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Generalized a New Class of Harmonic Univalent Functions

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ABSTRACT
In this present paper , we defined complex valued functions that are univalent of the form f = h+ḡ where h and g are analytic in the open unit disk Δ . we obtain a number of enough coefficient conditions for normalized harmonic functions that are starlike of order α , $0\leq \alpha \leq 1$. These coefficient circumstances are also show to essential when "h" has negative and g has positive coefficients.

Key Words: Harmonic function ,univalent function sense-preserving; starlike.convex combination

1. INTRODUCTION

A continuous function f = u+iv is a complex-valued harmonic function in a complex domain $\mathfrak{C}$ if both u and v are real harmonic in $\mathfrak{C}$. In any simply connected domain $\mathfrak{D} \in \mathfrak{C}$ we can write f= h +$\bar{g}$ where h and g are analytic in $\mathfrak{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathfrak{D}$ is that in $\mathfrak{D}$. See Clunie and Sheil-Small [2] .

Denote by $\mathcal{H}$ the class of functions f= h +$\bar{g}$ that are harmonic univalent and sense-preserving in the unit disk $\Delta = \{ z: |z| < 1 \}$for which

h(0) = f(0) = $f_z(0) - 1$ Then for f= h +$\bar{g}$ $\in \mathcal{H}$ we may express the analytic functions h and g as \` \`

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n \qquad (1)$$

Note that $\mathcal{H}$ reduces to the class of normalized analytic univalent functions if the co-analytic part of its members is zero. In 1984 Clunie and Sheil-Small [2] investigated the class $\mathcal{H}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on $\mathcal{H}$ and its subclasses. For more references see Duren [3] . w x In this note, we look at two subclasses of $\mathcal{H}$ and provide univalence criteria, coefficient conditions, extreme points, and distortion bounds for functions in these classes.

For $0 \leq \alpha < 1$ we let $\mathcal{G}_{\mathcal{H}}(\alpha)$. a denote the subclass of $\mathcal{H}$ consisting of $\mathcal{H}$ harmonic starlike functions of order α. A function f of the form (1) is harmonic starlike of order α, $0 \leq \alpha < 1$ for $|z| = r < 1$ if

$$\operatorname{Re}\left\{\frac{z f'(z)+z^2 f''(z)}{\lambda z f'(z)+(1-\gamma)f(z)}\right\} > \beta\left|\frac{z f'(z)+z^2 f''(z)}{\lambda z f'(z)+(1-\gamma)f(z)} - 1\right| + \alpha \qquad (2)$$

$0 \leq \lambda < 1$, $0 \leq \alpha < 1$, $\beta \geq 0$

We further denote by $\mathcal{G}_{\mathcal{H}}(\alpha)$.. a the subclass of $\mathcal{G}_{\mathcal{H}}(\alpha)$.. a such that the functions h and g in

f= h + $\bar{g}$ are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n|\, z^n \quad , \quad g(z) = \sum_{n=1}^{\infty} |b_n|\, z^n \qquad (3)$$

It was shown by Sheil-Small [4,] that $|a_n| \leq (n+1)$ (2n+1)/6 and $|b_n| \leq (n-1)$ (2n-1)/6 if f= h + $\bar{g}$ $\in \mathcal{G}_{\mathcal{H}}{}^0(0)$.

The subclass of $\mathcal{G}_{\mathcal{H}}(\alpha)$. where $\alpha = b_1 = 0$ is denoted by $\mathcal{G}_{\mathcal{H}}{}^0(0)$. These bounds are sharp and thus give necessary coefficient conditions for the class $\mathcal{G}_{\mathcal{H}}{}^0(0)$.
Avci and Zlotkiewicz [1] proved that the coefficient condition is sufficient for functions f= h + $\bar{g}$ to be in $\mathcal{G}_{\mathcal{H}}{}^0(0)$.Silverman proved that this coefficient condition is also necessary if $b_1 = 0$ and a_n if a and b in 1 are negative.

We note that both results obtained in are subject to the restriction that $b_1 = 0$. The argument presented in this paper provides sufficient coefficient conditions for functions $\mathcal{G}_{\mathcal{H}}(\alpha)$ f= h + $\bar{g}$ of the form (1) to be in $\mathcal{G}_{\mathcal{H}}(\alpha)$ where $0 \leq \alpha < 1$ and b_1 not necessarily zero. It is shown that these conditions are also necessary when f $\in \mathcal{G}_{\mathcal{H}}(\alpha)$.

2.MAIN RESULTS :

THEOREM 1.

Let f= h +$\bar{g}$ be given by (1). Furthermore, let `$\sum_{n=1}^{\infty}[n^2(\beta+1)-(\beta+\alpha)(\lambda^n-\lambda+1)]\,|a_n|+\sum_{n=1}^{\infty}[n^2(\beta+1)-(\beta+\alpha)(\lambda^n-\lambda+1)]\,|b_n|\leq 1-\alpha$ (2.1)

Where $a_1=1$ and $0\leq\alpha<1$,then f is harmonic univalent in Δ and f $\in\mathcal{G}_{\mathcal{H}}(\alpha,)$

Proof :

we have by inequality so that $z_1\neq z_2$, then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right|$$

$$\geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|$$

$$=1-\left|\frac{\sum_{n=1}^{\infty}b_n(z_1^n-z_2^n)}{(z_1-z_2)-\sum_{n=2}^{\infty}a_n(z_1^n-z_2^n)}\right|$$

$$1-\left|\frac{\sum_{n=1}^{\infty}|b_n|\,n}{1-\sum_{n=2}^{\infty}|a_n|n}\right|$$

$$\geq 1-\frac{\frac{\sum_{n=1}^{\infty}[n^2(\beta+1)-(\beta+\alpha)(\lambda^n-\lambda+1)]\,|b_n|}{2-\alpha}}{\frac{\sum_{n=1}^{\infty}[n^2(\beta+1)-(\beta+\alpha)(\lambda^n-\lambda+1)]\,|a_n|}{2-\alpha}}$$

$$\geq 0$$

Which proves univalence . f is sense –preserving in U this is because

$$|h'(z)|\geq 1-\sum_{n=2}^{\infty}|a_n|n\,|z|^{n-1}$$

$$>1-\sum_{n=2}^{\infty}|a_n|n$$

$$\geq 1-\frac{\sum_{n=2}^{\infty}[n^2(\beta+1)-(\beta+\alpha)(\lambda^n-\lambda+1)]\,|a_n|}{2-\alpha}$$

$$\geq\frac{\sum_{n=1}^{\infty}[n^2(\beta+1)-(\beta+\alpha)(\lambda^n-\lambda+1)]\,|b_n|}{2-\alpha}$$

$$\geq \sum_{n=1}^{\infty} n|b_n| > \sum_{n=1}^{\infty} |a_n| n\, |z|^{n-1}$$

For proving f $\in \mathcal{G}_{\mathcal{H}}(\alpha,)$, we must show that (2) holds true . by using ;note that w = u +iv β, α are real Re(w) $\geq \beta|w - 1| + \alpha$ if and only if Re{w(1+ $\beta e^{i\theta}$) $-$ $- \beta e^{i\theta}$ } $> \alpha$ and show that

$$\text{Re } \left\{ \frac{z f'(z)+z^2 f''(z)}{\lambda z f'(z)+(1-\gamma)f(z)} (1+\beta e^{i\theta}) - \beta e^{i\theta} \right\} > \alpha \ (-\pi \leq \theta \leq \pi)$$

Or equivalently

$$\text{Re } \left\{ \frac{(1+\beta e^{i\theta})z f'(z)+z^2 f''(z)}{\lambda z f'(z)+(1-\gamma)f(z)} - \frac{\beta e^{i\theta}\lambda z f'(z)+(1-\gamma)f(z)}{\lambda z f'(z)+(1-\gamma)f(z)} \right\} > \alpha \quad (2.2)$$

If we put

A(z) = $\left(1+\beta e^{i\theta}\right)z f'(z) + z^2 f''(z) - \beta e^{i\theta}\lambda z f'(z) + (1-\gamma)f(z) \geq$
$0 \ , for \ 0 \leq \alpha < 1$

B(z) = $\lambda z f'(z) + (1-\gamma)f(z)$

Re(w) $\geq \alpha \ |w-(1+\alpha)| \leq |w+(1+\alpha)|$

$|A(Z)+(1-\alpha)B(z)| - |A(Z)+(1+\alpha)B(z)|$

$\geq 0 \ for \ 0 \leq \alpha \leq 1$

So $|A(Z)+(1-\alpha)B(z)|$

$=\left(1+\beta e^{i\theta}\right)(z+\sum_{n=2}^{\infty} n^2 a_n z^n + \sum_{n=1}^{\infty} n^2 b_n \overline{(z)}^n - \beta e^{i\theta}(z+\sum_{n=2}^{\infty} \lambda_n - \lambda +$
$1) a_n z^n + \sum_{n=2}^{\infty} \lambda_n - \lambda + 1) b_n \overline{(z)}^n$

$|(2-\alpha)z| + \sum_{n=2}^{\infty}[n^2(\left(1+\beta e^{i\theta}\right) - (\beta e^{i\theta}+\alpha-1)(\lambda^n - \lambda +$
$1)] a_n z^n + \sum_{n=2}^{\infty}[n^2(\left(1+\beta e^{i\theta}\right) - (\beta e^{i\theta}+\alpha-1)(\lambda^n - \lambda+1)] b_n \overline{(z)}^n|$

$|A(Z)+(1+\alpha)B(z)|$

$$= \left|\left(1+\beta e^{i\theta}\right)\left(z + \sum_{n=2}^{\infty} n^2 a_n z^n + \sum_{n=1}^{\infty} n^2 b_n \overline{(z)}^n \right.\right.$$
$$- \beta e^{i\theta}\left(z + \sum_{n=2}^{\infty} \lambda_n - \lambda + 1\right) a_n z^n + \sum_{n=2}^{\infty} \lambda_n - \lambda + 1\Bigg) b_n \overline{(z)}^n$$
$$- \beta e^{i\theta}[\lambda(\Bigg| z + \sum_{n=2}^{\infty} n^2 a_n z^n + \sum_{n=1}^{\infty} n^2 b_n \overline{(z)}^n + (1-\lambda)(z$$
$$+ \sum_{n=2}^{\infty} n^2 a_n z^n + \sum_{n=1}^{\infty} n^2 b_n \overline{(z)}^n)] - (1$$
$$+ \alpha)[\lambda\left(z + \sum_{n=2}^{\infty} n^2 a_n z^n + \sum_{n=1}^{\infty} n^2 b_n \overline{(z)}^n \right) + (1-\lambda)(z$$
$$+ \sum_{n=2}^{\infty} n^2 a_n z^n + \sum_{n=1}^{\infty} n^2 b_n \overline{(z)}^n)]$$

$$\left| -az + \sum_{n=2}^{\infty} n^2 \left(1 + \beta e^{i\theta}\right) - \left(\beta e^{i\theta} + \alpha + 1\right)(\lambda_n - \lambda + 1)]\, a_n z^n \right.$$
$$\left. + \sum_{n=1}^{\infty} n^2 \left(1 + \beta e^{i\theta}\right) - \left(\beta e^{i\theta} + \alpha + 1\right)(\lambda_n - \lambda + 1)]\, b_n \overline{(z)}^n \right|$$

There fore ,

$$|A(Z) + (1-\alpha)B(z)| - |A(Z) + (1+\alpha)B(z)|$$
$$\geq 2\left\{(1-\alpha) - \sum_{n=2}^{\infty} [n^2(\beta+1) - (\beta+\alpha)(\lambda_n - \lambda + 1)\right] |a_n|$$
$$- \{n^2(\beta+1) - (\beta+\alpha)(\lambda_n - \lambda + 1)]|b_n| \geq 0$$

By inequality (2.1) , which implies that f $\in \mathcal{G}_{\mathcal{H}}(\alpha,)$
The harmonic univalent function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{x_n}{n^2(\beta+1) - (\beta+\alpha)(\lambda_n - \lambda+1)} z^n + \sum_{n=1}^{\infty} \frac{\overline{y_n}}{n^2(\beta+1) - (\beta+\alpha)(\lambda_n - \lambda+1)} \overline{z^n}$$

where $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1-\alpha$

show that coefficient bound given by (2) is sharp.

The function of the form (2.3) are in the class f $\in \mathcal{G}_{\mathcal{H}}(\alpha,)$,because

$$\sum_{n=2}^{\infty} [|n^2(\beta+1) - (\beta+\alpha)(\lambda_n - \lambda+1)|] \frac{|x_n|}{n^2(\beta+1) - (\beta+\alpha)(\lambda_n - \lambda+1)}$$
$$+ \sum_{n=1}^{\infty} [|n^2(\beta+1) - (\beta+\alpha)(\lambda_n - \lambda+1)|] \frac{|y_n|}{n^2(\beta+1) - (\beta+\alpha)(\lambda_n - \lambda+1)}$$

$\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1-\alpha$

The restriction placed in Theorem (2.1) on the moduli of the coefficients of f= h $+\bar{g}$ enables us to conclude for arbitrary rotation of the coefficients of f that the resulting functions would still be harmonic univalent and f $\in \mathcal{G}_{\mathcal{H}}(\alpha,)$

In the following theorem, it is shown that the condition (2.1) is also necessary for functions in f $\in \mathcal{G}_{\mathcal{H}}(\alpha,)$

THEOREM 2.

Let f= h + $\bar{g}$ with h and g be given by (1.2) .then f $\in \mathcal{G}_{\mathcal{H}}(\alpha,)$ *if and only if*

$$`\sum_{n=1}^{\infty}[n^2(\beta+1)-(\beta+\alpha)(\lambda^n-\lambda+1)]\,|a_n|+\sum_{n=1}^{\infty}[n^2(\beta+1)-(\beta+\alpha)(\lambda^n-\lambda+1)]\,|b_n|\leq 2\text{-}\alpha \quad (2.4)$$

.Where a_1 =1, 0≤ $\alpha < 1$ then f is harmonic univalent in Δ and f $\in \mathcal{G}_{\mathcal{H}}(\alpha,)$

3.**Extreme points**

In the following theorem, we obtain the extreme points of the class f $\in \mathcal{G}_{\mathcal{H}}(\lambda,\alpha,\beta)$

Theorem 3.. Let be given by (3). Thenif and only if can be expressed as

f(z) = $\sum_{n=1}^{\infty}(\mu_n h_n(z)+\delta_n g_n(z))$ $(z\ belongs\ to\ U)$ where$h_1(z)=z$

$$h_n(z)=z$$

$$h_n(z)=z-\frac{1-\alpha}{[n^2(\beta+1)-(\beta+\alpha)(\lambda^n-\lambda+1)}z^2(\text{ n }=2,3,\ldots\ldots)$$

And

$$h_n(z)=z-\frac{1-\alpha}{[n^2(\beta+1)-(\beta+\alpha)(\lambda^n-\lambda+1)}\overline{(z)}^n(\text{ n }=2,3,\ldots\ldots)$$

$$\sum_{n=1}^{\infty}(\mu_n+\delta_n)=1\,,h_n(z)$$

$\mu_n \geq 0\ and\ \delta_n \geq 0$

In particular , the extreme of f $\in \mathcal{G}_{\mathcal{H}}(\alpha,)$are $\{h_n\}and\ \{g_n\}$

4. Convex combination

Now, we show f $\in \mathcal{G}_{\mathcal{H}}(\alpha,)$ is closed under convex combination of its members.

Theorem (1) The class f$\in \mathcal{G}_{\mathcal{H}}(\alpha,)$ is closed underconvex combination.
Corollary (2).. The class f$\in \mathcal{G}_{\mathcal{H}}(\lambda,\alpha,\beta)$ is a convex set.

5. Distortion and growth theorems We introduce the distortion theorems for the functions in the class . f∈ $\mathcal{G}_{\mathcal{H}}(\alpha,)$

Theorem 1.let f∈ $\mathcal{G}_{\mathcal{H}}(\alpha,)$. then for $|z| = r < 1$, we have

$$|f(z)| \geq (1-b_1)r - \frac{\left(1-\alpha(1-b_1)\right)}{4(\beta+1)-(\beta+\alpha)(\lambda+1)}r^2$$

And

$$|f(z)| \leq (1-b_1)r + \frac{\left(1-\alpha(1-b_1)\right)}{4(\beta+1)-(\beta+\alpha)(\lambda+1)}r^2$$

Theorem 2.let f∈ $\mathcal{G}_{\mathcal{H}}(\alpha,)$. then for $|z| = r < 1$, we have

$$|f'(z)| \geq (1-b_1)r - \frac{2\left(1-\alpha(1-b_1)\right)}{4(\beta+1)-(\beta+\alpha)(\lambda+1)}r$$

And

$$|f'(z)| \leq (1-b_1)r + \frac{2\left(1-\alpha(1-b_1)\right)}{4(\beta+1)-(\beta+\alpha)(\lambda+1)}r$$

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