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Domination Polynomial of a Corona Graph

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Abstract: *Let the graph $G = (V, E)$ be a simple undirected graph of order $|V(G)| = n$. A set $S \subseteq V(G)$ is a dominating set of G, if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. A domination polynomial of a graph G is $D(G,x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G,i)x^i$, where $d(G,i)$ is the number of all dominating sets of G with size i and $\gamma(G)$ is the minimum domination number of G. In this paper, we construct the dominating set of $P_n o\, mK_1$, and obtain the domination polynomial $D(P_n \circ mK_1, x) = Pn\, i = \sum_{i=\gamma(P_n o\, mK_1)}^{|V(G)|} d(P_n \circ$ mK1, ixi, which we call an ordinary generating function for the dominating sets of graph Pn∘ mK1.*

Keywords: *Dominating sets, Domination polynomials, Corona graph, Ordinary generating function*
Mathematics subject classification: *05C30, 05C31, 05C69*

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I. Introduction and preliminary results

All of the graphs examined in this paper are finite and simple graph. In an undirected graph $G = (V, E)$, let $V(G)$ denote the set of all vertices of G and let $E(G)$ denote the set of all edges of G. The collected works on the subject of domination parameters in graphs has been discussed in this book [8]. A partition of $V(G)$ such that each class is a dominating in G is called a dogmatic partition of G. We have invariant polynomials for graphs in graph theory. For any vertex $v \in V(G)$, the open neighbourhood of v is the set $N(v) = u \in V(G)|uv \in E(G)$ and the closed neighbourhood of v is the set $N[v] = N(v) \cup v$. For a set $S \subseteq V(G)$, the open neighborhood of S is $N(S) = \cup_{v \in S} N(V)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A subset $S \subseteq V(G)$ is a dominating set if $N[S] = V$. In a simple graph G a dominating set S is a minimal dominating set in G if no proper subset $S' \subset S$ is a dominating set. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G [7]. We call such a set a $\gamma(G)$ −set of G. Let $D(G, i)$ be the family of dominating sets of a graph G with cardinality i and $d(G, i) = |D(G, i)|$. A domination polynomial of a graph G is $D(G, x) = \sum_{i=\gamma(G)}^{n} d(G, i)x^i$, where $d(G, i)$ is the number of all dominating sets of G with size i and $\gamma(G)$ is the minimum domination number of G [1, 6]. Our objective in this paper is to study the domination sets and its polynomial of the graph $P_n \circ mK_1$. The domination polynomial $D(G, x)$ is an ordinary generating function for the dominating sets of undirected graph G with respect to their cardinalities. Domination sets and its polynomials of the path, cycles, cubic graphs and some of special graphs were studied by Alikhani and Peng [1, 2, 6]. The mean value for the matching and graph products of dominating polynomial were studied in [4, 8]. The corona of two graphs

G_1 and G_1 as defined by Frucht and Harary[10] is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_1 where the ith vertex of G1G_1 is adjacent to every vertex in the i[th] copy of G_2. In the following section for the graph $P_n \circ mK_1$ the dominating set and its polynomial is driven.

Theorem 1.1. [1] Let S_n be a star graph n. Then the domination polynomial is $D(S_n, x) = x^n + (1 + x)^n$.

Theorem 1.2. [1] If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then
$D(G; x) = D(G_1, x)D(G_2, x)\ldots D(G_m, x)$

Theorem 1.3. [6] For every natural number n, $D(K_n, x) = (1 + x)^{n-1}$

Theorem 1.4 [2, 6] Let G be a graph with $|V(G)| = n$. Then

(i). If G is connected, then $d(G, n) = 1$ and $d(G, n - 1) = n$, where $d(G, n)$ is number of dominating set in graph G

(ii). $d(G, i) = 0$ if and only if $i < \gamma(G)$ or $i > n$.

(iii). $D(G, x)$ has no constant term.

(iv). $D(G, x)$ is a strictly increasing function in [0, ∞].

(v) Zero is a root of $D(G, x)$, with multiplicity $\gamma(G)$.

II. Main Results

Definition: 2.1. A $P_n \circ mK_1$ is a graph obtained by taking the corona of a path P_n with mK_1 and has $n(m+1)$ vertices, where m is the number of pendant vertices in each vertex of the path P_n.

Lemma 2.1. If $P_n \circ mK_1$ be a graph with $n(m+1)$ vertices, then $\gamma(P_n \circ mK_1) = n$, for all $m, n \in N$.

Proof: Let S be the dominating set of $P_n \circ mK_1$. Then by definition, for all i, $v_i \in S$, where $n \leq i \leq mn$. This implies that $|S| \geq n$. If S is γ-set of $P_n \circ mK_1$, then S exactly contain all vertices of the path P_n. Therefore, $\gamma(P_n \circ mK_1) = n$, for $m, n \in N$.

Theorem 2.1. Let $P_2 \circ mK_1$ be a graph with $2(m+1)$ vertices. Then $D(P_2 \circ mK_1, x) = x^2(1+x)^{2m} + 2x^{m+1}[1 + x((1+x)^m - 1)] + x^{2m}$.

Proof: Let $P_2 \circ mK_1$ be a graph and $D(P_2 \circ mK_1, i)$ be a family of dominating set with cardinality $i, 2 \leq i \leq 2(m+1)$. Let $V = \{v_1, v_2, u_{1,1}, u_{1,2}, \dots, u_{1,m}, u_{2,1}, u_{2,2}, \dots, u_{2,m}\}$ be the vertex set of G, where v_1, and v_2 are the vertices of P_2 and m is the number of pendant vertices to the vertices of path P_2. Since every pendant vertices of $P_2 \circ mK_1$ are adjacent to either v_1, or v_2. Hence, for $i = 2$ or $\{v_1, v_2\}$ the $\gamma(P_2 \circ mK_1) = 2$ and its polynomial is x^2. For the cardinality $i = 3$ the family of dominating set $D(P_2 \circ mK_1, 3)$ is obtained by selecting vertices v_1, v_2 and one vertex from $\{u_{1,1}, u_{1,2}, \dots, u_{1,m}, u_{2,1}, u_{2,2}, \dots, u_{2,m}\}$. Thus $d(P_2 \circ mK_1, 3) = \binom{2m}{1}$ and the domination polynomial term $\binom{2m}{1}x^3$. If we proceed like this for the remaining cardinality $i, 4 \leq i \leq 2(m+1)$. Then the domination polynomial of the graph is

$$\begin{aligned} D(G_1, x) &= x^2 + \binom{2m}{1}x^3 + \dots + \binom{2m}{2m}x^{2m+2} && (1) \\ &= x^2\left[\binom{2m}{1}x + \dots + \binom{2m}{2m}x^{2m}\right] && (2) \\ &= x^2(1+x)^{2m} && (3) \end{aligned}$$

The family of dominating set $D(P_2 \circ mK_1, i)$ is obtained also by selecting one of vertex from path P_2 and the remaining vertices taken from the either of the pendant vertices. Let us take vertex v_1, and all pendant vertices to v_2, which are dominating the graph $P_2 \circ mK_1$ and choose the remaining dominating vertices from the $\{v_2, u_{1,1}, u_{1,2}, \dots, u_{1,m}\}$ and similarly we can take for vertex v_2 and all pendant vertices to v_1 which are dominating the graph $P_2 \circ mK_1$. For the cardinality i, such that $m+1 \leq i \leq 2(m+1)$ the number of dominating set for $i = m+1, m+2$ are 1 and $\binom{m}{1}$ respectively. In general, the domination polynomial of the graph is

$$\begin{aligned} D(G_2, x) &= 2\left[x^{m+1} + \binom{m}{1}x^{m+2} + \dots + \binom{m}{m}x^{2m+2}\right] && (4) \\ &= 2x^{m+1} + 2x^{m+2}\left[(1+x)^m - 1\right] && (5) \\ &= 2x^{m+1}\left[1 + x((1+x)^m - 1)\right] && (6) \end{aligned}$$

For none of the vertices of the path P_2 are dominating set and for the cardinality $i = 2m$. Hence the domination polynomial of the graph is

$$D(G_3, x) = x^{2m} \qquad (7)$$

Therefore, from the equation (3), (6) and (7) we get the domination polynomial of $D(P_2 \circ mK_1) = D(G_1, x) + D(G_2, x) + D(G_3, x)$ and this completes the proof.

Theorem 2.2 Let $P_n \circ mK_1$ be graph with $n(m+1)$ vertices. Then

$$D(P_n \circ mK_1, x) = x^n(1+x)^{nm} + (1+x)x^{m+n-1}\sum_{i=1}^{n}(1+x)^{m(n-i)} + \dots + x^{nm}$$

Proof: Let $G = P_n \circ mK_1$ be a graph and $D(G, i)$ be a family of dominating set with cardinality $i, n \leq i \leq n(m+1)$. Let the vertex set V(G) is

$\{\{v_1, v_2, \ldots, v_n\}, \{u_{1,1}, u_{1,2}, \ldots, u_{1,m}\}, \{u_{2,1}, u_{2,2}, \ldots, u_{2,m}\}, \ldots, \{u_{n,1}, u_{n,2}, \ldots, u_{n,m}\}\}$, where $\{v_1, v_2, \ldots, v_n\}$ are the vertices of the path P_n and nm is the number of pendant vertices to the vertices of path P_n. Since every nm number of pendant vertices of G are adjacent to either vertices of P_n. Hence, for the cardinality $i = n$ the $\gamma(G) = n$ and its polynomial is x^n. The next family of dominating set is $D(G, n+1) = \{v_1, v_2, \ldots, v_n, v_{j,k}\}$ such that the vertex $u_{j,k}$ is chosen from nm numbers of pendant vertices of the path P_n, where $1 \leq j \leq n$ and $1 \leq k \leq m$. Hence $d(G, n+1) = \binom{nm}{1}$ and its domination polynomial term is $\binom{nm}{1} x^{n+1}$.
By proceeding this way for the remaining cardinality i such that $n + 2 \leq i \leq n(m+1)$.
Then the domination polynomial of the graph yields,
In the other case one of the vertices of the path P_n is not contained in the domination set of G. Suppose $v_1 \in$

$$D(G_1, x) = x^n + \binom{nm}{1} x^{n+1} + \ldots + \binom{nm}{nm} x^{nm+n} \quad (8)$$
$$= x^n \left[1 + \binom{nm}{1} x + \ldots + \binom{nm}{nm} x^{nm}\right] \quad (9)$$
$$= x^n (1+x)^{nm} \quad (10)$$

$V(P_n)$ is not contained in dominating sets of the graph G. This is provided by combining $\{v_2, v_3, \ldots, v_n\}$ with m pendant vertices which has minimum cardinality $i = m + n - 1$ and hence $d(G_2, m + n - 1) = 1$, and the domination polynomial of the term x^{m+n-1}. For the proceeding cardinality $i = m + n$ we choose one

$$D(G_2^*, x) = x^{m+n-1} + \binom{m(n-1)+1}{1} x^{m+n} + \ldots + \binom{m(n-1)+1}{m(n-1)+1} x^{m(n+1)} \quad (11)$$
$$= x^{m+n-1} \left[1 + \binom{m(n-1)+1}{1} x + \ldots + \binom{m(n-1)+1}{m(n-1)+1} x^{m(n-1)+1}\right] \quad (12)$$
$$= x^{m+n-1} (1+x)^{m(n-1)+1} \quad (13)$$

vertex from the pendant vertices except the vertices pendant to the vertex of v_1. Thus, the domination polynomial of the term is $\binom{nm-m}{1} x^{m+n}$. Therefore, the domination polynomial for v_1 is

From equation (13), we generate for the overall domination polynomial cases that are obtained by investigating the situations of each n vertices of the path $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ Therefore, the general domination polynomial of the second case is given as,
For the remaining third case we suppose that none of the vertices of $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ is contained in the dominating set of the graph G. Hence, for cardinality $i = nm$ the family of dominating set $D(G, nm) = 1$ and

$$D(G_2, x) = (1+x) x^{m+n-1} \sum_{i=1}^{n} (1+x)^{m(n-i)} \quad (14)$$

the domination polynomial term of this case is

Therefore, the equation (10), (14) and (15) above

$$D(G_3, x) = x^{nm} \quad (15)$$

completes the proof.

Definition: 2.2. [7] A graph $C_n o\, mK_1$ is a simple corona graph with $m(n+1)$ order of vertices. The vertex set $\{v_1, v_2, v_3, \ldots, v_n, v_1\}$ of a cycle C_n is joined with m number of pendant vertices to each vertex of the cycle C_n, where $m > 1$ and the sets $\{u_{1,1}, u_{1,2}, \ldots, u_{1,m}\}, \{u_{2,1}, u_{2,2}, \ldots, u_{2,m}\}, \ldots, \{u_{n,1}, u_{n,2}, \ldots, u_{n,m}\}$ are pendant vertices

Theorem 2.3 If $C_3 o\, mK_1$ is the graph with $3(m+1)$ vertices, then $D(C_3 o\, mK_1, x) = x^3(1+x)^{3m} + 3x^{m+2}[(1+x)^{2(m+1)} + x^{m-1}(1+x)^{m+1}] + x^{3m}$

Proof: Let $H = (C_3 o\, mK_1)$ be the family of dominating set with cardinality i such that $3 \leq i \leq 3(n + 1)$.

Let the vertex set $V(H)$ of the graph H is $\{\{v_1,v_2,v_3\},\{u_{1,1},u_{1,2},\ldots,u_{1,m}\},\{u_{2,1},u_{2,2},\ldots,u_{2,m}\},\ldots,\{u_{n,1},u_{n,2},\ldots,u_{n,m}\}\}$ and $3m$ number of pendant vertices to the graph of H. Herewith, we proceed four cases to prove the theorem.

Case-1. Let consider the vertices v_1, v_2, and v_3 of the cycle C_3 and it belongs to the dominating set of the graph H. Hence, for the size $i = 3$ the $d(H,3) = 1$ and its polynomial term is x^3. For the preceding size $i = 4$, the $d(H,4)$ is obtained by considering all vertices of the cycle C_3 and selecting one vertex from $3m$ number of pendant vertices to the vertices of cycle C_3. Thus, the dominating sets and its polynomial are $|D(H,4)| = \binom{3m}{1}$ and $\binom{3m}{1}x^4$ respectively. Similarly, we can proceed for the remaining cardinality i such that $5 \le i \le 3(m+1)$. Therefore, the ordinary generating function of the graph case-1 yields,

$$x^3 + \binom{3m}{1}x^4 + \ldots + \binom{3m}{3m}x^{3m+3} \tag{16}$$

$$= x^3\left[1+\sum_{i=1}^{3m}\binom{3m}{i}x^i\right] \tag{17}$$

$$= x^3(1+x)^{3m} \tag{18}$$

Case-2. Let for the vertex set $\{v_1,v_2,v_3\}$ of the cycle C_3 one of the vertex is not contained in the dominating sets of the graph H. Suppose v_1 is not contained in the dominating sets. Now by combining the vertex set $\{v_2,v_3\}$ and the m number of pedant vertices to the vertex v_1 to find the dominating set for the cardinality $m+2 \le i \le 3(m+1)$. Let for $i = m+2$ the $d(H,m+2) = 1$ and its polynomial term is x^{m+2}. For the size $i = m+3$ the $d(H,m+3)$ is computed by choosing one vertex from $\{v_2,v_3,\{u_{2,1},u_{2,2},\ldots,u_{2,m}\},\ldots,\{u_{3,1},u_{3,2},\ldots,u_{3,m}\}$. Hence the $d(H,m+3) = \binom{2m+2}{1}$. In the same way we can find the family dominating sets by avoiding the vertex v_2 or v_3. Therefore, the domination polynomial given by;

$$3\left[x^{m+2} + \binom{2m+2}{1}x^{m+3} + \ldots + \binom{2m+2}{2m+2}x^{3(m+1)}\right] \tag{19}$$

$$= 3x^{m+2}\left[1+\sum_{i=1}^{2(m+1)}\binom{2(m+1)}{i}x^i\right] \tag{20}$$

$$= 3x^{m+2}(1+x)^{2(m+1)} \tag{21}$$

$$3\left[x^{2m+1} + \binom{m+1}{1}x^{2m+2} + \ldots + \binom{m+1}{m+1}x^{3m+3}\right] \tag{22}$$

$$= 3x^{2m+1}\left[1+\sum_{i=1}^{m+1}\binom{m+1}{i}x^i\right] \tag{23}$$

$$= 3x^{2m+1}(1+x)^{m+1} \tag{24}$$

Case-3: For the vertex set $\{v_1,v_2,v_3\}$ of the cycle C_3, two of the vertex are not contained in the dominating sets of the graph H. Suppose v_1 and v_2 are not contained in the dominating sets. Now considering in the domination sets the vertex v_3 and the $2m$ number of pedant vertices to the vertices of v_1 and v_2 for the cardinality $2m+1 \le i \le 3(m+1)$. Let for size $i = 2m+1$ obviously, the $d(H,2m+1) = 1$ and its polynomial term is x^{2m+1}. But for the size $i = 2m+2$ the dominating number is obtained by choosing one vertex from $\{v_3,u_{3,1},u_{3,2},\ldots,u_{3,m}\}$. Thus, the $d(H,2m+2) = \binom{m+1}{1}$ and its term of polynomial is $\binom{m+1}{1}x^{2m+2}$. Since we have three distinct possible ways to avoid two vertices from the vertex set $\{v_1,v_2,v_3\}$ in the dominating sets. Therefore, the domination polynomial is given as,

Case-4. None of the vertex set $\{v_1, v_2, v_3\}$ of the cycle C_3 are contained in the dominating sets of the graph H. Hence, for cardinality $i = 3m$ the family of dominating set $D(H, 3m) = 1$ and the domination polynomial term of this case is;

$$D(H', x) = x^{3m} \qquad (25)$$

Therefore, from the equation (18), (21), (24) and (25) we get the domination polynomial of the graph $H = C_3 o\, mK_1$ and this completes the proof.

III. Conclusion

In this paper, we have shown that for a simple connected graph $P_n \circ mK_1$ the dominating sets and its ordinary generating function. Along the way, we found the domination polynomial for the corona product of the graphs $P_2 \circ mK_1$ and $C_3 \circ mK_1$.

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