*IOSR Journal of Mathematics (IOSR-JM)*
*e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 18, Issue 4 Ser. II (Jul. – Aug. 2022), PP 23-32*
*www.iosrjournals.org*

# Adomian's Decomposition Series and Variational Iteration Method on Burger's Equation

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***Abstract:****Adomian's decomposition method (ADM) and the variational iteration method (VIM) are well documented in the literature and have been found to be effective tools for obtaining an exact solution for initial boundary partial differential equations or, if this is not possible, for obtaining a highly accurate numerical solution. Various authors have compared the two approaches and concluded that, in general, both approaches accomplish the same result. In this study, both approaches are applied to Burgers' equation with three distinct initial boundary conditions. Adomian's decomposition method is typically computationally more challenging than the variational iteration method. VIM is typically applied to situations in which the initial condition is set to zero, and the elimination of the so-called "small terms or noisy terms" is a vital step.*
***Key Word****:Adomian's decomposition method; Variational iteration method; Burger's equation.*

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Date of Submission: 05-07-2022 Date of Acceptance: 19-07-2022

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## I. Introduction

Nonlinear phenomena play a crucial role in many areas and, in this context, knowledge of the explicit solution of the nonlinear differential equations are of fundamental importance. However, it is frequently impossible to find the exact solution to these situations.Even so, the ability to accurately solve these kinds of equations has seen significant advancement in the field of numerical analysis. The Adomian's decomposition method (ADM) and the variational iteration approach (VIM) are two well-known techniques that are highlighted in this study.

The first one (ADM) was developed by G. Adomian[1]. It involves expressing the solution function as a series of functions and determining the iterative formula that allows one to compute the subsequent components of the series using the initial boundary conditions provided[2]. In the literature, the convergence and the advantages of the ADM applied to many linear and nonlinear problems have been emphasized by several authors[3,4,5,6,7,8].

The later method (VIM) was developed by J. H. He[9,10]. It consists on constructing an appropriate correction functional connected to the considered equation, which, with the Lagrange multiplier, allows one to determine the iteration solution formula[2]. VIM has been applied to solve partial differential equations in several papers[11,12,13,14,15,16,17].

Both methods do not require discretization of the variables (they are not affected by errors associated to discretization), as well linearization, or perturbation (not changing the true solution of the problem). Hence, they are very efficient on determining an approximate or exact solution in a closed form, for both linear and nonlinear problems.

Numerous authors have compared these two approaches, applied to ordinary or partial differential equations, such as,to homogeneous and non-homogeneous partial differential equations[18],to solve a "moving boundary problem"[2], tothe modified Kortweg-de Vries equation[19], to the Sawada-Kotera-Ito seventh-order equation[20], to solve fourth order differential equations[21], to integro-differential equations[22], to fractional integro-differential equations[23]as well as many others.The primary findings of these studies were that the two techniques are effective, efficient, and yield more precise solution approximations, and so,for solving linear or nonlinear,ordinary or partial differential equations, the two approaches are useful tools.In general, however, VIM performs better since it does not involve the evaluation of Adomian polynomials, and therefore converges faster to the exact solution for certain nonlinear equations.

The Adomian's decomposition approach and the variational iteration method are introduced in section 2 with a brief description of each.Section 3 presents their application to Burger's equation using the $t$-direction and $x$-direction iteration formulas.The next step is to apply these methods to three different initial boundary problems for which the precise solution is already known. ADM has alreadybeen applied to these examples[5], but the study that follows looks at the same cases and compares the results of the two approaches(section 4).

## II. Description of the methods

**Adomian's decomposition method**

Consider a nonlinear differential equation $Fu = g$, where $F$ represents a nonlinear differential operator. ADM consists on decomposing the linear part of $F$ into $L + R$, where $L$ is an operator easily invertible and $R$ is the remaining part. Representing the nonlinear term by $N$, the equation $Fu = g$ can be written as

$$Lu + Ru + Nu = g. \tag{1}$$

Representing the inverse of the operator $L$ as $L^{-1}$, the following equivalent equation is achieved:

$$L^{-1}Lu = g - L^{-1}Ru - L^{-1}Nu. \tag{2}$$

Since $L^{-1}Lu = u + a$, where $a$ is the term emerging from the integration, equation (2) becomes

$$u = g - a - L^{-1}Ru - L^{-1}Nu. \tag{3}$$

The method looks for a series solution $u = \sum_{n=0}^{\infty} u_n$. Identifying $u_0$ as $g - a$, the rest of the terms $u_n$, $n > 0$, will further be settled by a recursive relation. The key of ADM is to develop the nonlinear term $Nu$ into a series of polynomials $Nu = \sum_{n=0}^{\infty} A_n$. These polynomials are the so called *Adomian polynomials*. Each polynomial $A_n$ depends only on the terms of the series soluction $u_0, u_1, \dots, u_n$. Adomian introduced the formula to generate these polynomials for all kinds of nonlinearities[1,5]. Additionally, it has been demonstrated that, rather than in the proximity of a point, the sum of the Adomian polynomials generalizes the Taylor series in the neighbourhood of a function,

$$Nu = \sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} \frac{1}{n!}(u - u_0)^n N^{(n)}(u_0) \tag{4}$$

The general term of the series tends to zero very fast, as $\frac{1}{(mq)!}$, according to the optimal choice of the initial term, for $m$ terms and $q$ the order of the linear operator $L$[1,24].

Substituting $u = \sum_{n=0}^{\infty} u_n$ and $Nu = \sum_{n=0}^{\infty} A_n$ into equation (3) one gets

$$\sum_{n=0}^{\infty} u_n = g - a - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n. \tag{5}$$

To determine the terms $u_n(x,t)$, $n = 0,1,2,\dots$, of the series solution, the recursive relation is employed:

$$\begin{aligned} u_0 &= g - a, \\ u_1 &= -L^{-1}R\, u_0 - L^{-1}A_0, \\ u_2 &= -L^{-1}R\, u_1 - L^{-1}A_1, \\ &\vdots \\ u_{n+1} &= -L^{-1}R\, u_n - L^{-1}A_{n,}, \\ &\vdots \end{aligned} \tag{6}$$

The Adomian polynomials are obtained by introducing a parameter $\lambda$:

$$u(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n, \tag{7}$$

$$N\big(u(\lambda)\big) = \sum_{n=0}^{\infty} \lambda^n A_n. \tag{8}$$

Adomian polynomials are then expressed as[1,25]:

$$A_n(u_0, u_1, \dots, u_n) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}. \tag{9}$$

Then, a Taylor series of $N \circ u$ in a neighborhood of $\lambda = 0$ is performed, and one gets:

$$N\big(u(\lambda)\big) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\big(u(\lambda)\big)\right]_{\lambda=0} \lambda^n = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0} \lambda^n. \tag{10}$$

Other methods have been developed for the calculation of Adomian polynomials[26,27,28].The infinite sum in equation (10) can be substituted by a finite sum, which facilitates its computation[26,27,28,29]. In this way, the Adomian polynomials can be calculated using the following formula:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}. \tag{11}$$

**Variational iteration method**

As described in ADM, the VIM is herein applied to a nonlinear differential equation $Fu = g$, where $F$ represents a nonlinear differential operator. The method consists on decomposing $F$ into $L + N$, where $L$ is an linear operator and $N$ is a nonlinear operator. Thus, the equation in canonical form is

$$Lu + Nu = g. \tag{12}$$

According to He[9,10], the expression

$$u_{cor}(1) = u_0(1) + \int_0^1 \lambda\,(Lu_0 + Nu_0 - g)dx \tag{13}$$

corrects the value for some special point, in this case for $x = 1$. The function $u_0(x)$ is the solution of $Lu = 0$, and $\lambda$ is the general Lagrange multiplier, which can be identified optimally via the variational theory. He[9,10] modified the expression (13) in the following way

$$u_{n+1}(x_0) = u_n(x_0) + \int_0^{x_0} \lambda\,(Lu_n + N\tilde{u}_n - g)dx, \tag{14}$$

being $u_0$ an initial approximation and $\widetilde{u_n}$ a restricted variation, i.e. $\delta\tilde{u} = 0$. The correctional functional, for any arbitrary value of $x$,is then obtained

$$u_{n+1}(x) = u_n(x) + \int_0^{x} \lambda\,(Lu_n(s) + N\tilde{u}_n(s) - g(s))ds. \tag{15}$$

With the determination of $\lambda$, the approximations $(u_n(x))_{n\in\mathbb{N}}$ follow immediately and the exact solution is achieved by taking the limit:

$$u(x) = \lim_{n\to+\infty} u_n(x).$$

## III. Applying ADM and VIM to Burger's equation

Consider Burger's equation

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} \tag{16}$$

with the following initial and Dirichlet boundary conditions:

$$u(x,0) = u_0(x)$$
$$u(0,t) = f_0(t), \quad u(1,t) = f_1(t).$$

**Application of Adomian's decomposition method to Burger's equation**

The linear operators, according to ADM, are as follows:

$$L_t(\cdot) = \frac{\partial}{\partial t}(\cdot), \quad L_{xx}(\cdot) = \frac{\partial^2}{\partial x^2}(\cdot). \tag{17}$$

Applying the inverse operator of $L_t(\cdot) = \frac{\partial}{\partial t}(\cdot)$, $L_t^{-1}(\cdot) = \int_0^t (\cdot)\, dt'$, to both sides of equation (16) one obtains:

$$u(x,t) = u_0(x) + L_t^{-1}\left(-u\frac{\partial u}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2}\right). \tag{18}$$

According to ADM, the solution can be expressed by the series $u(x,t)$

$$u(x,t) = \sum_{n=0}^{\infty} u_n\,(x,t) \tag{19}$$

and the nonlinear term $u\frac{\partial u}{\partial x}$ is represented by a series of Adomian polynomials

$$u\frac{\partial u}{\partial x} = \sum_{n=0}^{\infty} A_n. \tag{20}$$

When equations (19) and (20) are placed into equation (18), the following result is obtained:

$$\sum_{n=0}^{\infty} u_n = u_0(x) + L_t^{-1}\left(\nu\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} u_n - \sum_{n=0}^{\infty} A_n\right). \tag{21}$$

To compute the terms of the series solution $u_n(x,t)$, ADM employs the recursive relation:

$$u_0 = u_0(x),$$

$$
\begin{aligned}
u_1 &= L_t^{-1}\left(\nu \frac{\partial^2}{\partial x^2} u_0 - A_0\right),\\
u_2 &= L_t^{-1}\left(\nu \frac{\partial^2}{\partial x^2} u_1 - A_1\right),\\
&\vdots\\
u_n &= L_t^{-1}\left(\nu \frac{\partial^2}{\partial x^2} u_{n-1} - A_{n-1}\right),\\
&\vdots
\end{aligned} \tag{22}
$$

The Adomian polynomials depend on the particular nonlinearity. In this case, the $A_n$ polynomials are given by:

$$
\begin{aligned}
A_0 &= u_0 \frac{\partial}{\partial x} u_0,\\
A_1 &= u_0 \frac{\partial}{\partial x} u_1 + u_1 \frac{\partial}{\partial x} u_0,\\
A_2 &= u_0 \frac{\partial}{\partial x} u_2 + u_1 \frac{\partial}{\partial x} u_1 + u_2 \frac{\partial}{\partial x} u_0,\\
A_3 &= u_0 \frac{\partial}{\partial x} u_3 + u_1 \frac{\partial}{\partial x} u_2 + u_2 \frac{\partial}{\partial x} u_1 + u_3 \frac{\partial}{\partial x} u_0,
\end{aligned} \tag{23}
$$

Likewise, when ADM is applied in the $x$-direction, one yields

$$
u(x,t) = u_0 + \frac{1}{\nu} L_{xx}^{-1}\left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}\right] \tag{24}
$$

with

$$
u_0 = u(0,t) + x \frac{\partial u}{\partial x}(0,t), \tag{25}
$$

being

$$
L_{xx}^{-1}(\cdot) = \int_0^x d\,x' \int_0^{x'} (\cdot)\, dx''.
$$

The series solution is then given by:

$$
\sum_{n=0}^{\infty} u_n = u(0,t) + x \frac{\partial}{\partial x} u(0,t) + \frac{1}{\nu} L_{xx}^{-1}\left(\frac{\partial}{\partial t}\sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n\right). \tag{26}
$$

To compute the terms of $u_n(x,t)$, $n = 0,1,2,\ldots$, ADM employs the following recursive relation:

$$
\begin{aligned}
u_0 &= u(0,t) + x \frac{\partial u}{\partial x}(0,t),\\
u_1 &= \frac{1}{\nu} L_{xx}^{-1}\left[\frac{\partial}{\partial t} u_0 + A_0\right],\\
u_2 &= \frac{1}{\nu} L_{xx}^{-1}\left[\frac{\partial}{\partial t} u_1 + A_1\right],\\
&\vdots\\
u_n &= \frac{1}{\nu} L_{xx}^{-1}\left[\frac{\partial}{\partial t} u_{n-1} + A_{n-1}\right],\\
&\vdots
\end{aligned} \tag{27}
$$

**Application of variational iteration method to Burger's equation**

To solve the Burger's equation (16), according to VIM, the correction functional is constructed:

$$
u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(x,s)\left(\frac{\partial}{\partial s} u_n(x,s) + N(\tilde{u}_n(x,s))\right) ds, \quad n \geq 0 \tag{28}
$$

being

$$
N(\tilde{u}_n(x,t)) = \tilde{u}_n(x,t) \frac{\partial}{\partial x} \tilde{u}_n(x,t) - \nu \frac{\partial^2}{\partial x^2} \tilde{u}_n(x,t),
$$

where $\tilde{u}_n$ is a non-variational variation, meaning that $\delta N(\tilde{u}_n(x,s)) = 0$.

The value of $\lambda$ is chosen to make the correction functional (28) stationary, i.e. one must have $\delta u_{n+1} = 0$. Therefore,

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(x,s) \left( \frac{\partial}{\partial s} u_n(x,s) + N(\tilde{u}_n(x,s)) \right) ds \tag{29}$$

$$= \delta u_n(x,t) + \lambda(x,s)\delta u_n(x,s)|_{s=t} - \int_0^t \frac{\partial \lambda(x,s)}{\partial s} \delta u_n(x,s) +$$

$$+\lambda(x,s)\delta \left( u_n(x,s) \frac{\partial}{\partial x} u_n(x,s) - \nu \frac{\partial^2}{\partial x^2} u_n(x,s) \right) ds = 0.$$

Then, their stationary conditions are written as

$$-\frac{\partial \lambda(x,s)}{\partial s} = 0 \quad \text{and} \quad 1 + \lambda(x,s)|_{s=t} = 0. \tag{30}$$

Consequently, the Lagrange multiplier can be established as

$$\lambda(x,s) = -1. \tag{31}$$

Substituting the value of the Lagrange multiplier (31) into the functional (28), one achieves the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \frac{\partial}{\partial s} u_n(x,s) + u_n(x,s) \frac{\partial}{\partial x} u_n(x,s) - \nu \frac{\partial^2}{\partial x^2} u_n(x,s) ds, n \geq 0. \tag{32}$$

Similarly, when VIM is applied in the $x$-direction, the correctional functional of equation (16) is expressed as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda(s,t) \left( -\nu \frac{\partial^2}{\partial s^2} u_n(s,t) + N(\tilde{u}_n(s,t)) \right) ds, \quad n \geq 0, \tag{33}$$

where

$$N(\tilde{u}_n(x,t)) = \frac{\partial}{\partial t} \tilde{u}_n(x,t) + \tilde{u}_n(x,t) \frac{\partial}{\partial x} \tilde{u}_n(x,t).$$

and $\tilde{u}_n$ is such that $\delta N(\tilde{u}_n(s,t)) = 0$.

Likewise, the value of $\lambda$ is established to make the correction functional (33) stationary, i.e., $\delta u_{n+1} = 0$,

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^x \lambda(s,t) \left( -\nu \frac{\partial^2}{\partial s^2} u_n(s,t) + N(\tilde{u}_n(s,t)) \right) ds = \tag{34}$$

$$= \delta u_n(x,t) - \lambda(s,t)\nu \frac{\partial}{\partial s} \delta u_n(s,t)|_{s=x} + \nu \frac{\partial}{\partial s} \lambda(s,t) \delta u_n(s,t)|_{s=x} -$$

$$-\int_0^x \nu \frac{\partial^2 \lambda(s,t)}{\partial s^2} \delta u_n(s,t) + \lambda(s,t)\delta \left( u_n(s,t) \frac{\partial}{\partial s} u_n(s,t) + \frac{\partial}{\partial t} u_n(s,t) \right) ds = 0.$$

Therefore, the stationary conditions are obtained

$$-\nu \frac{\partial^2 \lambda(s,t)}{\partial s^2} = 0, \qquad 1 + \nu \frac{\partial \lambda(s,t)}{\partial s} |_{s=x} = 0, \qquad \text{and } -\nu \lambda(s,t)|_{s=x} = 0. \tag{35}$$

and the Lagrange multiplier can be identified

$$\lambda(s,t) = \frac{x-s}{\nu}. \tag{36}$$

Substituting the value of the Lagrange multiplier (36) into the functional (33), the iteration solution formula is achieved:

$$u_{n+1}(x,t)$$
$$= u_n(x,t) + \int_0^x \frac{x-s}{\nu} \left( -\nu \frac{\partial^2}{\partial s^2} u_n(s,t) + u_n(s,t) \frac{\partial}{\partial s} u_n(s,t) + \frac{\partial}{\partial t} u_n(s,t) \right) ds, \qquad n \geq 0. \tag{37}$$

## IV. Numerical Examples

**Example 1**

Take Burger's equation (16) with the particular initial and Dirichlet homogeneous boundary conditions, given by:

$$u_0(x) = \sin(\pi x), \tag{38}$$

$$u(0,t) = u(1,t) = 0. \tag{39}$$

The exact solution is given by[30,31]

$$u(x,t) = \frac{4\pi\nu \sum_{k=1}^{\infty} k\, I_k\left(\frac{1}{2\pi\nu}\right) sin\,(k\pi x)\, exp\,(-k^2\pi^2\nu t)}{I_0\left(\frac{1}{2\pi\nu}\right) + 2\sum_{k=1}^{\infty} I_k\left(\frac{1}{2\pi\nu}\right) cos\,(k\pi x)\, exp\,(-k^2\pi^2\nu t)}, \quad (40)$$

being $I_k$ the modified Bessel functions of first kind. Consider the solution for $\nu = 1$.

***Adomian's decomposition method***

Using equations (22) and (23), the first four terms of the Adomian's series solution are[5]:

$$u_0 = \sin(\pi x),$$
$$u_1 = -\sin(\pi x)(\pi + \cos(\pi x))\pi t,$$
$$u_2 = \frac{1}{2}\sin(\pi x)(\pi^2 + 6\cos(\pi x)\pi + 3(\cos(\pi x))^2 - 1)\pi^2 t^2,$$
$$u_3 = -\frac{1}{6}\sin(\pi x)\pi^3 t^3[28\pi^2\cos(\pi x) + \pi^3 - 15\pi$$
$$+51(\cos(\pi x))^2\pi + 16(\cos(\pi x))^3 - 10\cos(\pi x).$$

***Variational iteration method***

Substituting (38) into (32), the following iterative approximations are obtained:

$$u_0 = \sin(\pi x),$$
$$u_1 = \sin(\pi x) - \sin(\pi x)(\pi + \cos(\pi x))\pi t,$$
$$u_2 = u_0 + u_1 + \frac{1}{2}\sin(\pi x)(\pi^2 + 6\cos(\pi x)\pi + 3(\cos(\pi x))^2 - 1)\pi^2 t^2,$$
$$-\frac{1}{3}\sin(\pi x)\pi^3 t^3(2\cos(\pi x)^3 + 3\cos(\pi x)^2\pi + \cos(\pi x)\pi^2 - \cos(\pi x) - \pi),$$
$$\vdots$$

Early on in the second iteration, a large expression starts to emerge, making the computational job quite challenging.The computational evaluation becomes quicker and more efficient if the so-called "small terms," as stated in the literature[18,32], are removed. As a result, the terms with order larger than $n$are eliminatedin each iteration, and the following terms are attained:

$$u_0 = \sin(\pi x),$$
$$u_1 = u_0 - \sin(\pi x),$$
$$u_2 = u_0 + u_1 + \frac{1}{2}\sin(\pi x)(\pi^2 + 6\cos(\pi x)\pi + 3(\cos(\pi x))^2 - 1)\pi^2 t^2,$$
$$u_3 = u_0 + u_1 + u_2 - \frac{1}{6}\sin(\pi x)\pi^3 t^3[28\pi^2\cos(\pi x) + \pi^3 - 15\pi$$
$$+51(\cos(\pi x))^2\pi + 16(\cos(\pi x))^3 - 10\cos(\pi x)],$$
$$\vdots$$

This yields a solution that can be easily verified is the same for both approaches ADM and VIM. Both approaches seem effective and powerful in this case.However, VIM reduces the amount of calculations since it does not call for the evaluation of the Adomian polynomials.Each iteration's assessment using VIM is simple and direct.

**Example 2**

Take Burger's equation (16) with initial and boundary conditions given by:

$$u(x,1) = x - \pi\tanh\left(\frac{\pi x}{2\nu}\right), \quad (41)$$

$$u(0,t) = 0, \quad \frac{\partial}{\partial x}u(0,t) = \frac{1}{t} - \frac{\pi^2}{2\nu t^2}. \quad (42)$$

The exact solution is[5]

$$u(x,t) = \frac{x}{t} - \frac{\pi}{t}\tanh\left(\frac{\pi x}{2\nu t}\right). \quad (43)$$

***Adomian's decomposition method***

By Adomian's decomposition method applied in the $x$-direction, the terms of the series solutionare given by (27)[5,6]

$$u_0 = \frac{x}{t} - \frac{x\pi^2}{2\nu\, t^2},$$
$$u_1 = \frac{\pi^4 x^3}{24\nu^3 t^4},$$
$$u_2 = -\frac{\pi^6 x^5}{240\nu^5 t^6},$$
$$u_3 = \frac{17\pi^8 x^7}{40320\nu^7 t^8},$$
$$u_4 = -\frac{31\pi^{10} x^9}{725760\nu^9 t^{10}},$$
$$\vdots$$

The series obtained is the Taylor expansion of the real solution (43) around $x = 0$. The expansion of the function $\tanh(x)$is provided by the following series:

$$\tanh(x) = \sum_{n=1}^{\infty} B_{2n} \frac{4^n(4^n - 1)}{(2n)!} x^{2n-1}, \qquad |x| < \frac{\pi}{2}, \tag{44}$$

where $B_n$ are the Bernoulli numbers.

Therefore,the real solution (43) expanded in Taylor series around $x = 0$ is given by:

$$\frac{x}{t} - \frac{\pi}{t}\tanh\left(\frac{\pi x}{2\nu t}\right) = \frac{x}{t} - \sum_{n=1}^{\infty} B_{2n} \frac{4^n(4^n - 1)}{(2n)!}\left(\frac{\pi}{t}\right)^{2n}\left(\frac{x}{2\nu}\right)^{2n-1}, \qquad |x| < \nu t \tag{45}$$

This expansion coincides with the one obtained by Adomian's method

$$\frac{x}{t} - \frac{\pi}{t}\tanh\left(\frac{\pi x}{2\nu t}\right) = \frac{x}{t} - \frac{\pi^2 x}{2\nu\, t^2} + \frac{\pi^4}{24\nu^3 t^4}x^3 - \frac{\pi^6}{240\nu^5 t^6}x^5 + \frac{17\pi^8}{40320\nu^7 t^8}x^7 - \frac{31\pi^{10}}{725760\nu^9 t^{10}}x^9 + \dots . \tag{46}$$

***Variational iteration method***

Applying VIM in the $x$-direction, as with ADM, and substituting equations (42) and (25) into (37), the following successive approximations are computed:

$$u_0 = \frac{x}{t} - \frac{x\pi^2}{2\nu\, t^2},$$
$$u_1 = u_0 + \frac{\pi^4 x^3}{24\nu^3 t^4},$$
$$u_2 = u_0 + u_1 - \frac{\pi^6 x^5}{240\nu^5 t^6} + \frac{x^7\pi^8}{8064\nu^7 t^8},$$
$$u_3 = u_0 + u_1 + u_2 + \frac{17\pi^8 x^7}{40320\nu^7 t^8} - \frac{19}{725760}\frac{x^9\pi^{10}}{\nu^9 t^{10}} + \frac{67}{153222400}\frac{x^{11}\pi^{12}}{\nu^{11} t^{12}} - \frac{1}{25159680}\frac{x^{13}\pi^{14}}{\nu^{13} t^{14}} + \frac{1}{1950842880}\frac{x^{15}\pi^{16}}{\nu^{15} t^{16}},$$
$$\vdots$$

As in the previous example, the computational work to evaluate each iteration is hard. In this case, the terms of order larger than $n$ were also eliminated, and the following result was obtained:

$$u_0 = \frac{x}{t} - \frac{x\pi^2}{2\nu\, t^2},$$
$$u_1 = u_0,$$
$$u_2 = u_0,$$
$$u_3 = u_0 + \frac{\pi^4 x^3}{24\nu^3 t^4},$$
$$u_4 = u_3,$$
$$u_5 = u_0 + u_3 - \frac{\pi^6 x^5}{240\nu^5 t^6},$$
$$u_6 = u_5,$$
$$u_7 = u_0 + u_3 + u_5 + \frac{17\pi^8 x^7}{40320\nu^7 t^8},$$
$$\vdots$$

It is obvious from looking at the previous outcomes that the even iterations are equivalent to the prior one. Consequently, the terms of order lager than $2n+1$ were deleted. This way, both methods agree in each iteration. Furthermore,VIM reduces the volume of calculations by not requiring the evaluation of the Adomian polynomials.

**Example 3**

Consider once more Burger's equation (16) with the following particular initial boundary conditions:

$$u(x,1) = \frac{x}{1+e^{\frac{x^2}{4\nu}}}, \tag{47}$$

$$u(0,t) = 0, \quad u(1,t) = \frac{1}{t\left(1+\sqrt{t}e^{\frac{1}{4\nu t}}\right)}. \tag{48}$$

The exact solution, which can be verified by direct substitution in Burger's equation (16), is:

$$u(x,t) = \frac{x}{t\left(1+\sqrt{t}e^{\frac{x^2}{4\nu t}}\right)}. \tag{49}$$

***Adomian's decomposition method***

ADMhas already been applied successfully[5]in the$t$-direction with the inverse operator$L_t^{-1}(\cdot) = \int_1^t (\cdot)\, dt'$. The method converges in a neighborhood of $t = 1$ to the exact solution[5].

***Variational iteration method***

This example exhibits a feature not commonly seen in the literature, where the initial condition is defined at $t = 1$, instead of $t = 0$. He[9] approaches this problem by resorting to a trick, adjusting the initial solution to$t = 0$.

However, in this example, the same approach is not possible, since the solution has a singularity at $t = 0$.The integration operator for the iteration formula (32), has to start at $t = 1$and the following recursive relation is achieved:

$$u_{n+1}(x,t) = u_n(x,t) - \int_1^t \frac{\partial}{\partial s}u_n(x,s) + u_n(x,s)\frac{\partial}{\partial x}u_n(x,s) - \nu\frac{\partial^2}{\partial x^2}u_n(x,s)ds,\ n \geq 0. \tag{50}$$

In this case, in the third iteration a huge expression begins to appear, which turns the computational work very hard. The "small terms" must be deleted.Unfortunately, there is no universal rule in the literature for determining which "small terms" should be eliminatedmainly when $t \neq 0$.Rangkuti and Noorani[33]suggest utilizing the Taylor series to expand each iteration before removing the "small terms". In this paper each iteration was expanded using the Taylor series about $t = 1$, and the terms larger than $n$ were eliminated(corresponding "small terms").Consequently, the computational evaluation becomes quicker and more efficient, yielding the subsequent results:

$$u_0 = \frac{x}{1+e^{\frac{x^2}{4\nu}}},$$

$$u_1 = u_0 - \frac{\left((6\nu - x^2)e^{\frac{x^2}{4\nu}} + 4\nu\right)x}{4\nu\left(1+e^{\frac{x^2}{4\nu}}\right)^2}(t-1),$$

$$u_2 = u_1 + \frac{\left((-x^4 - 12\nu x^2 + 84\nu^2)e^{\frac{x^2}{4\nu}} + (x^4 - 20\nu x^2 + 60\nu^2)e^{\frac{x^2}{4\nu}} + 32\nu^2\right)x}{32\nu^2\left(1+e^{\frac{x^2}{4\nu}}\right)^3}(t-1)^2,$$

$$u_3 = u_2 - \left[\frac{\left((-x^6 - 30\nu x^4 - 180\nu^2 x^2 + 1416\nu^3)e^{\frac{x^2}{4\nu}}\right)x}{384\nu^3\left(1+e^{\frac{x^2}{4\nu}}\right)^4} + \right.$$

$$+\frac{\left((4x^6-24\nu x^4-528\nu^2x^2+1824\nu^3)e^{\frac{x^2}{4\nu}}\right)x}{384\nu^3\left(1+e^{\frac{x^2}{4\nu}}\right)^4}+$$

$$\left.+\frac{\left((-x^6+42\nu x^4-420\nu^2x^2+840\nu^3)e^{\frac{x^2}{4\nu}}+384\nu^3\right)x}{384\nu^3\left(1+e^{\frac{x^2}{4\nu}}\right)^4}\right](t-1)^3.$$

$$\vdots$$

The VIM applied to this example with the elimination of the "small terms" at each iteration arrives to the Taylor expansion of the real solution around $t = 1$.

## V. Discussion

Initial boundary nonlinear partial differential equations typically lack an exact solution. The ADM and VIM are well-known and widely studied tools that are useful for obtaining an exact solution or, when this is not possible, a highly accurate numerical solution in the neighborhood of the initial condition.Numerous authors have compared the two approaches in their studies, and in most cases, the same solutionisachieved.As described in the literature, when some of the conditions for the initial boundary problem are set to zero and the integration for the ADM and VIM starts at that point, the two methods behave in a very similar way, and both are powerful in achieving an exact solution near that point.

Some of this study's findings confirm what is already known about the ADM and VIM, specifically that both approaches can lead to the same result and that the VIM typically requires less computing power than the ADM because each iteration of the approach eliminates the convenient "small terms" and it does not involve the evaluation of Adomian polynomials.

For the first and second examples discussed in this paper, the two methods gave the same results as expected, with an advantage for the VIM in computational terms. For the third example, the VIM could not be so useful, because the integration must startat a non-zero point (the zero point is a singularity). Although in the literatureVIM is usually applied with the integration startingat$t = 0$, with the help of this example, one shows that VIM can still be practical and computationally efficienteven when the initial condition is a non-zero point, and that the solution can be achieved with the Taylor expansion at each iteration around the initial condition and with the elimination of the convenient "small terms". This way the VIM gets quicker and more effective.

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Teresa Abreu, et. al. "Adomian's Decomposition Series and Variational Iteration Method on Burger's Equation." *IOSR Journal of Mathematics (IOSR-JM)*, 18(4), (2022): pp. 23-32.