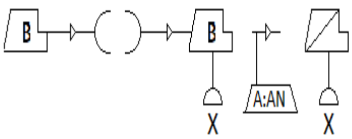
We introduce the following axioms of how Stoppers and Attractors behave:

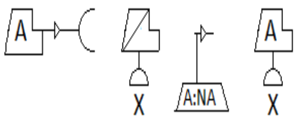

A
B
C
A:AtI
A
B
C
A
B
C

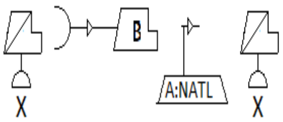

A
B
A:AD1
A
B

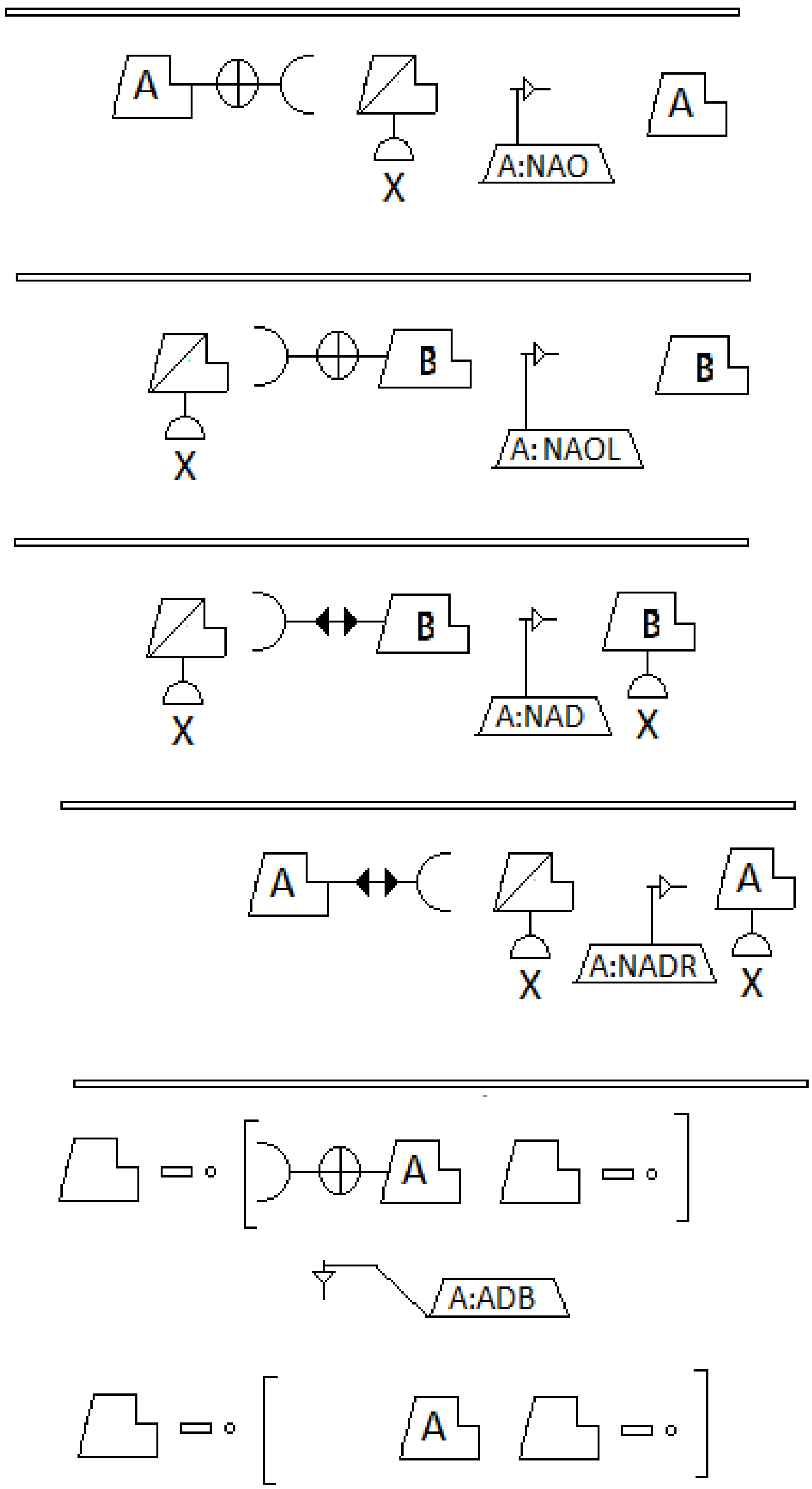

A
A
A:AA


A
B
A:AL
A
B


B
B
X
A:AN
X


A
A
X
A:NA
X


B
X
A:NATL
X



Structure 1.2

where the operators in A:AtI are Attractors carrying a "relevant to" relation, the operator in A:AD1 connected to B at the left is a Stopper carrying a "therefore" relation and the operator attached to B in A:AN is an Introductor and the X below it specifies that negation is introduced into B. A:AD2 is the same as A:AD1 except the Stopper is attached to the other structure. For A:AA the structure with the stripe through it signifies the empty structure i.e. an empty page. The empty structure has got truth value: always false. Note that A:NA just applies to Attractors carrying a "Therefore" relation. The negated empty structure has truth value: always true. Note that the intuition for A:NATL comes from the truth table for "Therefore". The axiom for "Relevance", "If and only if" and "Equivalence" is the same as: A:NAD and A:NADR. See third paragraph after Structure 2.0.11.1. A:AN holds for Attractors carrying any relation symbol. The intuition for A:WA is that the Attractor wrapped around the sentence. The intuition for A:ADB is that the Attractor can't link to the inside of a bracket.

Another axiom: A:ASS is stated in words: in a structure Stoppers can be excanged for Attractors and vice versa. A:SD is stated in words: a Stopper at either end of a line of structures may be dropped. A:OP is the axiom that you can choose operator priority in a statement of just Attractors and Stoppers. A:SUBST reads: if you have on a line in a proof structures including structure A, and on another line you have Structure A = Structure B then you can substitute A for B in the first line and conclude it.

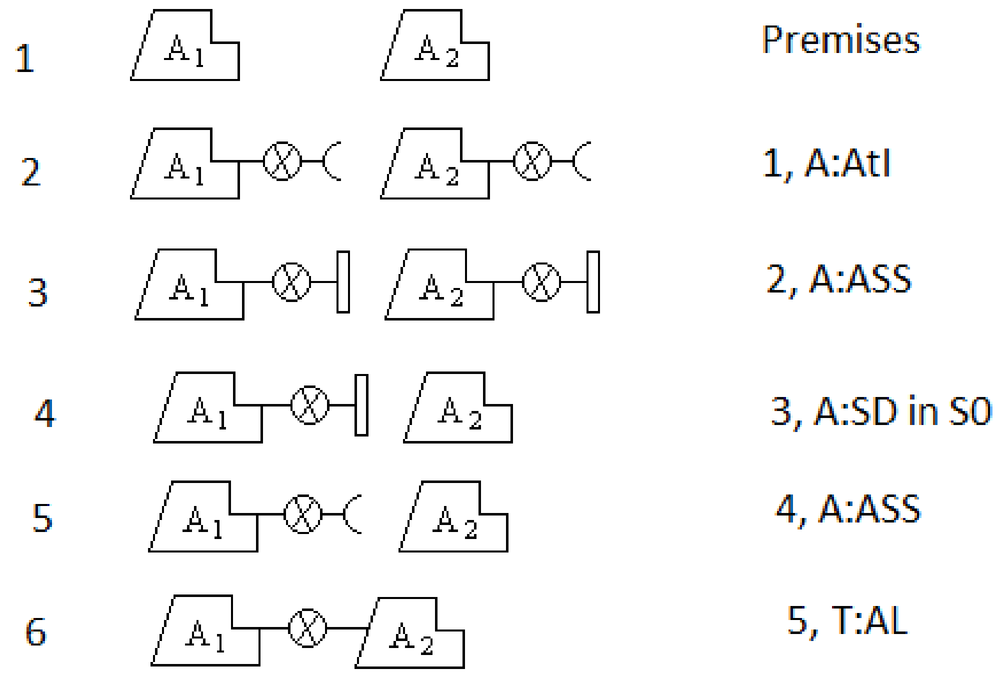
We will prove Propositional Logic is a superlanguage of SrL.

Chapter 2: Proof of Some "Axioms" of Propositional Logic

We first prove AND introduction using two structures:

2.0 Theorem (T:ANDI):

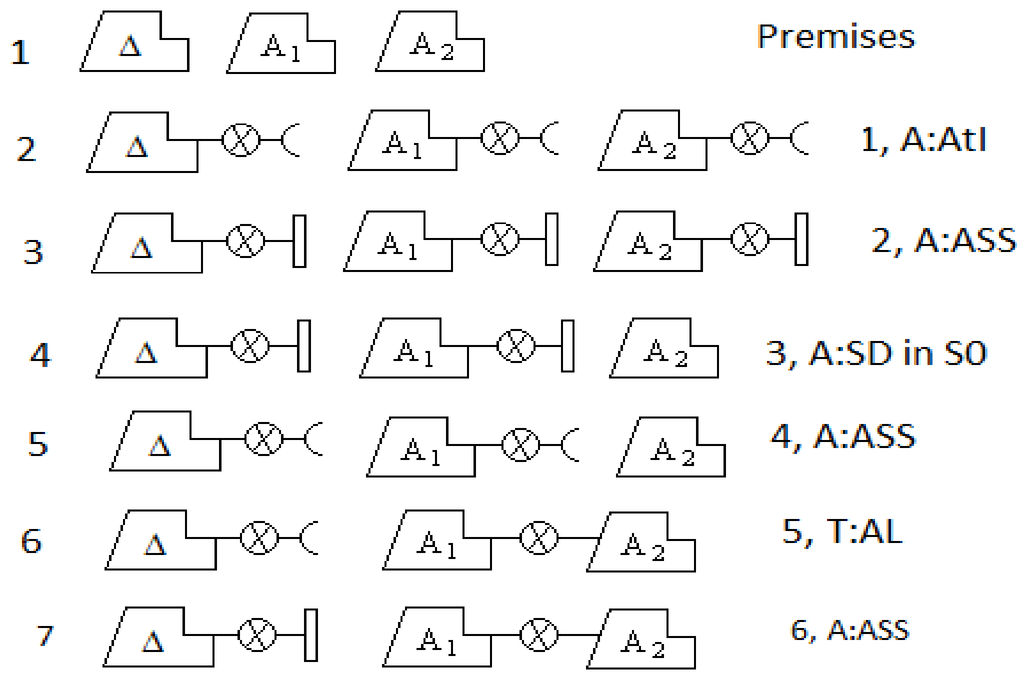
Line # Statement Reason



Structure 2.0

where for T:AL we refer the reader to the paragraph following Structure 2.0.1.

Note what else can be proven regarding "AND Introductoin":



Structue 2.0.0.1

We can do this validly by choosing not to link the Attractor on Delta in line 5.

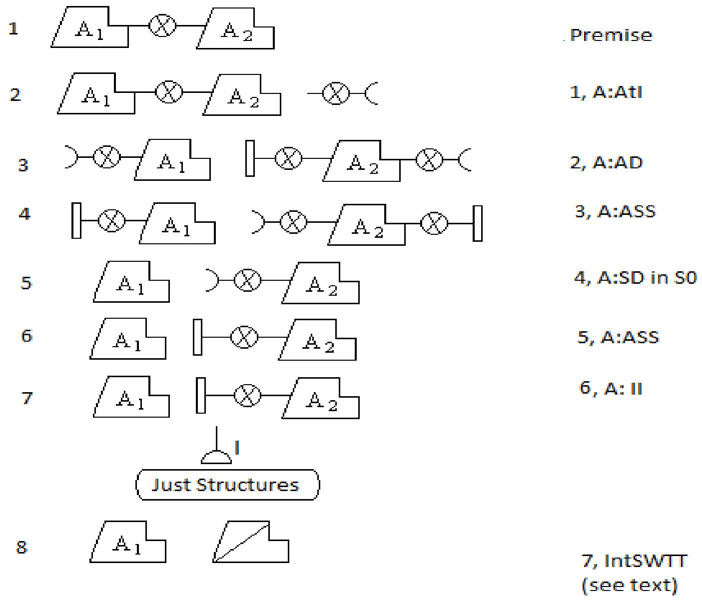
The Stopper on Delta thus remains as a reminder of what we did. We cannot legally remove this Stopper. On introducing interpretation uunder the model "Just Structures": Delta and the Stopper would disappear, which is not what the axiom in my source states, so this doesn't quite prove the axiom.

We state that OR introduction is valid. We need to accept OR introduction because it is needed to prove: P OR ~P.

We prove AND-elimination as follows:

2.0.1 Theorem (T:ANDE):

Line # Statement Reason



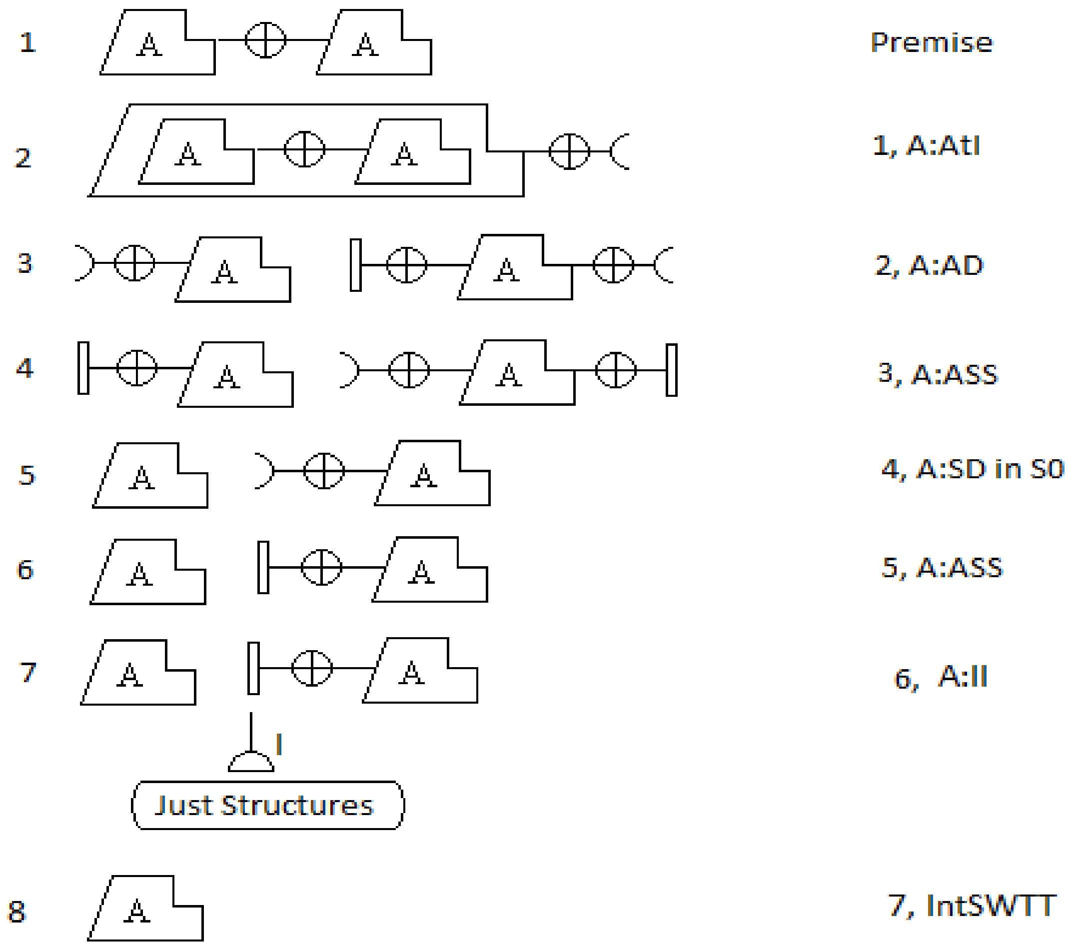
Structure 2.0.1

where line 8 is because, under our interpretation, we just take structures that we can write a truth table for. Note that the "I" next to the Introductor alters it's functioning a little into: "Introduce Model under Interpretation". Similarly we can also conclude: "structure A_2" by choosing to put the Stopper on the other structure in line 3.The same can be done with OR instead of AND but it is not valid since the truth table says it is not necessarily a logical consequence. 5 therefore 1 is also inferable (T:AL). Although we throw away information on interpretation we may also conclude A2 on another line in the proof so potentially no information is lost.

We can prove the "axiom": (p OR p) -> p. There is a problem with this "axiom" since it requires two operations.

We prove Theorem (T:AOA): (A OR A) -> A as follows:

Line # Statement Reason



Structure 2.0.2

Line 8 is since the part of the structure that has a truth table is structure A.

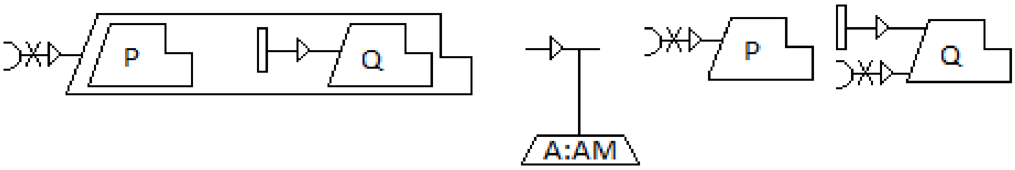
If the second A was another structure we would be losing information on interpretation so the same is not valid for this case.

We try to prove "Contradiction":

1 (p AND (p -> (q AND (not q))))	Premise
2 (q AND (not q))	1, T:MP
3 not p	2, folows from contradiction

where we used the theorem: Modus Ponens (proved in ref. [1] and later in this article). We work in letters if the concepts don't need clarification by symbols. Ref. [1] proves MP with assuming just 5 axioms. Now we have that line 3 does not follow since "Or introduction" is not allowed.

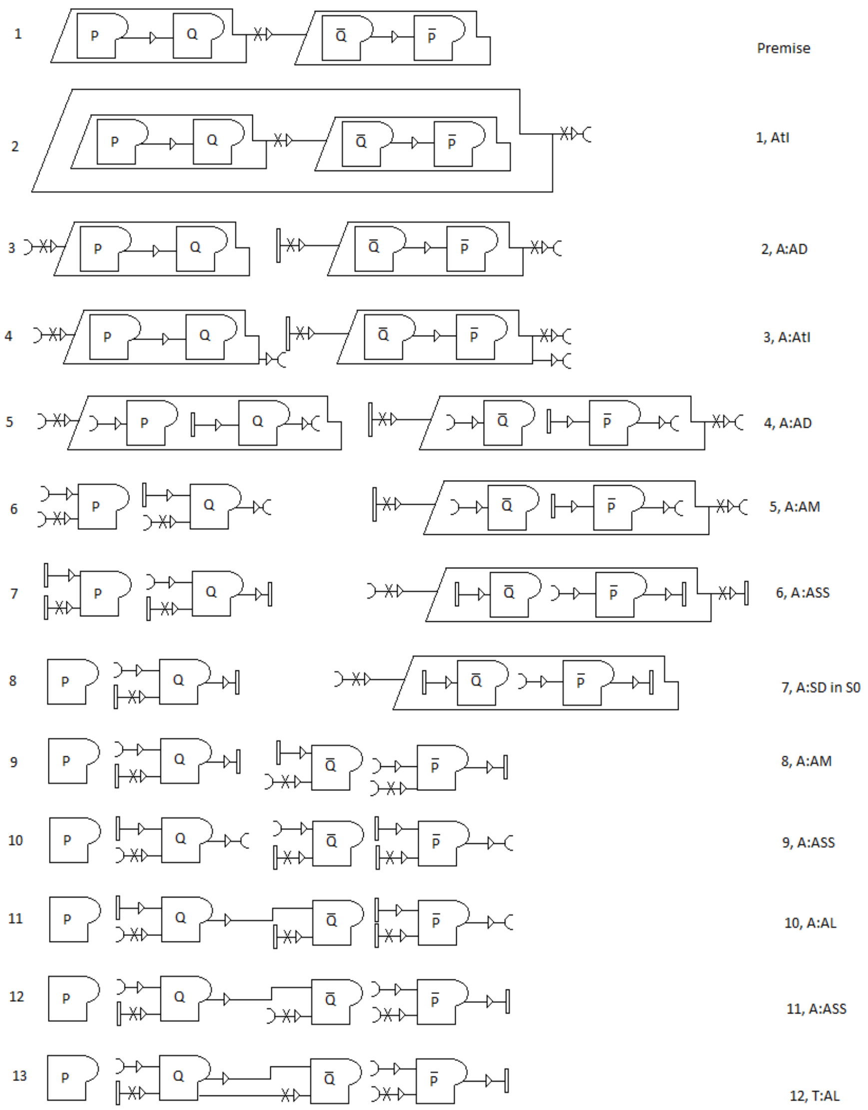
With the following axiom (how attractors go into structures in a structure):



Structure 2.0.3

we can prove contraposition ((P -> Q) -> (not Q -> not P)), by contradiction. Note that the symbol carried by the attractor must be "Therefore" or "Not Therefore" and the direction must match.. We assume the negative and proceed as follows:

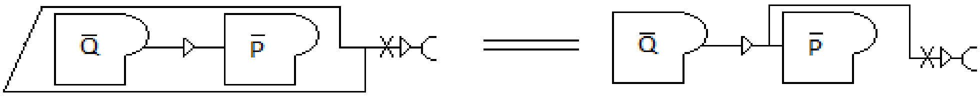
Line # Statement Reason



Structure 2.0.4

and we have a contradiction in line 13 (inside a sentence), so this proves contraposition.
We have: "Q therefore and not therefore not Q", and this is a contradiction. Here the enclosure of P is called a Proposition Enclosure. The other Attractors and Stoppers won't take away the contradiction. Isn't it amazing that we could derive a contradiction?

Note the following:



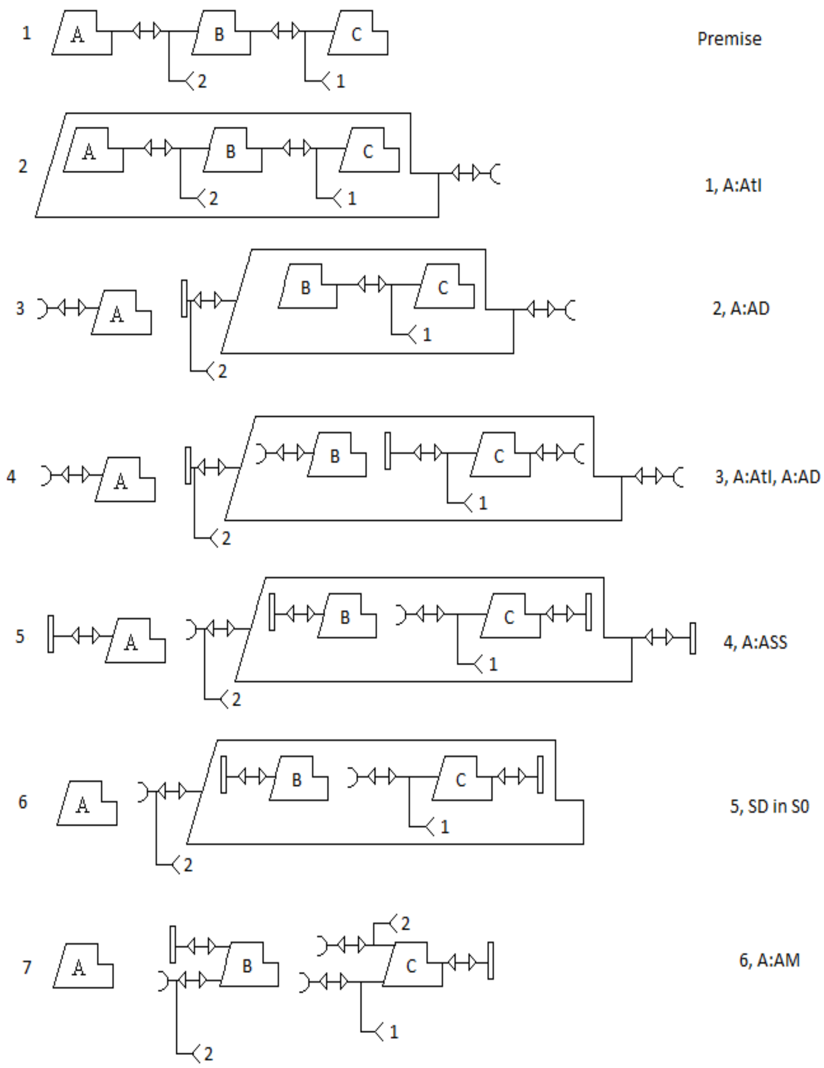
Structure 2.0.4.1

were the Attractor connects to the main connective.

Theorem, T:ASSOC:

We can prove association (A <> (B <> C)) <> ((A <> B) <> C) as follows:

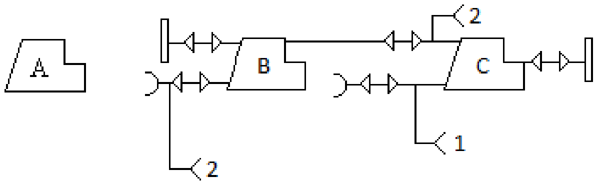
Line # Statement Reason



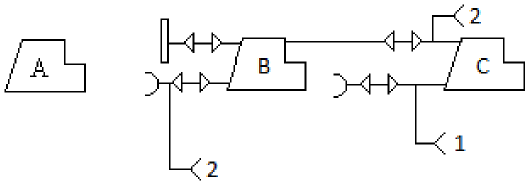
Structure 2.0.5

Now in line 7 we can choose the Attractor of the top Operator Priority Operator (carrying the 2) to connect B with C. We can then do an A:ASS and link B using the resulting Attractor to A: this operator has Operator Priority 1 (implicitly: from it's source).

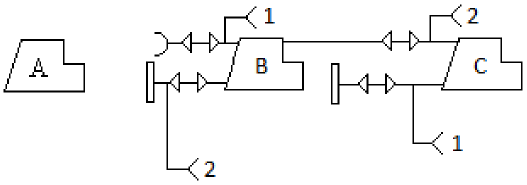
The relations are "bi-directional implication". The proof continues:

8 

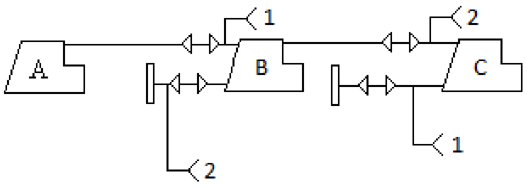
7, T:AL

9 

8, A:SD in S0

10 

9, A:ASS

11 

10, T:AL

Structure 2.0.6

Now we may drop the Stoppers because they would just give us the premise anded together with the conclusion. This is equivalent to doing T:ANDE to prove the identity.

Note that the same applies for all the other relations that we will define. We can see that in line 11, after doing A:ASS and T:AL twice we have a statement that cannot be stated with letter Propositional Logic using brackets.

Syllogism is proven in ref. [1]. For the case where there is an explicit "AND" between the two subpremise structures we reach a stage in the proof thus:

n   (A)->-| -(x)-| |->-(C)   n-1, A:AA

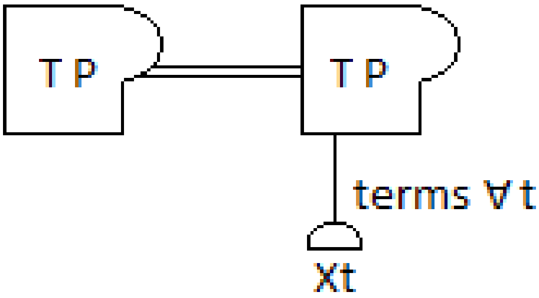
then link the Stopper carrying "AND" with A, do an A:ASS and then an A:AL and

T:AL, then do a T:ANDE to leave out the relation carrying "AND".

The "axiom": p -> (p OR q) can be proven using "Contraposition", "AND elimination", "De

Morgans Law" and the following law (Structure 2.0.7).

A rule not easy to express in letter Propositional Logic is:



Structure 2.0.7

It states that for all terms t in a proposition TP if we replace t with not-t the formula truth table stays the same if TP is a tautology. Here the operator attached to TP is called an Introductor and the relation is "equivalence". The proof is:

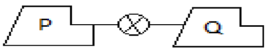
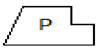
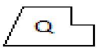
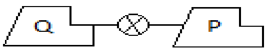
To prove: ~(p OR q) -> ~p

1	~(p OR q)	Assumption
2.	~p AND ~q	De Morgan
3.	~p	T:ANDE

where p and q are propositions and AND and OR is typed out.

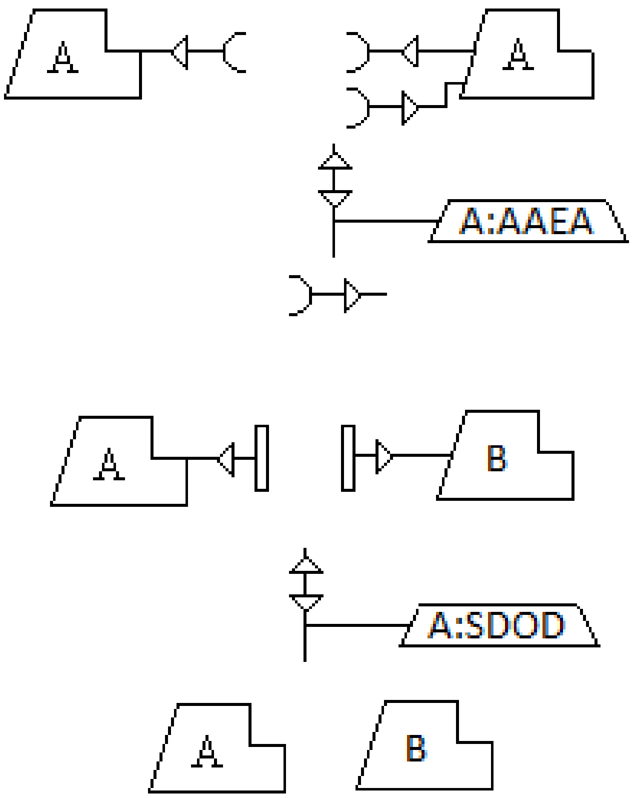
We can try to prove commutation: (p AND q) -> (q AND p) by assuming the

premise and then doing T:ANDE and then T:ANDI in the opposite order as follows:.

Line #	Statement	Reason
1		Premise
2		1, T:ANDE
3		1, T:ANDE
4		3,2, Unsplit
5		4, T:ANDI

Structure 2.0.8.

There are two more axioms of how Attractors and Stoppers behave:

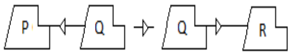
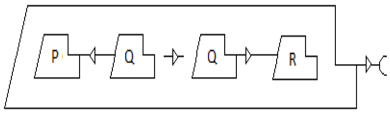
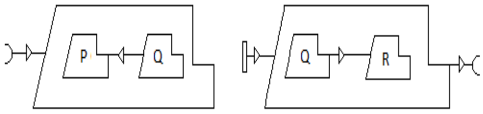
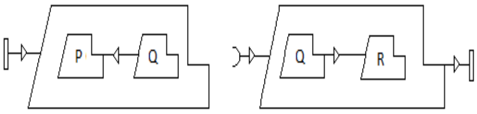
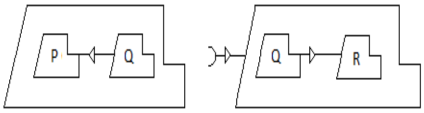
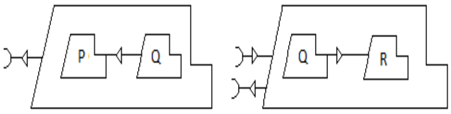
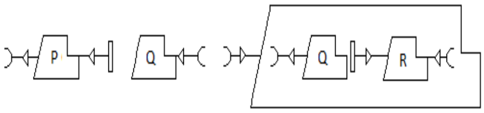
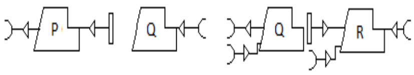
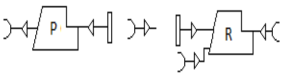
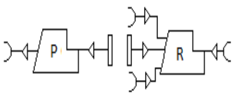
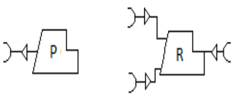
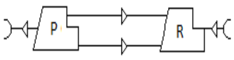
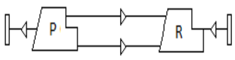
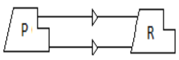
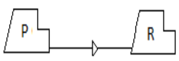


Structure 2.0.9.

Note well the direction of the "therefore" symbols. The labels read: "Attractor Annihilation Extra Attractor" and "Stropper Drop Other Direction".

With these two axioms we can prove the "axiom":

(p -> r) -> ((q -> p) -> (q -> r)) by reasoning backwards through the following proof:

Line #	Statement	Reason
1		Premise
2		1, A:AtI
3		2, A:AD
4		3, A:ASS
5		4, A:SD
6		5, A:AtI
7		6, A:AD
8		7, A:AM
9		8, A:AAEA
10		9, A:AM
11		10, A:SDOD
12		11, T:AL x 2
13		12, A:ASS
14		13, A:SD
15		14, T:ANDE

Structure 2.0.10

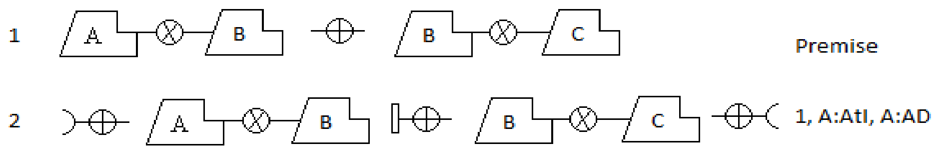
Line 12 follows since structure R with its two Attractors constitutes a potential structure.

We notice that we used the suspected A:SDOD, but we could just as well have kept the two Stoppers, these would have no effect on the result since the "therefore" symbols they carry faces in opposite directions. We can choose to interpret them as Stoppers and then they have no influence on the truth table of the result.

We next show that one cannot derive: A AND C from (A AND B) OR

(B AND C):

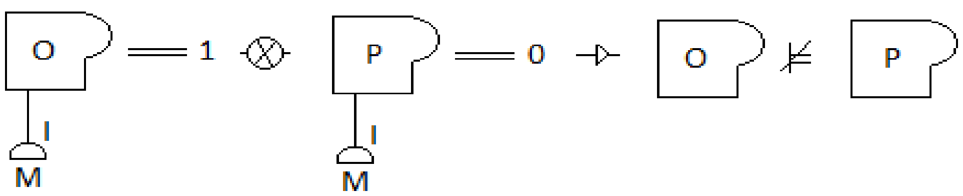
Line # Statement Reason



Structure 2.0.11

and we see that even though we can get rid of structure B we can not get rid of the bothersome Stopper carrying an "OR" relation.

Another axiom I have come across reads:



Structure 2.0.11.1

Where the symbols on the right reads: "P does not semanticly follow from O".

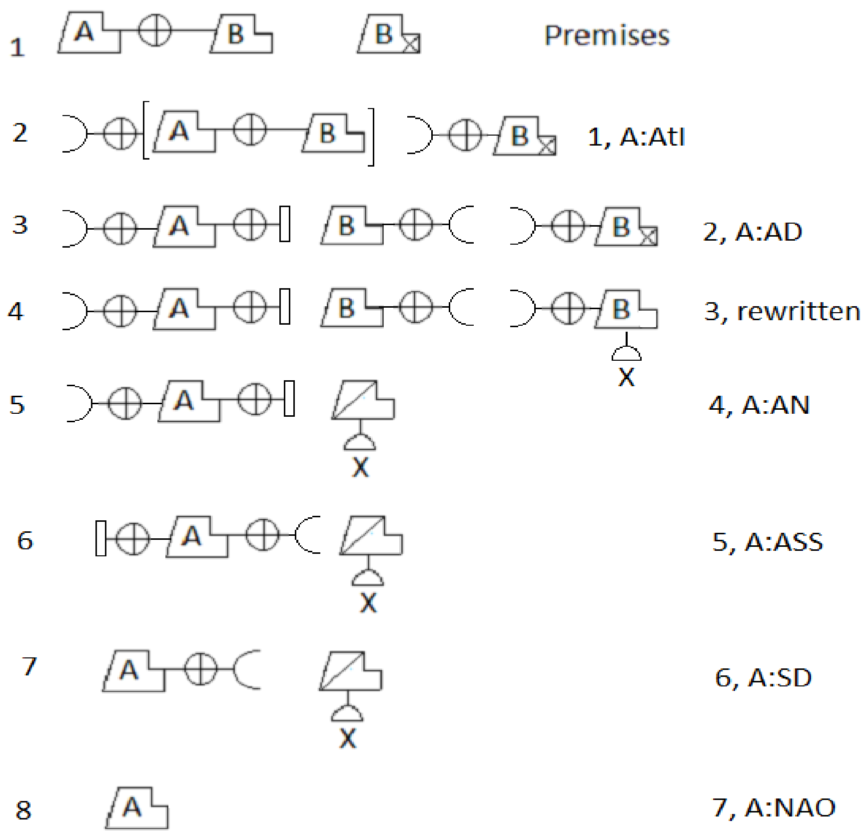
The same applies if O or P is a set of propositions.

This is obviously because one cannot write in a proof:

Index Statement Reason
...
n O ...
n+1 P n
...

since P under interpretation model M is false. With just letters, you must remember what object it represents, with my logic the object is displayed everywhere it occurs.We work out A:NAO in order to be able to prove true things. Now note that we can prove: Theorem (T:OANS): ~B AND (A OR B) -> A.

Proof:



Structure 2.0.16.

And from the same premises and T:ANDE we can prove: "structure B is false". Note that AND elimination is not a meaning preserving operation. The premise means: "structure A or structure B in association with not structure B." and this does not mean the same as "structure B is false". The fact that we can derive two different conclusions from the same premises might seem strange because we are used to meaning preserving operations, but this is what the symbols show. It is my opinion that the first derivation is meaning preserving (the premises means implicitly that: "structure A is true") and AND elimination not. The proof uses "AND" and "Exist Together" interchangedly. Note that we could equally well have included "AND" explicitly in which case we would have had a nonsense Stopper and Attractor carrying "AND" attached to B. So we see that nonsense Stoppers and Attractors can be dropped using A:AN (not explicitly included in the Axiom).

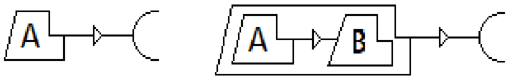
Theorem (T:OANS2): ~A AND (A OR B) -> B can be proven similarly.

We prove Modus Ponens:

These are two structures. We make this explicit:

Now we may introduce Attractors carrying a therefore symbol:

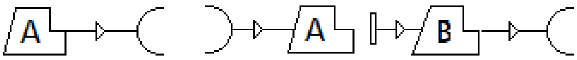
3



2, A:AI

We apply Attractor distribution to this to get:

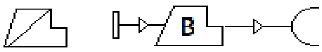
4



3, A: AD

Because we have matching structures A and the arrow carried by the Attractors point in the same direction the structures A annihilate:

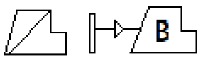
5



4, A:AA

Because there is no structure to the right of structure B, we may drop the Attractor:

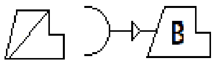
5



4, A:ADR

Now do Attractor-Stopper exchange:

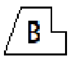
6



5, A:ASE

and drop the Attractor, to get:

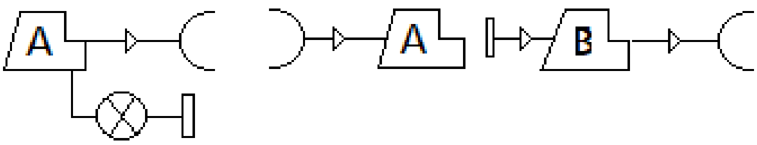
7



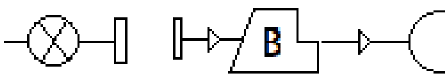
6, A:ADR

Where we could drop the Attractors because of A:ASS and A:SD. We therefore see that it seems like our Logic mixes object language and metalogical language since metalogical language is needed to prove Modus Ponens by other writers.

With our premise the same except for an AND in between P and P -> Q we can also prove MP. We reach a stage in the proof:

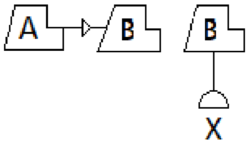


and then we can apply A:AA to get:

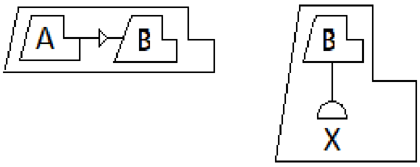


and then we can drop the nonsense Stopper carrying "AND", the other Stopper and Attractor to get: "structure B is true".

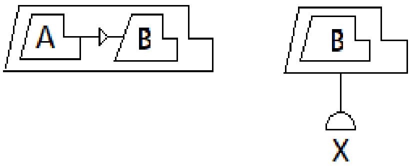
We prove Modus Tollens:



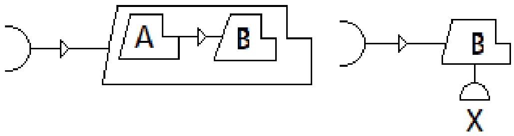
Premise

2 

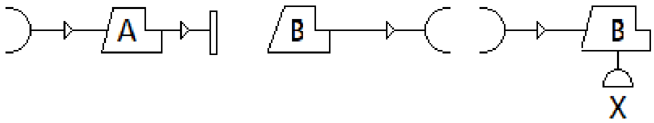
1, restated.

3 

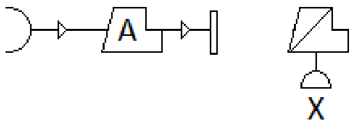
2, restated.

5 

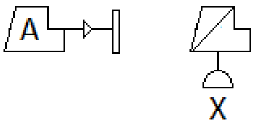
3, A:AI

6 

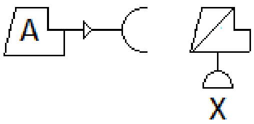
5, A:AD

7 

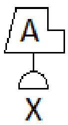
6, A:AN

8 

7, A:ADR

9 

8, A:ASE

10 

9, A:NA

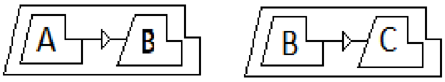
Note that the object first appearing in line 7 reads: "the negated empty structure" and does not read: "the structure of everything". This is evident in going from line 9 to 10. Also note that this proof do not use Contraposition.

Next we prove Syllogism. The premises are:

1 

Premises

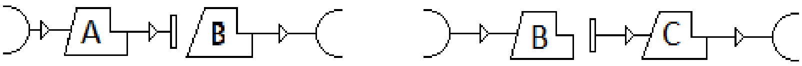
Rearrange this:

2  1, rearrange

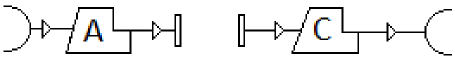
Now we may introduce Attractors carrying therefore symbols:

3  2, A:AI

Distribute the Attractors:

4  3, A:AD

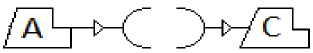
After Attractor annihilation we get:

5  4, A:AA

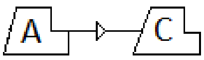
Dropping the nonsense Attractors we get:

6  5, A:ADR

Now we do Attractor-Stopper exchange to get:

7  6, A:ASE

Executing the Attractors we get:

8  7, A:AE

This proves Syllogism. We state that the same would apply if the second B had a universal quantifier. Thus a syllogism is proveable if we can write the premise in the form of line 2. We have to exercise a little care for universal vs. existential quantification.

We see if our axioms produces the correct inferrence when one of the B's is negated:

1	(A)->-(B) (~B)->(C)	Premise
2	(A)->-(B) (~B)->(C) ->-(	1, A:AtI
3	)->-(A)->-| (B)->-( )->-(~B) |->(C)->-(	2, A:AD
4	(A)->-| (B)->-( )->-(~B) |->(C)	3, A:ASS, A:SD, A:ASS
5	(A)->-| (~_) |->(C)	4, A:AN
6	(A)->-((~_))->(C)	5, A:ASS

Structure 2.0.17

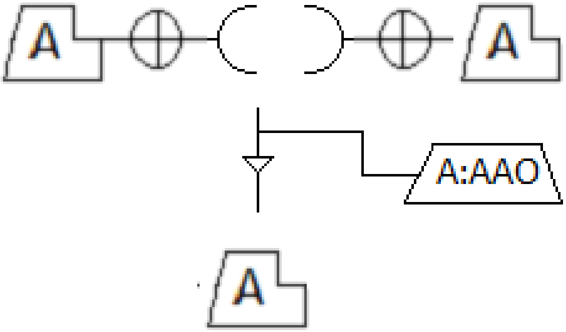
where (~_) is a negated empty structure and using A:NA we can conclude:

(~A)->-(C)

OR (metalogicly) using A:NATL we can conclude:

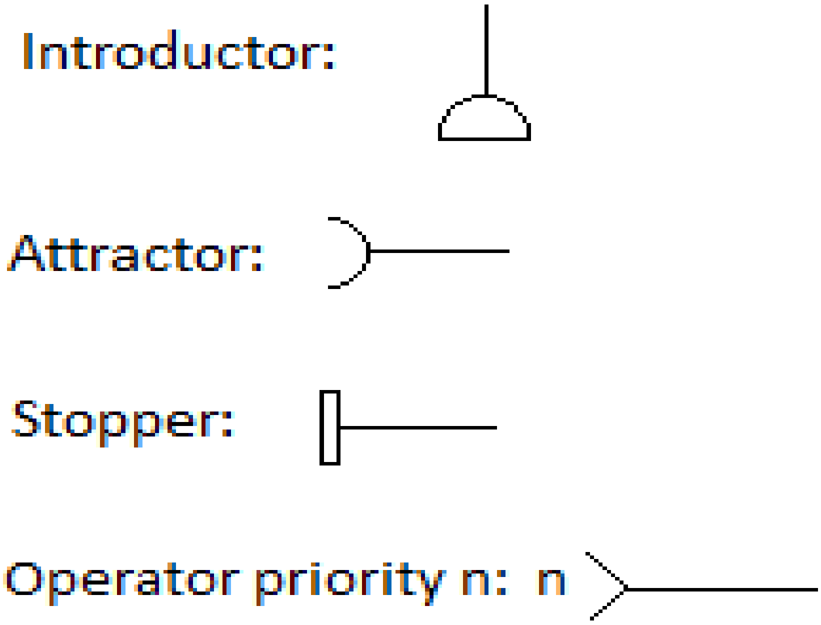
(A)->-((~_) or: (~A).

To prevent us inferring B from A, A OR B, we must allow the following axiom:

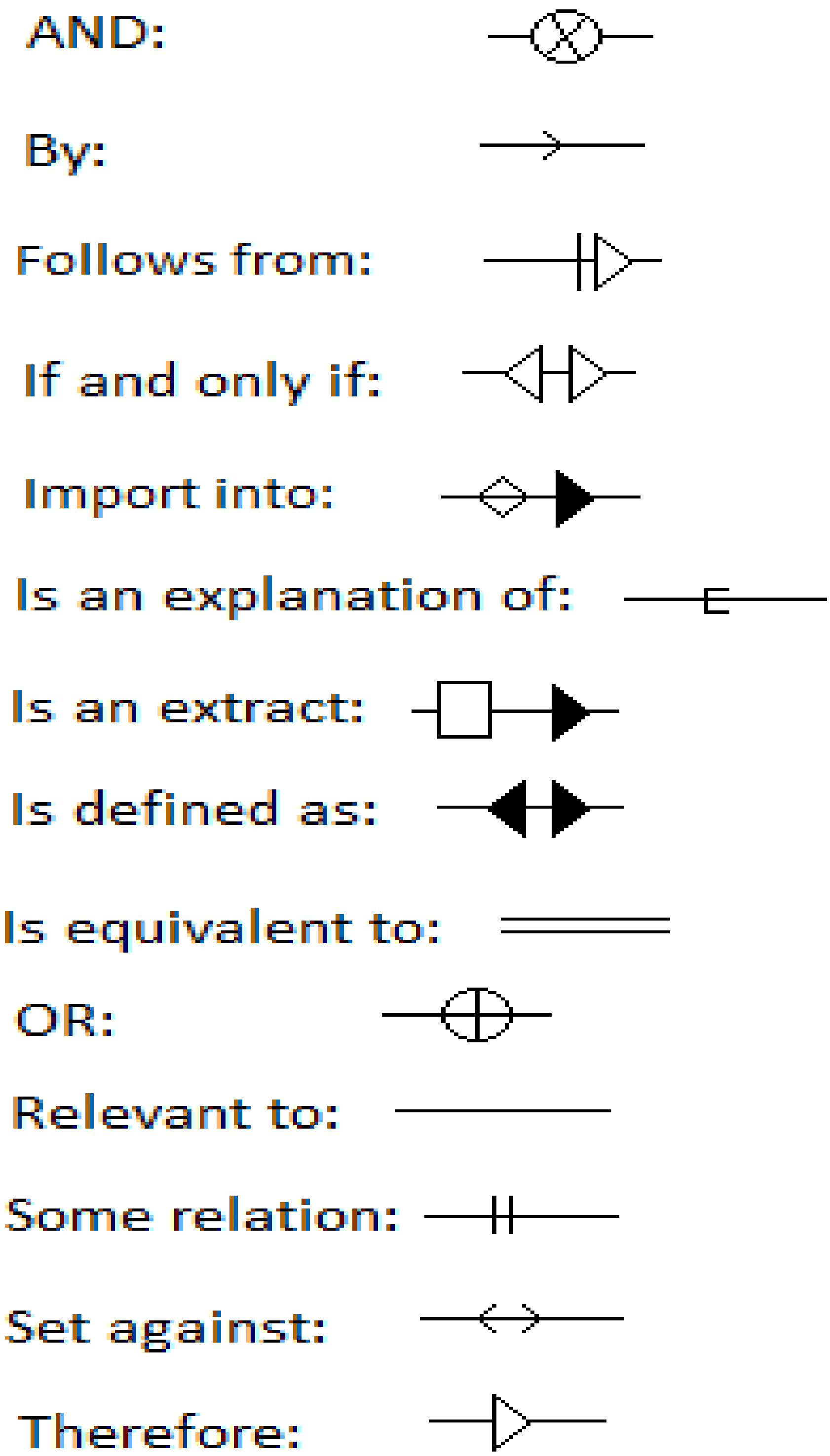


Structure 2.0.17

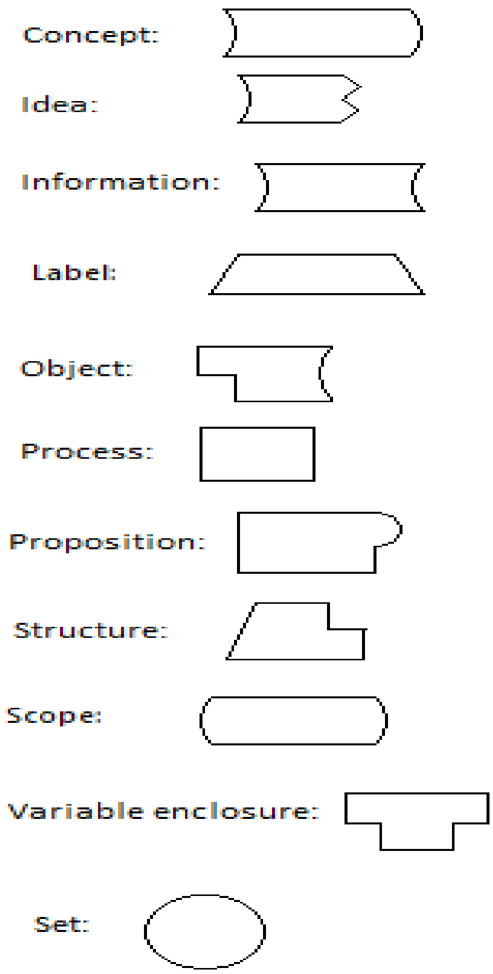
Appendix A: Operator List.



Appendix B: Relation List.



Appendix C: Enclosure List.



Conclusion: we conclude that SrL is usefull in a variety of settings.

Complience with Ethical Standards:

Funding: The writer did not receive any funding.

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

Bibliography

[1]. W. F. Esterhuyse, Introduction to Logical Structures, Scribd. 2012.
[2]. S. K. Langer, An Introduction to Symbolic Logic, Dover Publications, 2011
[3]. T. Button, Level Theory, Part 1: Axiomatizing the Bare Idea of a Cumulative Hierarchy of Sets. The Bulletin of Symbolic Logic. Cambridge. 2021. DOI: 10.1017/bsl.2021.13.
[4]. F. P. Schuler. The Geometric Anatomy of Theoretical Physics. Internet. 2015
[5]. H.O. van Rooyen. Relation Nets in Learning and Teaching: A Knowledge Representation Approach. Academia. Internet. Undated.
[6]. Stanford Encyclopedia of Philosophy. Model Theory. Internet. https://plato.stanford.edu/entries/model-theory/. 2020.
[7]. A.N. Witehead, B. Russell. Principia Mathematica.
[8]. Willem F. Esterhuyse. "Proof of "Axioms" of Propositional Logic" *IOSR Journal of Mathematics (IOSR-JM)*, 18(4), (2022): pp. 20-41.
[9]. J. Castro-Manzano. Distribution Tableaux, Distribution Models. Axiom. MDPI. 2020. pages: 31-39