IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 18, Issue 2 Ser. III (Mar. – Apr. 2022), PP 60-68
www.iosrjournals.org

The Collatz Conjecture: A Case Study In Mathematical Problem Solving

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ABSTRACT
Most of the answers so far have been along the general lines of 'Why hard problems are important', rather than 'Why the Collatz conjecture is important'; I will try to address the latter.
*The Collatz conjecture is **the** simplest open problem in mathematics. You can explain it to all your non-mathematical friends, and even to small children who have just learned to divide by 2. It doesn't require understanding divisibility, just evenness.*
The lack of connections between this conjecture and existing mathematical theories (as complained of in some other answers) is not an inadequacy of this conjecture, but of our theories.
*This problem has led directly to theoretical work by Conway showing that **very similar questions are formally undecidable**, certainly a surprising result.*

Date of Submission: 01-04-2022 Date of Acceptance: 14-04-2022

I. Introduction :

[I3] Mathematical "experiments": computer visualisations are the best known example. Although in several respects unreliable, as it often involves the reduction of the (infinite) continuum to a finite, discrete set, they do produce "clues" that serve as a guide for a proof.

[I4] Probabilistic considerations: although proofs in the genuine sense of the word, what they establish is not that a mathematical object (say, a natural number) has a certain property (say, being a prime), but has that property with a certain probability.

[I5] Computer proofs: to be distinguished from computer visualisations, these proofs involve the checking of a finite, though huge amount of separate cases such that human checking is either impossible or too prone to errors and hence a computer program performs the task. The result is not a proof in the classical sense, since unavoidably a human cannot check the proof, one of the basic standards to call a proof a proof.

[I6] Metamathematical considerations: although one has a proof sat- isfying the required standards, the result is seen as paradoxical, counter- intuitive, in conflict with expectations, and hence it is questioned. It can also involve formal metamathematical results, e.g., in showing that a partic- ular problem is unsolvable.

Usually given a specific case, i.e., a particular theorem and its proof history, one will see that one item or a few of the above list will actually be used in the proof search. It is rather exceptional to have a case where (nearly) all these elements are present. The topic of this paper is quite simply the presentation (to a certain depth) of one such case study. All elements, save [I1], of the list are present in one way or another. It can thus be considered an *exemplar* (in the Kuhnian sense), and, perhaps more importantly, as far as I know, a *new* exemplar. As is so often the case, in many philosophical discussions, the same typical example keeps coming back, wrongly suggesting that no other examples are available[1]. In addition, the problem is fairly easy to state, although the mathematics that are used in search of a proof reach formidable heights. And, finally, it is also a problem that many mathematicians consider absolutely *not* interesting. As will be shown here, the problem definitely is interesting, but then the question is

[1]Think, e.g., about thought experiments. A tiny set of examples keeps coming back over and over again: Galileo's thought experiment about heavy and light masses, Newton's bucket experiment concerning absolute properties such as acceleration, and Einstein's thought experiment about travelling on a light wave. It has led some philosophers to mistakenly claim that there is no real problem about thought experiments as they are exceptional and, hence, not important.

The Collatz Conjecture 9 why so many think otherwise. In Section 3, I will provide some suggestions, relating to this matter.

This paper is primarily based on the overview article of Jeffrey Lagarias [2004][2] that provides an extremely detailed presentation of the problem and the attempts to deal with it. Additional sources are used to highlight details of the main story. The contrast between Lagarias' presentation and mine is that I focus on the philosophically interesting features, not necessarily the "pure" mathematical aspects. However, as should be clear, this paper is heavily indebted to the excellent work done by him.

II. The problem

Consider a function from N_0 to N_0, defined as follows: T(n) =

®
n/2 if n is even

(3n+1)/2 if n is odd Next define the iterate of T as usual:

®
$T^{(0)}(n) = n$ $T^{(i+1)}(n) = T(T^i(n))$

The question is now to show that for every $n \in N_0$, there is a finite k, such that

$T^{(k)}(n) = 1$.

A straightforward example: take n=7, then we have the following sequence 7→11→17→26→13→20→10→5→8→4→2→101234567891011
therefore $T^{(11)}(7) = 1$ and k = 11.
2. The origin of the problem

It is easy to understand why, if one has only the above information and is asked whether or not this is an *interesting* problem, the answer will most likely be negative. Why?

[2]This paper available on the Internet is an update of a previous webpaper from 1996, see Lagarias [1996], and itself a further elaboration of Lagarias [1985]. The most recent paper is an annotated bibliography whereas Lagarias [1996] retraces the history of the problem, proofs included.

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Firstly, it is quite easy to "invent" similar problems, so why should this particular case attract our attention? As a matter of fact, this type of argument has been used on several occasions by mathematicians, the most famous case no doubt Gauss' comment on the problem that was to become Fermat's Last Theorem. In 1816 he wrote to Heinrich Olbers (known as the originator of the Olbers' paradox) that "he could easily lay down a multitude of such propositions, which one could neither prove nor dispose of" (see Ribenboim [1979], p.3).

Secondly, suppose we do manage to show the theorem to be correct, what have we gained? Are there other problems around that would get solved in the process as well? At first sight not.

Thirdly, on the level of proof methods, it is not guaranteed at all that interesting things will come out of it. Is it likely that some ingenious new proof method could solve this problem, but is it to be expected? These are all very good reasons to consider the problem not interesting (as the author of this paper believed for a very long time, up to the point that he actually wrote that because the problem has no connections with other problems, it was perfectly acceptable to consider it uninteresting; so this paper is at the same time a correction on one of my former views).

In fact, notwithstanding the observation that not that many mathemati- cians are actually involved with this problem, it is definitely an interesting problem. Let me say a few words about its origin. When one is dealing with number- theoretic functions, say functions f from N_0 to N_0, then one of the particular problems one has to deal with is *notation* and *representation*. What I mean is the following.

Suppose that the function f from N_0 to N_0 is a permutation. Then there are several ways to represent this function:

(a) One of the classical forms is in tabular form:

® ®
12345... f(1) f(2) f(3) f(4) f(5) . . .

Note that this representation supposes to have the necessary knowledge on how to continue the table.

(b) Obviously,asforanyfunction,wecanhaveanexplicitform:f(n)=some symbolic expression involving n.
(c) Avariationon(b)isafunctiondefinedimplicitlybysomerecurrence relations:

$f(n) = g(f(n − 1), f(n − 2), . . . , f(3), f(2), f(1))$,

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wherenotalloff(i),n−1®i®1needoccurandwheregissomespecified function.

(d) Anotherformthatdiffersradicallyfromthethreeabove,butjustlike(a) supposesthatonehassufficientknowledgeonhow tocontinuethefigure,isa graphical representation.

1→2→3 4→5→6 7→... ←−−−−− ←−−−−−

where an arrow represents an application of the function f (in this case, the simple function f(n), defined by f(3n+1) = 3n+2, f(3n+2) = 3(n+1) and f(3(n + 1)) = 3n + 1)). Although this example is rather trivial, the importance of a well- chosen representation must be obvious. The graphical representation shows immediately that f is composed of an infinite number of 3-cycles. One could very well imagine that if f becomes more complex, the graph can tell more things than an algebraic of analytical expression. (Note at the same time the connection with visualisations; although there is no computer involvement here, it does show the importance of an image).

Note also that different graph representations are possible. Instead of simply listing the natural numbers and drawing the appropriate arrows, we can start with 1 and list the iterates of 1:

$1 \rightarrow f(1) \rightarrow f^2(1) \rightarrow f^3(1) \rightarrow ...$

All of this shows that if we want to understand what permutations are all about, what their properties are, then it is a useful approach to examine the graphs of such functions. In addition, it allows to rephrase some questions into graph- theoretical questions. This is actually the area that the “creator” of the problem, Lothar Collatz, was working on. Although his examples are different from what is now known as the Collatz Conjecture (CC), they raise the same problems. His original question was whether, for a particular function f, the trajectory starting with 8 and the iterates of 8, contains 1 or not. (I use here the term “trajectory” because it need not be a cycle). One nowseestherelationtotheCC.Rephrased in terms of trajectories, the CC claims:

For any natural number n, the trajectory starting with n, contains the number 1.

Of course, no mathematician doubts the importance of permutation theory. It is sodeeplyentrenchedinnumbertheoryandbeyond, thatismustbecon- sideredoneofthecorepartsofmathematics.Althoughonemightperhaps

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consider the CC as a “spin-off”, it is clear that the general question that is raised by it is an interesting one. What seems to have been at play is that there are several gaps in the research of the CC. The problem disappears for some years only to reappear at some other moment in the hands of another mathematician. Thefactthatitwasnoteasytolocatethe“true”originsoftheproblemsis supported bytheobservationthattheverysameproblemisknownunder differentnames: Hasse’s algorithm, the Syracuse problem, Kakutani’s problem, Ulam’s problem, and sometimes it is even referred to as the Hailstone problem. Thelastnameisareferencetothebehaviourof thesequenceof$T^i(n)$.Ittends tomoveupwardsanddownwardsmuchinthewaythathailstoneshitthe ground andbounce backup again.

3. Mathematicalinduction,numbercrunchingandpictures

An important feature to notice in the search for a proof of the CC is that, at first sight, it seems not very useful to invoke mathematical induction as a proof method. One of the obvious problems is that it does not help to start from the assumption that the CC has been proven for all cases up to a number n in order to prove the case for n+1, as the iterates for n+1 can go well beyond n + 1. In the above exampleforn=7,thehighestvalueonereachesis26.Thiswouldshift the problem to the question whether one can show that:

For all n, there is a finite number N(n), such that for all i, $T^{(i)}(n)$ ® N(n).

Inaddition,onewouldneedsomeconnectionbetweenN(n)andN(n+1)tobe abletogettheinductionprocessworking.However,itis clearthatthisnew tasklookseverybitasdifficultastheoriginaltask.Ofcourse,onemighttryan inductiononsomeotherparameterof

theproblem,butitbecomessoonclear thateitheronekeepscomingbacktotheoriginalproblemitselforoneendsup worseoff.E.g.,one mighttryaninductiononk,suchthat$T^{(k)}(n)=1$,ifatall.
However,oneneedsawaytoenumeratethensuchthatkformsasequence1,

2,3,...(withorwithoutgaps?).Butthatseemsanevenharderquestionto answer:

Given a natural number k, what are the numbers n such that $T^{(k)}(n) = 1$?

Ifwehadananswertothisquestionand,foreveryk,wecouldlistthenumbers n,thenofcourseifwecouldprovethatsomenumbern ismissing

forallk,thenwewouldhaveadisproofoftheCC.Clearly,thisisnotan interesting strategy and so, in short, one does well (initially) to forget about mathematical induction.

As one might expect with this kind of problem, it is very tempting to collect *numerical evidence*, corresponding to a mixture of careerinduction[I2]and computerproof(amixof[I3]and[I5]).TheCChasbeencheckeduptoa staggering 3.24×10^{17}.One might wonder what the relevance of suchevidence could possiblybe.

One argument is rather trivial: one might come up with a counterexam- ple, thereby settling the problem by producing a disproof. However, oddly enough, in many cases where such evidence is collected, the mathematicians tend to believe that there are no counterexamples.Sowhydotheydoit?

A possible answer is that mathematicians sometimes do what scientists in general do: you collect evidence hoping that some pattern appears that tells yousomethingabouttheproblemyourstudying.Asithappensinthiscase,the onlythingthatappearsiscomplexity andmorecomplexity.Table1showsthe maximumvaluereachedofthenumbern,(indicatedby the variable N) as n ranges from 1 to 100.000. Note, e.g., that between 1.819 and 4.254, the highest value remains 1.276.936 but at 4.255 it jumps straight away to 6.810.136. Even in this case, however, it is clear that the numerical evidence is interesting for it is shows that we are most likely dealing with a problemthatisintrinsically complexandthereforeweshouldnotbesurprisedthattheproblemsresists attempts to prove it.

Astothecomputeraspectofthisnumericalsearch,itisclearthatweare dealingherenotwithamereenumerationofcases;thesizeof the set of checked cases is simply too large to be checked one by one. Hence a whole range of mathematical techniques and computer engineering is involved and, therefore, it becomes interesting. Note that for the computer checking a dis- tributed network had to be createdtohavesufficientcomputationalpower.

4. Enterprobabilitiesandstatistics5.1.Aprobabilisticargument

What is more interesting is the fact that there exists a probabilistic heuristic argument, a perfect illustration of [I4], that (at least some) mathematicians seemtofindconvincingenoughtobelievetheCCtobeprovable.Thisisthe argument:

N	Path length	Maximum value
1 2 3 7 15 272554476397031,8194,255 4,5919,66320,89526,62331,911 60,975 77,671	0 1 7 16 17 111 47 97 131 170 161 201 170 184 255 307 160	1 2 16 52 160 9,232 13,120 39,364 41,524 250,504 1,276,9 8,153,620 27,114,424 50,143,264 106,358,020 121,012,8 593,279,152 1,570,824,736

	334 231	

Table 1. Sequence of peak values up to N = 100,000 (© *Scientific American*, see Hayes [1984])

(a) You do not have to worry about even numbers 2n, because in the next step, you will have n, so you go "down", i.e., the numbersarebecomingsmaller.

(b) Thereforelookatwhathappenswhenyoustartwithanoddnumber2n+1. Eitherinthenextstepyouwillhaveanoddnumber oranevennumber.*Assume that the probability is* 1/2*in both cases*.

(c) Repeattheprocess.Thisproducesthefollowingpicture:

n n/2
(3n + 1)/2

3(3n + 1)/2 + 1)/2 (3n + 1)/4

(eacharrowhasaprobability1/2andnotethat3(3n+1)/2+1)/2isaneven number,sinceby construction(3n+1)/2isodd).

(d) Considernowatrajectoryfromoneoddnumbertoanotheroddnumber. SupposethatinbetweenthereareN−1odd numbers.Intotal

thisproducesNtransitionsfromanoddnumbertothenext.Whatweexpectis thatN/2ofthesetransitionswillhappeninonestep,N/4in twosteps,andso on.This leads to a growth factor:

$(3/2)^{N/2}.(3/4)^{N/4}.(3/8)^{N/8}$ ... So the average growth factor per transition is: $(3/2)^{1/2}.(3/4)^{1/4}.(3/8)^{1/8}$ ...

(e) Asimplecalculationshowsthatthenumeratorisnothingbut3tothepower
$1/2 + 1/4 + 1/8 + \cdots = 1$, therefore 3; and the denominator is $2^{1/2}.4^{1/4}.8^{1/8}\cdots$
$=2^2=4$.(Hereasimpleinductivereasoningwilldothetrick).Hencethe averagegrowthfactorpertransitionis3/4whichissmallerthan1, soon averagethenumbers"shrink",thereforetheCCshouldbecorrect.

Ofcourse,thisbeautifulargumentstandsorfallswiththeassumptionmadein
(b)(initalics).Istherereasontoassumethatthereisjustasmuchchanceto haveanoddoranevennumberinthenextstep?Actuallynot and,inaddition, therearemanyinterestingproblemsinnumbertheorywhereoneexpects certain probabilities but amazingly enough, the mathe- matical "facts" show otherwise. A famous example to illustrate this point concerns a conjecture put forward by Georg Polya.Think about the prime decomposition of natural numbers.Count the number of primes, that need not be distinct.Call r(n) = numberofprimesinn.Theneitherr(n)isevenorodd.Doesitnotseemlikely thatifwepickanarbitrarynumbertheprobabilitythatr(n) isevenoroddis 1/2?Asithappensthisisnotthecase,andthebehaviourofthefunctionr(n) turnsouttobequitecomplex.Inthatsense,it isquiteunderstandablethatfor some mathematicians these probabilistic considerations carry little weight.

5.2. Gathering statistical evidence

Related to the above are what one might call *statistical analyses* of the prob- lem. Here the objective is to explore and hopefully to understand and explain particularfeaturesthatappearinthenumericaltables,notnecessarilyto

find arguments for or against the correctness of the conjecture. 8k + 4 4k + 2 2k + 1

Consider, e.g., the fact that consecutive numbers have trajectories of the same length(andotherproperties).Insomecasesthis phenomenoncanbeeasily explained.Thediagramshowswhynumbersofthe8k+4end8k+5musthave the same trajectory length.

Although,assaid,itisnotclearinwhatwaysuchresultscouldcontributetoa finalanswer,i.e.,aproofsatisfyingtheusualstandards, thereseemstobea veryclearanalogytobedrawnwithscientificpractice.Ifitismeaningfulto speakofa*Collatz-universe*,meaning therebyallthenumericalmaterialrelated to the conjecture, then these probabilistic and statistical analyses correspond to anexploration ofthatuniverse.Oneisnotreallyexpectingtofindlawsorthe like,butratherindicationsthatsuggestwhatpossiblelawsonecouldlook or aimfor.Inasensethemathematicianistryingtogeta"grip"ontheproblemby wandering through the territory.

5. Digression: generating concepts to tackle the problem

The heading of this section seems to suggest that its topic is of minor im- portance. Such is definitely not the case, but there are two reasons why I want to treat it separately: firstly, because it is a common feature of the whole mathematical enterprise and in that sense it occurs in [I1] up to and including [I6], and, secondly, because the topic and its related literature is too vast to treat here in a thorough way. What is this feature? For want of a better notion, I propose to call it *generating concepts* (GC). Let me first of all illustrate what I mean using CC.

Take a look at the original problem. What concepts occur in the problem formulation? We talk about functions, natural numbers, about elementary arithmetical operations (addition, multiplication, division) and about iter- ation. Those are roughly the "ingredients" of the problem. The striking feature when one goes through the history of CC is that the concepts as

3k + 2

8k + 5 12k + 8 6k + 4

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formulated in the original problem statement play hardly any role at all. Instead, and techniques such as listed in [I1]-[I6] promote this process, a whole range of derived concepts is introduced and in some theorems none of the original concepts actually occur. For CC, what follows are some of the derived concepts:

1 (a) The notion of iteration leads rather naturally to the idea of a *trajectory*, i.e., the sequence of numbers, starting with n, and ending with the first 1 to occur.

2 (b) An obvious correlate of (a) is the *length of the trajectory*.

3 (c) Given a trajectory, let k be the least positive numbers such that T
$^{(k)}(n) <$

n, then k is called the *stopping time* of n, or, $\sigma(n) = k$.

4 (d) Derived from (c) is $\sigma_\infty(n)$, this is the *total stopping time*, i.e., that k such that $T^{(k)}(n) = 1$, (this relates of course to (b)).

5 (e) The *expansion factor* s(n) is defined as the division of the largest value

reached in a trajectory by n, i.e., s(n) = $^{\sup}k®0\ T^{(k)}(n)$. n

6 (f) The *parity vector* $v_k(n)$, basically corresponding to the trajectory, where all the numbers are reduced modulo 2.

As an illustration, consider once more the example n = 7, then the properties are:

(a) *Trajectory of* n=7: ⟨ 7,11,17,26,13,20,10,5,8,4,2,1⟩ ,(b) *Length of the trajectory* = 12,

(c) $\sigma(7) = 7$, (d) $\sigma_\infty(7) = 11$,

(e) s(n) = 26/7 ≈ 3,7

(f) $v_{11}(7)$ = ⟨ 1,1,1,0,1,0,0,1,0,0,0,1⟩

On the one hand, it seems obvious that these new concepts should emerge, as it is easy to see how they are related to the original problem and, hence, how they can be helpful in the search for a proof. However, this is only part of the story. Besides the concepts mentioned above, many others could have been proposed, but apparently have not been proposed. As an example, take this personally thought-up concept:

M_7=thesetofalltrajectoriessuchthatthelengthofthetrajectoryisamultiple of 7

and related to that:
N_7 = those numbers that belong to a trajectory in M_7.

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Itismyestimatedguessthatnomathematicianwillfindthenotionsof M_7 and N_7 theleastbitinteresting.Butthenthequestionmustbe: why?Nodoubtthe answerwillbe:themathematicians'practice,butthatdoesnothelptofillinthe details.Whatisitinthatpracticethat allowsmathematicianstomakesucha selection?Letmereformulatethatquestioninslightlymoreabstractterms.

Suppose that:

(a) we are given a set X, and

(b) a property corresponds to a subset of X, then,

(c) we have a total of $2^{|X|}$ possible properties.

If X is of infinite size, so is 2^X . Hence we are faced with a double question:

(Q1)Howisa*finite*subsetoftheinterestingpropertieschosen?(Q2)Howare uninteresting propertiesavoided?

Notetheimportanceof(Q1).Computerprogramssuchas*Automatic Mathe- matician*, developed in the eighties by Douglas Lenat, were indeed capable of generating interesting concepts, but, as time went on, they tended to drown in them. Somehow, real-life mathematicians seem to avoid this pitfall. Apart from general considerations about concept generation and selection as stud- ied in cognitive psychology[3] (involving the study of metaphors, analogies, conceptual blending,andthelike),mathematicsisinthissensea specialcase in that conceptgenerationandproofaretiedtogether.E.g.,inthecaseofCC, $\sigma(n)$ is moreinterestingthan $\sigma_\infty(n)$ because the first theorems one could prove about CC involved the stopping time function and not the total stopping time function. Thereby the conceptisreinforcedandallcon-ceptsthatcanbeeasilylinkedto it. If a derived concept does not turn up somewhere in a proof, then it will most likely disappear. As the production of proofs is a rather difficult and often slow process, it explains why so few derived concepts survive.

As a further support of this thesis—the link between concept generation and proof production—it is worthwhile to look at so-called "seminal" papers in the history of mathematics, i.e., those contributions that either set in motion a new branch of the mathematical tree or relaunched a research that had arrived at a standstill. One such famous example is Bernhard Riemann's paper "Uber die Anzahl der Primzahlen unter einer gegebenen

[3]The literature in this field is too extensive and too varied to be reported here, but, obviously, for mathematics a fine example (although many, such as myself, tend to disagree with the authors) is the recent work of Lakoff and Nunez [2000].

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Grösse"[1859],("OntheNumberofPrimeNumbersLessthanaGiven Quantity").Iwillnotgointodetailshere,butone,ifnotthe moststriking featureofthepaperisthattherearehardlyanyproofsandifso,theytendtobe "over-summarized", making it a tough job to reconstruct what the author might have meant[4]. On the other hand, what the paper does is to introduce a range of new functions that get connected to existing and well- studied functions, thereby offering a new range to explore. As the paper is generally acknowledged as a fundamental contribution, it is reasonable to conclude that such concept generation attempts are considered as important as proofs themselves.

However,letmenowreturntothemainstoryofthispaperandlookintoitem [I6] on the list.

7. Metalevel considerations

In1972JohnConwaypublishedashortpaperwithacuriousandimportant result:ageneralizationofCCisundecidable.Inthatsense,it isabeautiful illustrationofatype[I6]kindofargument.Itimpliesthat*perhaps* CCitselfis undecidable,althoughatpresentnosuch resulthasbeenfound[5].

The generalization is the following:

Consider a function g from integers to integers (note that this is not an essential extensionastheintegerscanalwaysbemappedone-to-oneontothenatural numbers[6]), suchthat

$g(n) = a_i n + b_i$ for $n \equiv i \pmod p$,

andwhere a_i and b_i arerationalnumberssuchthatg(n)isalwaysaninteger.

[4]OneofthebestsourcesaboutRiemann'spaperisEdwards[1974].Thestatementonthelow proofqualityofthepaperisbasedonthis

quoteofEdwards:"TherealcontributionofRiemann's 1859paperlaynotinitsresultsbutinitsmethods.Theprincipalresultwas aformula[...] However,Riemann'sproofofthisformulawasinadequate[…]". (p. 4)

[5]If CC would turn out to be undecidable, then it would most certainly replace the "busy beaver" as the simplest undecidable problem. The"busybeaver"concernsTuringmachinesproducinga stringof'1'-sonanemptytape.SeeBoolosetal.[2002],pp.41–44,for aclearandconcise exposition of the "busy beaver" problem.
[6]The reason for the extension from natural numbers to integers has to do with the problem of encoding a problem known to be undecidable into this generalization of *CC*. In that sense the construction can be reformulated restricted to natural numbers, however the result would be definitely 'ugly'.

20JeanPaulVanBendegemCCthencorrespondstothespecialcase,where: $g(n)=(1/2)n+0$ for $n\equiv 0 \pmod 2$, and $g(n)=(3/2)n+1/2$ for $n\equiv 1 \pmod 2$.

So $a_0=1/2, b_0=0, a_1=3/2$ and $b_1=1/2$.
The undecidability comes down to the fact that, given a function g, and

givenanumbern,thereisnoalgorithmthatdecideswhetherthereisanumber ksuchthat $g^{(k)}(n)=1$. Actually,Conwayprovedaneven strongerresult,viz.all rational numbers b_i may be equal to 0.

Obviously,whatthisresultimpliesis,atleast,thatoneshouldnotbeamazedby thecomplexityoftheoriginalproblem,theCC.Thefact that the statement resisted and continues to resist proof for quite some time now, is perhaps something to be expected, given Conway'sresult.Inthatsense,itdoeshavean influenceonmathematicians'expectations.However,thestorydoesnotend there.There are links between CC and ergodic theory (see Lagarias [1985], Section 2.8), thus introducing considerations about stochasticity and randomness into the proof search. These considerations are clearly not purely mathematical, witness this quote fromtheconclusionofLagarias[1985]:

Isthe 3x + 1 problemintractablyhard?The difficulty ofsettling the 3x + 1 problem seems connected to the fact that it is a deterministic process that simulates "random" behaviour.Wefacethisdilemma:Ontheonehand,totheextentthattheproblemhas structure,wecan analyseit—yetitispreciselythisstructurethatseemstopreventus fromprovingthatitbehaves"randomly."Ontheotherhand,tothe extent that the problem is structureless and "random," we have nothing to analyse and consequently cannot rigorously prove anything. Of course there remains the possibi1ity that someone wil1 findsomehiddenregularityinthe3x+1problemthatallows someof the conjec- tures about it to be settled. The existing general methods in number theory and ergodic theory do notseemtotouch the3x+1problem;inthissenseitseems intractable at present. Indeed all the conjectures made in this paper seem currently to

beoutofreachiftheyaretrue;Ithinkthereismorechanceofdisprovingthosethatare false.

It seems obvious, at least to me, that such statements do not only go beyond mathematics proper, but at the same time contain (a) philosophical ideas about thestructureofthemathematicaluniverse,(b)theexpectationsonemight reasonablyhaveconcerningthe likelihoodofprovingatheorem,and(c)the connection(s)betweenthesetwoelements.Inasensethiscould

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be considered a form of philosophy emerging out of mathematical practice itself, and hence, produced by mathematicians themselves. This explains to a certain extent the contrast with philosophical explanations by philosophers about mathematics, that tendtofocuson"end-products",i.e.,"finished"proofs. Letmeexplorethisideaabitfurtherintheconclusionofthispaper.

III. Conclusion

Afirstminorremarktomakeisthatthereadersurelywillhavenoticedthatan illustrationof[I1]ismissing.Thereareindeed,asfarasI know, no examples of "sketchy proofs" that could possibly be translated or trans- formed into an acceptable proof. On the whole, occurrencesof[I1]seemtoberatherrare.
However, the presence of all the other elements do show that the Collatz Conjecturedeservestobecalledan"exemplar".

Secondly,andmoreimportantly,thereaderwillalsohavenoticedthatIhave givenno"real"proofsofpartialresults.After all,seeLagarias[2004],asone mightexpect,thereisamultitudeofproofsdealingwithbitsandpiecesofthe CC,butIdidnotwanttopay attention to that part of the mathematical process. I did want to focus on all those elements that are at the same time not proofs, but essential to guide the search for a proof. My claim is that these considerations are part and parcel of mathematical practice and, by implication, that a philosophy of mathematics that claims to deal with the essential features of what mathematics is all about, shouldincludetheseelements.

Thirdly,asaconsequenceoftheobservationabove,itfollowsthatmath- ematics—orthemathematicalbuilding,touse thebestknownmetaphor— neednotbeanintegratedwholeoraunityinsomesense.Afterall,notonlywill proofmethodsdifferfrom mathematical domain to mathemati-cal domain— think, e.g., about the difference between “diagram chasing” in category theory and mathematicalinductioninnumbertheory(seeVanBendegem[2004])— buttheadditionalelements[I1]upto[I6]willmostcer- tainly differ from domain to domain—in number theory number crunching is obviously possible but visualisations, equally obviously, seem more suited to geometrical and topological problems. Note that this form of ‘disunity’ I am pleading for, is not in contradiction with the existence of the founda- tions of mathematics, such as settheory.Fromthefoundationalpointofview,welookat the end-products, i.e., mathematical theories, leave out the details of the process that has led to the theory, and then integrate these theoriesbyconstructingacommon language wherein these theories can be

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translated, thus creating a new universe that has a uniformity that the daily practice of mathematicians seems to be lacking. In terms of languages, foundationalworkcorrespondstodesigninganartificiallanguagesuchas Esperanto.WhereasinthispaperIam suggestingthatweshouldalsohavea lookatthelanguageswedailyspeak.InthesamemannerthatEsperantodid notbecometheworld language, working mathematicians know that there is this special group of “foundational speakers” that seem to have trouble to convince everyone else to speak as they do. In addition, the better we understand our daily languages, the more likely we will understand what kind of artificial languages will have any rate of success or not.

Asafinalclosingremark,letmejustmentionthatatthemomentofwriting— February 2005—the problem remains unsolved.

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