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Mathematical Modelling of Dynamical Systems: Plotyn Graphical Presentation of Results

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Abstract: *Results are the expected outcome of every experimental or empirical research. Mathematically, these results could be extracted through analytical evaluation, numerical analysis or simulation. The presentation and interpretation of these results are the information conveyed to the end users for implementation. Consequently, the importance and relevance of result presentation and interpretation cannot be overemphasized. This paper studies the graphical presentation of results extracted from the analysis of dynamical model, involving either ordinary or delay differential equations, using MATLAB application and derived a novel plotyn code through the concept of plotyy and ploty4. The code is useful for plotting the results from the numerical analysis or simulation of dynamical systems of n-equations.*

***Key Word**: Mathematical modeling; Dynamical system; Epidemiology; Graphical results; Plotyy; Plotyn.*

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I. Introduction

Empirical and experimental studies over the years have shown that the outcome of every research is meant to be interpreted and implemented for the overall benefits of mankind. Similarly, mathematical models are formulated and evaluated with the sole intend of providing relevant solution(s) to emanating human and societal problems. According to the words of Benyah[1], "Mathematical models and computer simulations are useful experimental tools for building and testing theories, assessing quantitative conjectures, answering specific questions, determining sensitivities to changes in parameter values, and estimating key parameters from data. Understanding the transmission characteristics of infectious diseases in communities, regions, and countries can lead to better approaches to decreasing the transmission of these diseases".

Consequent upon the significant and the utmost importance attached to results generated from model analysis in research projects, it is very pertinent that such outcome is well captured for accurate interpretation and subsequent implementation of such results. A good presentation of result is very significant because every generated outcome is expected to convey the required information for effective communication and also to provide the driving force required to ignite the implementation processes. In epidemiology, this is needful in controlling epidemic/pandemic prevailing situation ravaging a community of people.

Epidemiological models are often derived by segmenting an entire population into distinct classes also known as compartments. The most common compartments are the susceptible (S), exposed (E), infected/infectious (I) and recovery/recovered (R) compartments. Everyone in the given population is considered susceptible, that is, they are all prone to infection during outbreak of infectious diseases if they interact with or come in contact with an infected individual (someone carrying the virus). For many infectious diseases, there exist a certain period wherein the susceptible individuals get infected but not infectious. During this period known as the incubation period, they are considered to be exposed but not infectious hence classified as exposed individuals[2,3,4]. But once infectious, they migrate into the infectious compartment. The recovery compartments house all those that were infected with the virus but recovered after treatment (or vaccination).

There are also some cases of maternally derived immunity against infectious diseases, for instance measles. It is noted that in such cases, babies are given birth to as susceptible individuals but since they exhibits immunity against the disease for a few period of months due to their protection from maternal antibodies (passed across the placenta or through colostrum), they are classified separately in the maternal immune compartment denoted by M[1,5]. In essence, they are considered as susceptible individuals with temporary maternal immunity.

The choice of compartments to be used in model formulation depends on the characteristics of the particular disease being modeled and the overall purpose of the model being derived[1]. The flow pattern between compartments in any epidemiological model determines the acronym used in describing the model and that also aids in generating the system of equations governing the dynamics of the model. For instance we have the *SI, SIS, SIR, SIRS, SEIR, SEIRS, MSIR, MSEIR,* etc.

II. Mathematical Models of Basic Dynamical Systems

It is conventional to develop or work with systems of differential equations in mathematical modelling. These systems of equations comprises of two or more equations respectively. The standard systems of equations are those generated from the fundamental compartmental models which include the *SI*, *SIR*, *SEIR*, *MSIR*. Further extensions are often generated using these fundamentals as the basis.

System of Three Differential Equations

The mathematical epidemiological models leading to systems of three equations are in most cases the basic SIR and SIRS models. The SIRS model indicates there is a likelihood that an individual could be re-infected after recovery from an initial infection. That is, the recovered individual is still considered susceptible to the virus or infection. Whereas the SIR model indicates that once infected, the recovered individual develops permanent immunity against the virus or disease. It is pertinent to note that these models are also applicable in other related fields of study when adhered to the guiding principles (see related cases in Agaba *et al.*, 2020[6]; Asseng *et al.*, 2014[7]; Blyuss *et al.*, 2020[8] and Erneux, 2009[9]. The models below give the basic system of three equations respective.

$$S' = -\beta SI \qquad\qquad S' = -\beta SI + \lambda R$$
$$I' = \beta SI - rI \qquad\qquad I' = \beta SI - rI$$
$$R' = rI \qquad\qquad R' = rI - \lambda R$$

SIR model　　　　　　*SIRS* model

where β is the disease transmission rate, r is the recovery rate and λ is the rate at which the recovered losses their immunity. The basic reproduction number in both cases is obtained as $R_0 = \beta / r$.

Dynamical Systems with Four Differential Equations

The scenario where disease exposure or the incubation period of the disease is considered while formulating the model, the resulting system of equations gives the *SEIR* or *SEIRS* models. Similarly, in the case of temporary maternal immunity, the model compartments are *MSIR* or *MSEIR* as deemed fit. The *M* compartment portrays a situation where few susceptible infants exhibit maternal immunity against the infection. Though they are often considered in separate compartment, in some cases they could be distinguished by certain parameter while classified within the susceptible compartment. Basic systems of equations associated with such models are therefore given as:

$$M' = p\delta N - \eta M \qquad\qquad S' = -\beta SI \qquad\qquad S' = -\beta SI + \lambda R$$
$$S' = q\delta N + \eta M - \beta SI \qquad\qquad E' = \beta SI - \alpha E \qquad\qquad E' = \beta SI - \alpha E$$
$$I' = \beta SI - rI \qquad\qquad I' = \alpha E - rI \qquad\qquad I' = \alpha E - rI$$
$$R' = rI \qquad\qquad R' = rI \qquad\qquad R' = rI - \lambda R$$

MSIR model　　　　　　*SEIR* model　　　　　　*SEIRS* model

with δ denoting the annual incremental rate, p is a fraction of the population that exhibits maternal immunity while q represents the remaining fraction which are susceptible (note that $p + q = 1$), η is the rate at which the new babies loss their maternal immunity. The parameter α denotes the exposure rate of the susceptible to the virus. The basic reproduction number is still obtained as $R_0 = \beta / r$.

System of Dynamical Models Extended from the Basis Compartmental Models

Systems of dynamical models could be extended in different perspectives as demonstrated in several research papers[10-17]. For instance, the model in Agaba *et al.*, 2017[10] generated system of five differential equations whereas in a different paper[11] a system of six differential equations were generated using the SIRS epidemic model. The model with six differential equations entails the segregation of the aware and unaware population. Consequently, the first three equations were used to describe the dynamics of the unaware population and the last three equations to represent the aware population respectively. A similar case is found in Agaba, 2014[2] and Zhang *et al.*, 2011[18] where the first set of equations were used to describe the dynamics of the spread of rabies among dogs and the second set of equations described the transmission of rabies from dogs to human population. Their models applied the principle of SEIRS model having system of eight differential equations. The discourse above simply indicate that each of these extended models often takes its bearing from among the basic epidemiological models with added perspective that carter for other aspects of the realistic context being modelled.

III. MATLAB Graphical Representation of Results

MATLAB is a software with several application packages which in most cases output results in graphical interface. Results in the conventional graphical interface are output as two dimensional (2-D) line plot or in form of the three dimensional (3-D) line plot (see Figure 1).

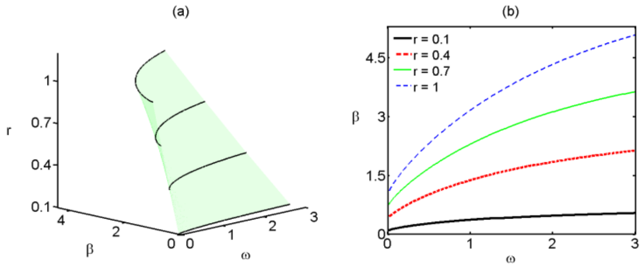

Figure 1: (a) A 3-D line plot and (b) 2-D projection respectively extracted from Agaba *et al*., 2017[11]

Plotting Standard Coordinates and the Plotyy Function

The 2-D line plot usually plots the columns of the y variables against the index of each value when y is a real number. The plot(x, y) in MATLAB, also referred to as the two coordinates plot, represent a plot of the vector y versus the vector x. While the 3-D analog of the 'plot' function is given as 'plot3'. The 3-D line plot involves a function that displays a three dimensional plot of a set of data points such as plot3(x, y, z) where x, y, z are vectors or matrices. If x, y and z are three vectors of the same length, plot3(x, y, z) generates a line in 3-D through the points whose coordinates are the elements of x, y and z and then produces a 2-D or 3-D projection of that line on the screen.

The plotyy of the MATLAB application uses the 2-D line plots having two y-axes plotted against the x-axis on both the left and the right side. The two y-axes help to display both sets of data on one graph even though the relative values of the data differ. For instance, the function plotyy(x_1, y_1, x_2, y_2) plots x_1 against y_1 with the y-axis labeling on the left and plots x_2 against y_2 with this y-axis labeling on the right. Similarly, it could be capture in the form [AX, H_1, H_2] = plotyy(x, y_1, x, y_2) which returns the handles of the two axes created in AX and the handles of the graphics objects from each plot in H_1 and H_2. AX(1) represents the left y-axis while AX(2) is the right y-axis generated from the plot. An example of the plotyy graph is the result generated and captured in Figure 3(a) for the dynamics of the susceptible and infected population extracted from the model in Agaba and Soomiyol, 2020[19].

The Extension of Plotyy in Generating Plotyyy and Ploty4

The MATLAB code and the general concept of the plotyy were used to generate extensions such as the plotyyy by Denis Gilbert, Ph.D and the ploty4 by Peter Bodin (for details see *https://www.mathworks.com/matlabcentral/fileexchange*) and their concepts of plotyyy and ploty4 were applied in obtaining the pictorial representation of results in some researches[6,10,12,19,20,21]. For instance, the concept of the plotyyy applied by Agaba and Soomiyol[19] generates the result captured in Figure 3(b).

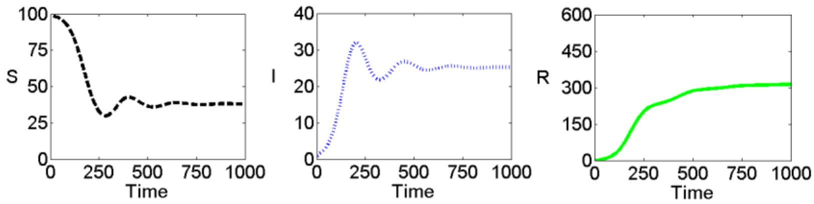

Figure 2: Plot showing the dynamics of the *SIR* model used in Agaba and Soomiyol, 2020[19] captured in a subplot. All parameter values remain unchanged with $\tau = 1$.

In modelling, compartmental models are often derived as models involving system of equations ranging from two and above. Consequently, considering the *SIR* model which comprises of three equations, plotting the dynamics of the system will require representing the classes of the population using three xy-plot interfaces either as separate figures or using the subplot command where these could be represented in a 1-by-3 subplot graph on a single figure as shown in Figure 2. However, this could be better represented on a single graph conveying the same message in a more enhanced pictorial view using the ploty3 which has 3 y-axes plotted against the *x*-axis as shown in Figure 3(b). The same process is applicable to the ploty4 with the addition of the 4th *y*-axis.

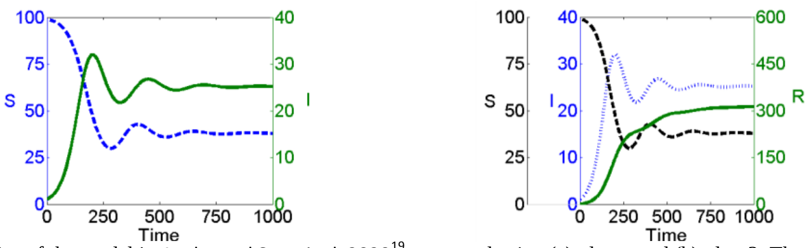

Figure 3: The dynamics of the model in Agaba and Soomiyol, 2020[19] extracted using (a) plotyy and (b) ploty3. The parameter values used are extracted as defined in the paper with $\tau = 1$.

IV. The Derivation of Plotyn Code

The concept of the plotyy and that of the extended ploty4 necessitate the derivation of the novel plotyn MATLAB code which happens to be a more coordinated code that facilitates easy application or usage. Though, the plotyn code could be applied generally on any system of equations, it was coded in this paper for plotting results emanating from the analysis of dynamical models comprising of systems of differential equations solved either using the *ode45* MATLAB application (for ordinary differential equations) or the *dde23* (for delay differential equations). It was derived to generally plot results obtained from the numerical simulation or analysis of a system consisting of *n*-equations.

Syntax for the Derivation of Plotyn Code

The steps enumerated below are the procedures taken in the derivation of the plotyn code.

1. Create a MATLAB file; for instance plotyn.m
2. Input the system of equations as a function code
3. Generate the function code for the analysis and then plot the solution obtain from the system of equations:
    a. input the parameters and their values,
    b. input the initial conditions and the time span,
    c. analyse and extract the solution for the system of equations using *ode45* or *dde23* for the ordinary or delay differential equations respectively,
    d. set the condition of determining system with odd or even number of equations using the *if*-statement,
    e. plot $j = n/2$ and $j+1$ population classes as a reference figure using the plotyy function,
    f. assign colours, positions, size of the figures and *x*-limits for the subsequent *y*-axes,
    g. plot the remaining classes of population on the *y*-axes and distribute them to the left and right of the reference figure in descending and ascending order respective from $j = n/2$ and $j+1$ alongside their respective colours and *x*-limits,
    h. label the *x* and *y* axes accordingly.

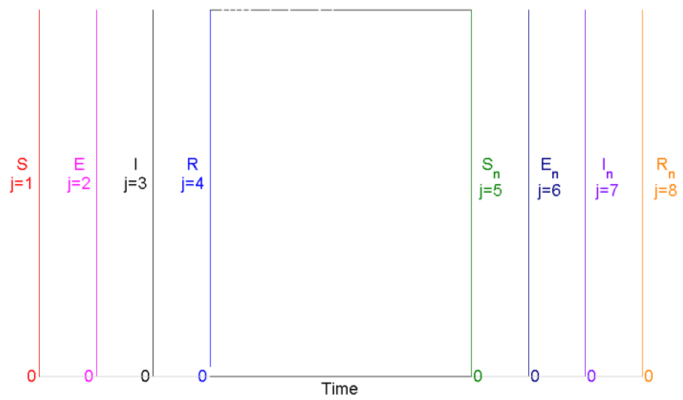

Figure 4: Showing the structure of the general output from the system of *n*-equations, illustrating with $n = 8$. Note that *j* connotes *n*, the number of equations in the dynamical system.

Structural Representation of Results Generated using Plotyn Code

A sample structure as captured in Figure 4 illustrates the pictorial outlook expected when using the novel plotyn code derived in this paper (taking $n = 8$). For example, the application of the code using the model derived in Agaba, 2014[2] gave the result shown in Figure 5 for a system with eight ordinary differential equations. Whereas, Figure 6 gives the result obtained by using the same code for the numerical simulation of a system consisting of five differential equations with delay[10].

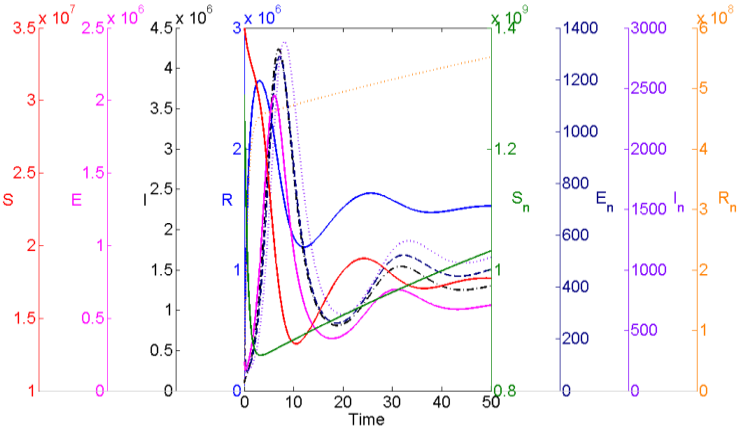

Figure 5: Showing the output for the model extracted from Agaba, 2014[2], a system consisting of eight differential equations ($j = 8$) using the parameter values and initial conditions as defined in the paper.

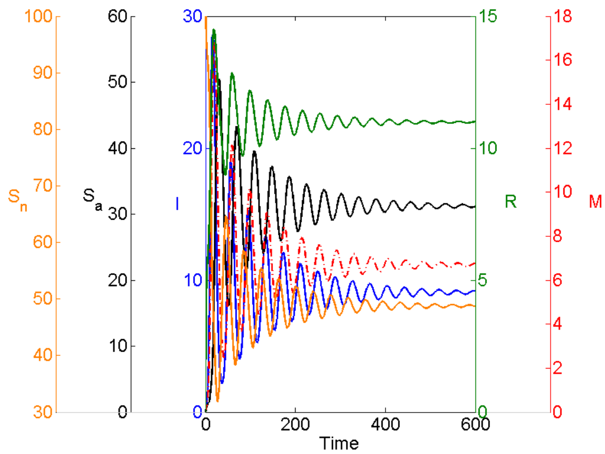

Figure 6: Showing the result obtained for the model extracted from Agaba *et al*., 2017[10]. A system of five differential equations ($j = 5$) with delay, $\tau = 8$. The remaining parameter values and initial conditions are as defined in the paper.

V. Discussion

This paper studies the presentation of graphical results generated from the analysis of dynamical models using derived MATLAB code referred to as plotyn. These codes were derived based on the concept of already existing standard plotyy code and its extension, the ploty4. Consequently, this paper developed a novel MATLAB code that could be used to represent results generated from the numerical analysis or simulation of dynamical systems involving either ordinary or delay differential equations on graphical interface using a single graphical representation irrespective of the total number of differential equations in the system.

The written code is structured such that it can be easily use by anyone with little background knowledge of MATLAB since it requires less user inputs. The application of the code requires that the user input the parameter values, initial conditions and the system of differential equations to be analysed. The appropriate tool required for the analysis is thereafter selected, that is, the user selects either *ode45* for ordinary differential equations or *dde23* for the delay differential equations and then run the analysis. The outcome is produced as a graphical display in the form described in Figure 4 which can be saved as a picture file.

Consequently, this research paper serves as additional well of knowledge, among existing documents and literatures, contributing enormously to knowledge acquisition on MATLAB coding, graphical display and also mathematical modelling of dynamical systems. The application of the derived novel plotyn code is not limited to the analysis of mathematical models of dynamical systems only but could be very useful in other related fields of study.

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