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Modeling of Default Risk by Jump Diffusion Process

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Abstract: *Jump diffusion is a stochastic process that involves jumps and diffusion. A jump process is a type of stochastic process that has discrete movements, called jumps, with random arrival times, rather than continuous movement, typically modeled as a simple or compound Poisson process. In this paper we use jump diffusion process to model default risk and compare results of the traditional Merton and Moody's Kealhofer, McQuown, and Vasicek (MKMV) models. Results show that, jump diffusion models perform better than both the traditional Merton and MKMV models.*
Key words: *Default Risk, Option Pricing, Jump Diffusion Model, Jump Process, Poisson Process.*

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I. Introduction

A diffusion process is a continuous-time Markov process with almost surely continuous sample paths (Kou and Wang[6]). Jump diffusion is a stochastic process that involves jumps and diffusion. A jump process is a type of stochastic process that has discrete movements, called jumps, with random arrival times, rather than continuous movement, typically modeled as a simple or compound Poisson process[8]. In option pricing, a jump-diffusion model is a form of mixture model, mixing a jump process and a diffusion process. Jump-diffusion models have been introduced by Robert C. Merton in 1976 as an extension of jump models. Due to their computational tractability, the special case of a basic affine jump diffusion is popular for credit risk and short-rate models[7].

II. The Diffusion process of a Stock price

The stock is a European call option and it follows a geometric Brownian motion throughout the life of the option ($T - t$). We assume that the stock price S, pays annual dividend q and has an expected return μ equal to the risk free rate $r - q$ and the constant volatility σ. Since stock prices do exhibit randomness, the governing equation that captures the randomness in stock markets is given by[1]:

$$dS = S\mu dt + S\sigma dW(t) \tag{1}$$

where W_t is a Wiener Process and the equation (1) above is in the form of an Ito process. Now, using Ito's Lemma, which states that, if a random variable follows an Ito Process, then another twice differentiable function G described by the stock price S and time t also follows an Ito process[5]:

$$dG = \left(\frac{\partial G}{\partial S} S\mu + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2} S^2\sigma^2 \right) dt + \frac{\partial G}{\partial S} S\sigma dW(t) \tag{2}$$

Using the lognormal property, we let $G = \ln S$ to ensure that the stock price is strictly greater than 0. Applying Ito's Lemma to $\ln S$ and calculate the partial derivatives with respect to S and t, we get:

$$G = \ln S$$

$$\frac{\partial G}{\partial S} = \frac{1}{S}, \quad \frac{\partial G}{\partial t} = 0, \quad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$$

Plugging the partial derivatives into Ito's Lemma gives:

$$dG = \left(\frac{1}{S} S \mu + 0 - \frac{1}{2}\frac{1}{S^2} S^2 \sigma^2\right) dt + \frac{1}{S} S \sigma \, dW(t)$$

$$= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma \, dW(t)$$

(3)

Therefore the distribution of $\ln S_T - \ln S_0 = \left(\mu - \frac{\sigma^2}{2}\right) T + \sigma \sqrt{T}$

(4)

Rearranging equation (4) and taking the exponential on both sides, we obtain the distribution of the stock price at expiration:

$$S_T = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma \, dW(t)}$$

(5)

Which can also be written as:

$$\ln S_T = \ln S_0 + \int_0^t \left(\mu - \frac{\sigma^2}{2}\right) dt + \int_0^t \sigma \, dW(t), \qquad for \ t \in [0, \ldots, T]$$

(6)

III. The Jump Diffusion process of a Stock Price

The jump diffusion process is a result of the work by Merton (1976). Merton suggested a model where jumps are combined with continuous changes. Merton extended the Black-Scholes model to incorporate more realistic assumptions and that deal with the fact that empirical studies of market returns, do not follow a constant variance log-normal distribution[7]. Define:

S = Current Stock Price,
K = Strike Price,
T = Time to maturity in years,
σ = Annual volatility,
m = Mean of jump size, measured as a percentage of the asset price,
v = Standard deviation of jump size,
λ = Mean number of jumps per year (intensity),
$dW(t)$ = Wiener Process,
$N(t)$ = Compound poison process,
V_{BS} = Value of option using Black-Scholes Formula,
V_{MJD} = Value of option using Merton Jump Diffusion Model.

The percentage jump size is assumed to be drawn from a probability distribution in the model. The probability of a jump in time Δt is $\lambda \Delta t$. The average growth rate in the asset price from the jumps is therefore λk . The risk-neutral process for the asset price is given by[9]:

$$\frac{dS}{S} = (\mu - \lambda k)\,dt + \sigma\,dW(t) + d\left(\sum_{i=1}^{N_t}(Q_i - 1)\right)$$

(7)

where μ is the instantaneous expected return per unit time, $W(t)$ is a standard Brownian motion, $N(t)$ is a Poisson process with rate λ, and $\{Q_i\}$ is a sequence of independent and identically distributed (i.i.d) nonnegative random variables such that $\gamma = \log(Q)$ has a normal distribution denoted as $N(\mu_j, \sigma_j^2)$ with the density function[3]:

$$f_\gamma(y) = \frac{1}{\sqrt{2\pi}\sigma_j} e^{-\frac{(y-\mu_j)^2}{2\sigma_j^2}}$$

All the sources of randomness and uncertainty, $N(t), W(t)$ and γ s are assumed to be independent. Solving (7) we obtain[7]:

$$\ln S_T = \ln S_0 + \int_0^t \left(\mu - \frac{\sigma^2}{2} - \lambda\left(m + \frac{v^2}{2}\right)\right)dt + \int_0^t \sigma\,dW(t) + \sum_{j=1}^{N_t}(Q_j - 1)$$

(8)

$$S_T = S(0)\,e^{\left(\mu - \frac{\sigma^2}{2} - \lambda\left(m + \frac{v^2}{2}\right)\right)t + \sigma W(t)} \prod_{i=1}^{N(t)} Q_i$$

(9)

where $N(t)$ is a Poisson Process with rate λ and probability k jumps occurring over the life of the option equal to[8]:

$$P_k(\lambda t) = P(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \text{ for all } k = 0,1,\ldots$$

(10)

and Q_j is a log-normally distributed random variable.

1. Simulating the Jumps

We use Monte Carlo simulation method for to simulate jumps. When jumps are generated by a Poisson process, the probability of exactly k jumps in time t is given by[4]:

$$P_k(\lambda t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Where λ is the average number of jumps per year. Equivalently, λt is the average number of jumps in time t. Suppose that on average 0.5 jumps happen per year. The probability of k jumps in 2 years is:

$$\frac{e^{-0.5\times 2}(0.5\times 2)^k}{k!}$$

To simulate this process following jumps over 2 years, we need to determine on each simulation trial:
i. The number of jumps
ii. The size of each jump

Table 1 below gives the probability and cumulative probability of $0, 1, 2, 3, 4, 5, 6, 7$ and jumps in 2 years. The probabilities have been calculated using python.

Table 1. Probabilities for number of jumps in 2 years

Number of jumps, k	Probability of exactly k jumps	Probability of k jumps or less
0	0.3679	0.3679
1	0.3679	0.7358
2	0.1839	0.9197
3	0.0613	0.9810
4	0.0153	0.9963
5	0.0031	0.9994
6	0.0005	0.9999
7	0.0001	1.0000
8	0.0000	1.0000

Source: (Hull 2003)

To determine the number of jumps, on each simulation trial we sample a random number between 0 and 1 and use Table 1 as a look-up table. If the random number is between 0 and 0.3679, no jumps occur; if the random number is between 0.3679 and 0.7358, one jump occurs; if the random number is between 0.7358 and 0.9197, two jumps occur; and so on. To determine the size of each jump, it is necessary on each simulation trial to sample from the probability distribution for the jump size once for each jump that occurs. Once the number of jumps and the jump sizes have been determined, the final value of the variable being simulated is known for the simulation trial.
An important particular case of Merton's model is where the logarithm of one plus the size of the percentage jump is normal ($Y = \ln(1+k)$). We write the jump component as a normal random variable and the resulting payoffs will be risk neutral. Merton shows that, the solution to a European price option that follows a jump diffusion process is given by[2]:

$$V_{MJD}(S, K, T, \mu, \sigma, m, v, \lambda) = \sum_{k=0}^{\infty} \frac{\exp(-m\lambda T)(m\lambda T)^k}{k!} V_{BS}(S, K, T, r_k, \sigma_k)$$

(11)

The variable V_{BS} is the Black-Scholes option price when the dividend yield is q. The volatility (σ_k) and the risk free rate (r_k) from equation (11), conditional on k jumps occurring is given by:

$$\sigma_k = \sqrt{\sigma^2 + \frac{kv^2}{T}}$$

(12)
and

$$r_k = \mu - \lambda(m-1) + \frac{k \ln m}{T}$$

(13)

2. *Calculation of Distances to Default (DD) under jump process*

The Distance to Default (*DD*) is the number of standard deviations between the expected asset value at maturity T and the debt threshold K. *DD* is the basis of credit evaluation. It is a standard index reflecting the company's credit quality, which can be compared for different companies and for different periods of time. The greater the value of *DD*, the more likely the company is to repay debts in due time, as a consequence the defaults will be less and the credit will be better. The DD scaled by asset volatility reflects how far a firm's asset value is from the value of obligations that would trigger a default[3].

The DD for the jump process is given by:

$$DD = \frac{\log\left(S/K\right) + \left(r_k - \sigma_k^2/2\right)T}{\sigma_k\sqrt{T}}$$

(14)

where σ_k is the volatility on k jumps and r_k is the risk-free rate on k jumps.

The probability of default (PD) defined as the probability of the asset value falling below the debt threshold at the end of the time horizon T is given by:

$$PD = 1 - N\left(DD\right) \quad (15)$$

3. *Estimation of DD and PD from Federal Reserve Economic Data by Jump diffusion model*

Table 2 shows the data on short term liabilities (STL), long term liabilities (LTL), and total asset values recorded from Federal Reserve Economic Data. Time *(T)* is the time in years where these data were recorded. We have taken a period of ten years from 2011/10/01 to 2020/10/01.

Table 2. Short and long term liabilities, average debts and total asset values

Time (T)	2011/10/01	2012/10/01	2013/10/01	2014/10/01	2015/10/01	2016/10/01	2017/10/01	2018/10/01	2019/10/01	2020/10/01
STL	3810	3829	3813	4177	5900	4336	3705	3585	4775	6003
LTL	16487	16947	19431	22299	30692	32037	29130	29690	28792	29921
Asset(S)	173063	171211	191450	205093	203037	198507	201953	211339	228884	253764

Source (Federal Reserve Economic Data, https://fred.stlouisfed.org, https://fredhelp.stlouisfed.org)

Table 3 shows the distances to default *(DD)* and probabilities of default *(PD)* calculated from Table 2 using mean jump size, $m = 1$. Averege asset value *S* and debt (liabilities) are used to calculate the Distance to Default *(DD)* in equation (14). *DD* is used to calculate the probability of default *PD* given by equation (15).

Table 3 DD and PD from Table 1 by Jump diffusion process, $m = 1$ (mean jump size)

Time (*T*)	1	2	3	4	5	6	7	8	9	10
DD_{STL}	1.1051	1.0989	1.0929	1.0869	1.0810	1.0752	1.0695	1.0640	1.0584	1.0530
PD_{STL}	0.1345	0.1359	0.1372	0.1385	0.1398	0.1411	0.1424	0.1437	0.1449	0.1462
DD_{LTL}	0.1709	0.1699	0.1690	0.1680	0.1671	0.1662	0.1654	0.1645	0.1637	0.1628
PD_{LTL}	0.4322	0.4325	0.4329	0.4332	0.4336	0.4340	0.4343	0.4346	0.4350	0.4353

Table 4 shows the distances to default *(DD)* and probabilities of default *(PD)* calculated from Table 2 using mean jump size, $m = 2$.

Table 4 DD and PD from Table 1 by Jump diffusion process, $m = 2$ (mean jump size)

Time (*T*)	1	2	3	4	5	6	7	8	9	10
DD_{STL}	14.9219	14.3101	13.7068	13.1099	12.5198	11.9365	11.3597	10.7892	10.2250	9.6668
PD_{STL}	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
DD_{LTL}	13.9876	13.3816	12.7829	12.1910	11.6059	11.0275	10.4555	9.8898	9.3302	8.7766
PD_{LTL}	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 5 shows the distances to default *(DD)* and probabilities of default *(PD)* calculated from Table 2 using mean jump size, $m = 3$.

Table 5 DD and PD from Table 1 by Jump diffusion process, $m = 3$ (mean jump size)

Time (T)	1	2	3	4	5	6	7	8	9	10
DD_{STL}	22.7839	21.6010	20.4319	19.2763	18.1338	17.0040	15.8867	14.7816	13.6883	12.6067
PD_{STL}	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
DD_{LTL}	21.8496	20.6720	19.5080	18.3574	17.2199	16.0950	14.9825	13.8821	12.7935	11.7165
PD_{LTL}	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 6 shows the comparison of distances to default (*DD)* calculated by the jump process, Merton and *MKMV* approaches.

Table 6. Comparison of *DDs* by Jump process, Merton and *MKMV* approaches

Time(T)	1	2	3	4	5	6	7	8	9	10
$JDD_{STL}(m = 1)$	1.105 1	1.0989	1.0929	1.0869	1.0810	1.0752	1.0695	1.0640	1.0584	1.0530
$JDD_{LTL}(m = 1)$	0.170 9	0.1699	0.1690	0.1680	0.1671	0.1662	0.1654	0.1645	0.1637	0.1628
$JDD_{STL}(m = 2)$	14.92 19	14.3101	13.7068	13.1099	12.5198	11.9365	11.3597	10.7892	10.2250	9.6668
$JDD_{LTL}(m = 2)$	13.98 76	13.3816	12.7829	12.1910	11.6059	11.0275	10.4555	9.8898	9.3302	8.7766
$JDD_{STL}(m = 3)$	22.78 39	21.6010	20.4319	19.2763	18.1338	17.0040	15.8867	14.7816	13.6883	12.6067
$JDD_{LTL}(m = 3)$	21.84 96	20.6720	19.5080	18.3574	17.2199	16.0950	14.9825	13.8821	12.7935	11.7165
DD_{STL} (Merton)	19.18 60	13.5666	11.0771	9.5930	8.5803	7.8327	7.2516	6.7833	6.3953	6.0672
DD_{LTL} (Merton)	10.38 47	7.3431	5.9956	5.1923	4.6442	4.2395	3.9250	3.6715	3.4616	3.2839
$DD_{k=0.3}$ (MKMV)	14.13 86	9.9975	8.1629	7.0693	6.3230	5.7720	5.3439	4.9987	4.7129	4.4710

Table 7 shows the comparison of Probabilities of (*PD)* calculated by the jump process, Merton and *MKMV* approaches.

Table 7. Comparison of *PDs* by Jump process, Merton and *EDFs* by *MKMV* approaches

Time(T)	1	2	3	4	5	6	7	8	9	10
$JPD_{STL}(m = 1)$	0.134 5	0.1359	0.1372	0.1385	0.1398	0.1411	0.1424	0.1437	0.1449	0.1462
$JPD_{LTL}(m = 1)$	0.432 2	0.4325	0.4329	0.4332	0.4336	0.4340	0.4343	0.4346	0.4350	0.4353
$JPD_{STL}(m = 2)$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$JPD_{LTL}(m = 2)$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$JPD_{STL}(m = 3)$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$JPD_{LTL}(m = 3)$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
PD_{STL} (Merton)	0.0	0.0	0.0	0.0	0.0	2.4e-15	2.0e-13	5.9e-12	8.0e-11	6.5e-10

PD_{LTL} (Merton)	0.0	1.0e-13	1.0e-09	1.0e-07	1.7e-06	1.1e-05	4.3e-05	0.0001	0.0003	0.0005
$EDF_{k=0.3}$ (MKMV)	0.0	0.0	1.1e-16	7.8e-13	1.3e-10	3.9e-09	4.5e-08	2.9e-07	1.2e-06	3.9e-06

IV. Discussion of Results

The data for calculation of *DDs*, *PDs* and *EDFs* was collected from Federal Reserve Economic Data. From Table 1, *DDs* and *PDs* were calculated using the mean jump size, $m = 1$. The maximum value for DD_{STL} **is** 1.1051 and PD_{STL} is 0.1345. While the maximum value for DD_{LTL} is 0.1709 and PD_{LTL} is 0.4322. From Table 2, *DDs* and *PDs* were calculated using the mean jump size, $m = 2$. The maximum value for DD_{STL} **is** 14.9219 and PD_{STL} is 0.0. While the maximum value for DD_{LTL} is 13.9876 and PD_{LTL} is 0.0. From Table 3, *DDs* and *PDs* were calculated using the mean jump size, $m = 3$. The maximum value for DD_{STL} **is** 22.7839and PD_{STL} is 0.0. While the maximum value for DD_{LTL} is 21.8496and PD_{LTL} is 0.0. In each case, STL stands for short term liabilities and LTL stands for long term liabilities. We see from the three tables, improved results were obtained as the number of mean jump size raised from 1. From table one we have small values of *DDs* with greater probabilities of default. While from table 2 and 3, we see greater values of *DDs* and smaller value of *PDs*. This finding indicate that, firm's will be tronger to default with mean jump size, $m > 1$.

V. Comparison of jump process with Merton and MKMV approaches.

From Table 6, we see that the *DDs* generated by the jump process are larger than both generated by the traditional Merton and *MKMV* approaches. Also **Table 7** shows smaller values to *PDs* generated by the jump process compared to those generated by the Merton and *MKMV* approches. This finding indicate that the jump diffusion process performs better to credit (default) risk estimation compared to the traditional Merton and *MKMV* approaches.

VI. Conclusion and Suggestion for Future Research

In this paper we have used the jump diffusion process to model credit risk (default risk). We have used the data from Federal Reserve Economic recorded from 2011/10/01 to 2020/10/01. We have calculated the distances to defaults (*DDs*) using both the short term and long term liabilities. In each case, the short term liabilities have provided stable results compared to long term liabilities. Firms looks more stable to default using short term liabilities than the long term, except only with *MKMV* approach which has a special case of combining both the short and long term liabilities at the same setting when calculating the *DDs* and Expected default frequencies (*EDFs*). Then we compared our results to the famous two models of credit risk, the traditional Merton and *MKMV*. Results show that, jump diffusion process perform better than both the traditional Merton and *MKMV* models. In future we will extend the jump diffusion process to other stock markets like banks and financial institutions.

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