IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 18, Issue 3 Ser. II (May. – June. 2022), PP 01-05
www.iosrjournals.org

Strongly 2-Nil Clean Fuzzy Rings

Muhammad Reza Maulana*, Noor Hidayat, Abdul Rouf Alghofari
Department of Mathematics, University of Brawijaya, Indonesia

Abstract:
Based on the definition of a strongly 2-nil clean ring and the concept of fuzzy subring, we introduce a new structure that is a strongly 2-nil clean fuzzy ring. Some of characteristics of a strongly 2-nil clean fuzzy ring are as follows: (i) If R is a strongly 2-nil clean fuzzy ring then μ(1-x)=μ((p-q)+g) for two idempotent elements p,q and a nilpotent element g; (ii) μ(x-x^3)=μ(z) for a nilpotent element z; (iii) μ(1-x-t)=μ(g) for a tripotent element t and a nilpotent element g; (iv) μ_t is a subring on R for t∈[0,μ(0)].
Key Word*: Ring; Strongly 2-nil clean ring; Fuzzy subring.*

Date of Submission: 03-06-2022 Date of Acceptance: 17-06-2022

I. Introduction

An element of a ring with a unit element is said to be clean if it can be written as the sum of an idempotent element and a unit element. A ring is called as a clean ring if every element of it is clean [6]. If R is a clean ring and commutative then R is called as a strongly clean ring [7]. If each element of a ring R can be written as the sum of idempotent and nilpotent elements, then R is called as a nil clean rings [2]. Furthermore, if a nil clean ring R is commutative then it is called a strongly nil clean ring [3]. The structure (strongly) nil clean rings has attracted a lot of attention. For example, Ying et al. [9] have discussed about a strong sum of idempotent and tripotent that commute. Chen and Sheibani [1] have introduced the strongly 2-nil clean rings including their related concepts. A ring is said to be strongly 2-nil clean ring if each element of the ring can be written as the sum of two idempotent elements and nilpotent elements that commute [1]. One of important properties is that R is strongly 2-nil clean rings if and only if every $a \in R$, then there are two idempotents elements $e, f \in R$ and a nilpotent element $w \in R$ that commute such that $a = e - f + w$.

In 1965, Zadeh [10] developed the concept of fuzzy sets. A fuzzy set on a set X is defined as a mapping from domain X into codomain [0,1]. The theory of fuzzy sets has evolved in many directions since its inception, and it now has applications in a wide range of domains. This concept was exploited by Rosenfeld [8] to construct the fuzzy subgroup theory. In 1982, Liu [4] proposed the fuzzy ring concept.

In this paper, we combine the concept of strongly 2-nil clean and fuzzy rings to develop a new structure, namely strongly 2-nil clean fuzzy rings. Some properties of strongly 2-nil clean fuzzy rings will also be derived.

II. Preliminaries

Before discussing the main results, we present some important properties of strongly 2-nil clean rings and fuzzy rings. The proof can be found in the given references.

Strongly 2-nil clean rings

Lemma 1. [9] Let $a \in R$. If $a^2 - a$ is nilpotent, then there exists a monic polynomial $\theta(t) \in \mathbb{Z}[t]$ such that $\theta(a)^2 = \theta(a)$ and $a - \theta(a)$ is nilpotent.

Lemma 2. [1] Let R be a ring. Then the following statement are equivalent:
1) R is strongly 2-nil clean
2) For any $a \in R$, there exist two idempotents $c, d \in R$ and nilpotent $w \in R$ that commute such that $a = c - d + w$.

Theorem 1. [1] Let R be a ring. Then the following are equivalent:
1) R is strongly 2-nil clean
2) For all $a \in R, a - a^3 \in N(R)$
3) For all $a \in R, a^2 \in R$ is strongly 2-nil clean.

Theorem 2. [1] Let R be a ring. Then the following are equivalent

1) R is strongly 2-nil clean
2) For any $a \in R$, there exists a tripotent $c \in R$ such that $a - c \in R$ is nilpotent and $ca = ac$.

Theorem 3. [1] A ring R is strongly nil clean if and only if

1) $2 \in R$ is nilpotent
2) R is strongly 2-nil clean.

Fuzzy Subring

In the following part, we provide some concepts of fuzzy subring.

Theorem 4. [5] Let μ be a fuzzy subset of R. Then μ is a fuzzy subring of R if and only if μ_t is a subring of R, for each $t \in [0, \mu(0)]$.

Theorem 5. [5] Let R be a ring with identity, and let μ be a fuzzy subset of R. If μ is a fuzzy subring of R, then $F(R)$ a ring.

Proposition 1. [5] Let R be a commutative ring. Let μ and v to be two fuzzy subrings of R such that $\mu \subset v$. Then $F_\mu(R)$ is a subring of $F_v(R)$.

III. Main Results

Definition 1. Let R be a ring with identity element 1, $Id(R)$ is the idempotent element set in R and $U(R)$ is the unit element set of R. A fuzzy ring μ of R is clean fuzzy rings if any $x, y \in R$ satisfies

a) $\mu(x) \geq \min\{\mu(p), \mu(q)\}$ for each $p \in Id(R), q \in U(R)$.
b) $\mu(x + y) \geq \min\{\mu(p_1 + q_1), \mu(p_2 + q_2)\}$ for each $p_1, p_2 \in Id(R), q_1, q_2 \in U(R)$.
c) $\mu(-x) = \mu\big(-(p + q)\big) = \mu(p + q)$ for each $p \in Id(R), q \in U(R)$.
d) $\mu(xy) \geq \min\big(\mu(p_1 + p_2) \cdot \mu(q_1 + q_2)\big)$ for each $p_1, p_2 \in Id(R), q_1, q_2 \in U(R)$.

Lemma 3. If R is a clean ring with identity element 1, $Id(R)$ is the idempotent element set of R and $U(R)$ is the unit element set of R, then fuzzy ring μ of R is a clean fuzzy rings of R.

Proof. Let $x, y \in R$. Since R is a clean ring, then there are elements $p_1, p_2 \in Id(R), q_1, q_2 \in U(R)$, such that $x = p_1 + q_1, y = p_2 + q_2$, and thus

a) $\mu(x) = \mu(p_1 + q_1) \geq \min\{\mu(p_1), \mu(q_1)\}$.
b) $\mu(xy) = \mu(p_1 + q_1)(p_2 + q_2) \geq \min\{\mu(p_1 + q_1), \mu(p_2 + q_2)\} = \min\{\mu(x), \mu(y)\}$.
c) $\mu(-x) = \mu\big(-(p_1 + q_1)\big) = \mu(p_1 + q_1) = \mu(x)$.

■

Definition 2. Let R be a ring, $Id(R)$ is the idempotent element set in R and $N(R)$ is the nilpotent element set in R. A fuzzy ring μ in R is a nil clean fuzzy ring if each $x, y \in R$ satisfies

a) $\mu(x) \geq \min\{\mu(p), \mu(q)\}$ for each $p \in Id(R), q \in N(R)$.
b) $\mu(x + y) \geq \min\{\mu(p_1 + q_1), \mu(p_2 + q_2)\}$ for each $p_1, p_2 \in Id(R), q_1, q_2 \in N(R)$.
c) $\mu(-x) = \mu\big(-(p + q)\big) = \mu(p + q)$ for each $p \in Id(R), q \in N(R)$.
d) $\mu(xy) \geq \min\{\mu(p + q) \cdot \mu(r + s)\}$ for each $p, r \in Id(R), q, s \in N(R)$.

Lemma 4. If R is a nil clean ring with idempotent element set $Id(R)$ and nilpotent element set $N(R)$, then fuzzy ring μ in R is a nil clean fuzzy ring in R.

Proof. Let $x, y \in R$. Because R is a nil clean ring, then there exist $p_1, p_2 \in Id(R), q_1, q_2 \in N(R)$, such that $x = p_1 + q_1, y = p_2 + q_2$. Hence, we obtain

a) $\mu(x) = \mu(p_1 + q_1) \geq \min\{\mu(p_1), \mu(q_1)\}$
b) $\mu(xy) = \mu\big((p_1 + q_1)(p_2 + q_2)\big) \geq \min\{\mu(p_1 + q_1), \mu(p_2 + q_2)\} = \min\{\mu(x), \mu(y)\}$
c) $\mu(-x) = \mu\big(-(p_1 + q_1)\big) = \mu(p_1 + q_1) = \mu(x)$.

■

Proposition 2. Let R be a ring with identity element 1, $Id(R)$ is the idempotent element set in R, $N(R)$ is the nilpotent element set in R, and $U(R)$ is the unit element set in R. If μ is a nil clean fuzzy ring on R, then μ is a clean fuzzy ring of R.
Proof. Let $x \in R$. Since μ is a clean fuzzy ring then $\mu(x) = \mu(p_1 + q_1) \geq \min\{\mu(p_1), \mu(q_1)\}$ for any $p_1 \in Id(R), q_1 \in N(R)$. Since $q_1 \in N(R)$, there is $n \in \mathbb{Z}^+$so that ${q_1}^n = 0$.
■

Definition 3. Let R be a ring and its idempotent element set and nilpotent element set are respectively denoted by $Id(R)$ and $N(R)$. A fuzzy ring μ in R is called a strongly nil clean fuzzy ring if any $x, y \in R$ satisfies

a) $\mu(x) \geq \min\{\mu(p), \mu(q)\}$ and $\mu(pq) = \mu(qp)$ for each $p \in Id(R), q \in N(R)$
b) $\mu(x + y) \geq \min\{\mu(p_1 + q_1), \mu(p_2 + q_2)\}$ and $\mu(p_1q_1) = \mu(q_1p_1), \mu(p_2, q_2) = \mu(q_2p_2)$ for each $p_1, p_2 \in Id(R), q_1, q_2 \in N(R)$.
c) $\mu(-x) = \mu(-(p + q)) = \mu(p + q) = \mu(pq) = \mu(qp)$ for each $p \in Id(R), q \in N(R)$.

Lemma 5. If R is a strongly nil clean ring, $Id(R)$ is the idempotent element set in R and $N(R)$ is the nilpotent element set in R, then a fuzzy ring μ in R is a strongly nil clean fuzzy ring on R.
Proof. Since R is a strongly nil clean ring, every $x \in R$ can be expressed in terms $x = a + b$, with $a \in Id(R)$ and $b \in N(R)$ that commute. A fuzzy ring μ on R must satisfy

a) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}\, \forall x, y \in R$
b) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}\, \forall x, y \in R$
c) $\mu(1) = 1$,

and $x, y \in R$ can be expressed in terms of $x = a + b$ and $y = c + d$ for $a, c \in Id(R)$ and $b, d \in N(R)$. Thus, a fuzzy ring μ in R is a strongly nil clean fuzzy ring.
■

Definition 4. Let R be a ring, $Id(R)$ is the idempotent element set in R, $N(R)$ is the nilpotent element set in R. A fuzzy ring on R is called a strongly 2-nil clean fuzzy ring if each $x, y \in R$ satisfies:

a) $\mu(x) \geq \min\{\mu(p), \mu(q), \mu(g)\}$ for each $p, q \in Id(R), g \in N(R)$.
b) $\mu(x + y) \geq \min\{\mu((p + q) + g), \mu((r + s) + h)\}$ for each $p, q, g \in Id(R), r, s, h \in N(R)$.
c) $\mu(-x) = \mu\left(-((p + q) + g)\right) = \mu((p + q) + g) = \mu(x)$ for each $p, q \in Id(R), g \in N(R)$
d) $\mu(xy) \geq \min\{\mu((p + q) + g) \cdot \mu((r + s) + h)\}$ for each $p, q, r, s \in Id(R), g, h \in N(R)$.

Lemma 6. If R is a strongly 2-nil clean fuzzy ring with identity element and $Id(R)$ is the idempotent element set in R. If μ is a fuzzy subring, then μ is a strongly 2-nil clean fuzzy ring.
Proof. Let $x, y \in R$. Then it can be shown that

a) $\mu(x) = \mu((p + q) + g) \geq \min\{\mu(p), \mu(q), \mu(g)\} \in R$
b) $\mu(x + y) = \mu[((p + q) + g) + ((r + s) + h)] \geq \min\{\mu((p + q) + g), \mu((r + s) + h)\}$
c) $\mu(-x) = \mu\left(-((p + q) + g)\right) = \mu((p + q) + g) = \mu(x), \forall x \in R$
d) $\mu(xy) = \mu[((p + q) + g)((r + s) + h)] \geq \min\{\mu((p + q) + g), \mu((r + s) + h)\}$.
■

Lemma 7. If R is a strongly 2-nil clean fuzzy ring then for each $x \in R$, it holds that
$$\mu(1 - x) = \mu((p - q) + g)$$
for idempotent $p, q \in R$ and nilpotent element $g \in R$.

Proof. Let $x \in R$. Because $x \in R$, x can be expressed in the form $x = s + q + g$ for idempotent elements s, q and nilpotent element g. Next we show that if s is an idempotent element, then $1 - s$ is also an idempotent element:
$$\begin{aligned}(1 - s)^2 &= (1 - s)(1 - s)\\ &= 1 - 2s + s^2\\ &= 1 - 2s + s\end{aligned}$$

$$= 1 - s.$$

Let $1 - s = p$, $q \in R$ is an idempotent element and $g \in R$ is a nilpotent element. Then we have that

$$x = (s + q) + g$$
$$1 - x = 1 - \left((s + q) + g\right)$$
$$1 - x = (1 - s) - q - g$$
$$1 - x = p - q - g$$
$$1 - x = p - q + g$$
$$\mu(1 - x) = \mu\left((p - q) + g\right).$$

■

Theorem 6. If R is a strongly 2-nil clean fuzzy ring, then for each $x \in R$, it holds that $\mu(x - x^3) = \mu(z)$, where $z \in N(R)$.

Proof. Let $x \in R$. Since R is a strongly 2-nil clean fuzzy ring, Lemma 3 says that there exist two idempotent elements p, q and nilpotent elements g that commute. Furthermore, we also have that $\mu(1 - x) = \mu\left((p - q) + g\right)$. Let $y=p-q$, then we have $\mu 1-x=\mu y+g$. By noting $pq=qp$, it holds that

$$\begin{aligned} y^3 &= (p - q)^3 \\ &= (p - q)^2(p - q) \\ &= (p - 2pq + q)(p - q) \\ &= p^2 - pq - 2p^2q + pq - q^2 \\ &= p - pq - 2pq + 2pq + pq - q \\ &= p - q \\ &= y. \end{aligned}$$

So, $\mu(x - x^3) = \mu[(y + g) - (y + g)^3]$, where $(y + g) - (y + g)^3 = z \in N(R)$.

■

Theorem 7. If R is a strongly 2-nil clean fuzzy ring then for each $x \in R$, there exists tripotent $t \in R$ such that $\mu(1 - x - t) = \mu(g)$ with $g \in N(R)$.

Proof. Let $x \in R$ with $p, q \in Id(R)$ and $g \in N(R)$. Due to R is a strongly 2-nil clean fuzzy ring, we have $\mu(1 - x) = \mu\left((p - q) + g\right)$. For $t = p - q$, it can be shown that

$$\begin{aligned} t^3 &= (p - q)^3 \\ &= (p - q)(p^2 - 2pq + q^2) \\ &= (p - q)(p - 2pq + q^2) \\ &= p^2 - 2p^2q + pq - pq + 2pq^2 - q^2 \\ &= p^2 - 2pq + pq - pq + 2pq - q^2 \\ &= p - q \\ &= t. \end{aligned}$$

So, t is a tripotent and we can show that $\mu(1 - x) = \mu\left((p - q) + g\right) = \mu(t - g)$. Furthermore, we also have $\mu(1 - x - t) = \mu(t + g - t) = \mu(g)$, where $g \in N(R)$.

■

Theorem 8. If is a R strongly 2-nil clean fuzzy ring with membership function μ, then μ_t is a subring on R for each $t \in [0, \mu(0)]$.

Proof. $\mu_t = \{x \in R, \mu(x) \geq t\}$ is not an empty set on R because there is $t = 0$ so that $\mu(0) \geq 0$. Let $x, y \in \mu_t$, then x, y can be expressed in term of the sum of two idempotent elements and a nilpotent element and $\mu(x) \geq t$ and $\mu(y) \geq t$. Since μ is a fuzzy subring on R, we get $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$. It leads to

a) $\mu(x - y) \geq t$, that $x - y \in \mu_t$

b) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ that $\mu(xy) \geq t$ and $xy \in \mu_t$

So μ_t is a subring on R.

■

References

[1]. Chen H, Sheibani M. Strongly 2-nil clean rings. Journal of Algebra and Its Applications. 2017;16(9): 1750178.

[2]. Diesl AJ. Nil clean rings. Journal of Algebra. 2013;383: 197-211.

[3]. Koşan T, Wang Z, Zhou Y. Nil-clean and strongly nil-clean rings. Journal of Pure and Applied Algebra. 2016;220(2):633-646.

[4]. Liu WJ. Fuzzy invariant subgroups and fuzzy ideals. Fuzzy Sets and Systems. 1981;8:133-139.

[5]. Melliani S, Backhadach I, Chadli LS. Fuzzy rings and fuzzy polynomial rings. Springer Proceedings in Mathematics & Statistics. 2018;228: 89-98.

[6]. Nicholson WK. Lifting idempotents and exchange rings. Transactions of the American Mathematical Society. 1977;229:269-278.

[7]. Nicholson WK. Strongly clean rings and fitting's lemma. Communications in Algebra. 1999;27(8): 3583-3592.

[8]. Rosenfeld A. Fuzzy groups. Journal of Mathematical Analysis and Applications. 1971;35(3): 512-517.

[9]. Ying Z, Koşan T, Zhou Y. Rings in which every element is sum of two tripotents, Canadian Mathematical Bulletin. 2016;59(3): 661-672.

[10]. Zadeh LA. Fuzzy Sets. Information and Control, 1965;8(3): 338-353.

Muhammad Reza Maulana, et. al. "Strongly 2-Nil Clean Fuzzy Rings." *IOSR Journal of Mathematics (IOSR-JM)*, 18(3), (2022): pp. 01-05.