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Numerical Approach of Fractional Integral Operators on Heat Flux and Temperature Distribution in Solid

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Abstract: *This paper is about the study of heat flux/thermal flux $\emptyset_q$ and temperature distribution in solid medium. The work was carried out by transforming the second order heat equation into fractional differential equation and solved by using the integral transform and their fractional derivatives. The results are conveyed in term of wright function $W(\alpha,\beta;z)$ which is in graceful compact form suitable for numerical computation. Some numerical results are also pointed out.*

Key Words: *Mittage Leffler function, Wright function, Error function, Fourier sine transform, Laplace transform, Heat flux*

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I. INTRODUCTION

In thermal engineering heat transfer is concern with exchange of thermal energy between physical systems. Heat flux is the rate of thermal energy flow per unit surface area of heat transfer on surface. A semi-infinite solid is an idealized body that has single plane surface and extend to infinity in all directions. The idealized body is used to indicate that the temperature change in the part of body due to the thermal conditions on a single surface [1]. For example, the earth, thick wall, steel piece of any shaped quenched rapidly etc. in determining variation of temperature near its surface and other surface being too far to have any impact on the region in short period of time since heat doesn't have sufficient time to penetrate deep into body thus thickness can be neglected [2].

A good understanding of temperature and heat flux is important for the fire test in building structures and aerospace industries [3].

II. PRELIMINARIES

Initially at temperature T_0 consider a semi-infinite solid. Suddenly the temperature of the one face of the solid is raised up to temperature T_s at time zero. Defining $V=\frac{T-T_0}{T_s-T_0}$. If we presume that thermal conductivity is constant, no internal heat generation and negligible temperature variations in the both y and z directions, then appropriate differential equation is given by classical non-homogenous heat equation defined by Mills and Ganesan [4]

$$\frac{\partial V}{\partial t}=\lambda\frac{\partial^2 V}{\partial x^2}$$

where λ is the thermal diffusivity, subject to suitable boundary conditions.

$$\left.\begin{aligned} &when\ time\ t=0; temperature\ V=0\\ &at\ the\ surface\ x=0; temperature\ V=1, and\\ &when\ surface\ x\to\infty; \text{temperature V} = 0.\end{aligned}\right\}$$

The following facts are considered to study the heat flux and temperature distribution in the semi infinite solid medium.

The Laplace transform for the function $f(x)$, for all real numbers $x\geq 0$ defined by [5]

$$L\{f(x),S\}=\int_0^\infty e^{-st}f(t)dt=F(s);\ \text{Re}(s)>0$$

Where s is a complex frequency parameter.

The Fourier sine transform is given by [6]

$$F_s(n,t)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x,t)sinnx\,dx$$

The Mittage-Leffler function $E_{\alpha,\beta}(z)$ is defined by [7, 8]

$$E_{\alpha,\beta}(z)=\sum_{k=0}^\infty\frac{z^k}{\Gamma(\beta+\alpha k)},\alpha\in C,z\in C,R(\alpha)>0,\beta>0$$

In series form the Wright function $W(\alpha,\beta;z)$ is represent by [10]

$W(\alpha,\beta;z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\beta+\alpha m)m!}$ ...(1)

The Laplace transform of fractional deferential operator ${}_0D_x^{\alpha,\beta} f(x)$ is given by [11]

$L[\,{}_0D_x^{\alpha,\beta} f(x); s] = s^{\alpha}\tilde{f}(s) - s^{\beta(\alpha-1)} I_{0+}^{(1-\alpha)(1-\beta)} f(0+); 0<\alpha<1$ ...(2)

The error function $erf(x)$ is given by [12]

$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$

The complementary error function $erf(x)$ is given by

$erfc\,(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$

The relation between complementary error function and Wright function is defined by

$W\left(-{}^1\!/_2, 1; z\right) = erfc\left(\frac{z}{2}\right)$ ...(3)

The following integral within limit $n \to 0, n \to \infty$ is required for solution [1]

$\int_0^{\infty} n \sin nx\, E_{\alpha,\alpha+1}(-n^2Kt^2)\, dn = W\left(-{}^{\alpha}\!/_2\,; 1; \frac{-x}{\sqrt{Kt^{\alpha}}}\right)$

III. SHAPE OF THE MATHEMATICAL MODELING

Now, introducing a new model based on the transfer of heat energy on solid surface by conjunction of fractional differential operator with heat equation. We establish an expression for the temperature and heat flux per unit area of solid surface. Initially temperature of surface is T_0 with condition $V = \frac{T-T_0}{T_s-T_0}$.

We present a method based on Laplace transform for deriving the solution of the unified fractional heat equation

$D_t^{\alpha,\beta} V(x,t) = \lambda \frac{\partial^2 V}{\partial x^2}\,;\ 0<\alpha<1, t>0, x \in \mathrm{R}$ ...(4)

Theorem 3.1: If $D_t^{\alpha,\beta} V\,(x,t) = \lambda\, V''(x), (\alpha>0, \beta>0, \lambda>0, x \in R)$ with boundary condition $V(x, t=0) = 0$ and $V(x=0,t) = 1\,, x \in R, \lim_{x\to 0} V(x,t) = 0$, then

$V(x,t) = 1 - \frac{\frac{x}{\sqrt{\lambda t^{\alpha}}}}{\Gamma\left(\frac{-\alpha}{2}+1\right)1!} + \frac{\left(\frac{x}{\sqrt{\lambda t^{\alpha}}}\right)^2}{\Gamma(-\alpha+1)2!} - \cdots = W\left(-{}^{\alpha}\!/_2, 1; \frac{-x}{\sqrt{\lambda t^{\alpha}}}\right)$ ...(5)

where W (.) is wright function.

Proof: We know that by applying Fourier sine transform on the equation (4), gives

$D_t^{\alpha,\beta} V_s(n,t) = \lambda \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial^2 V}{\partial x^2} sinnx\, dx$

Integrating by parts, and using boundary conditions

$$D_t^{\alpha,\beta} V_s(n,t) = \lambda \sqrt{\frac{2}{\pi}} \left\{sinnx \frac{\partial V}{\partial x}\right\}_0^{\infty} - n\lambda \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial V}{\partial x}\, cosnx\, dx$$

$$= \lambda \sqrt{\frac{2}{\pi}}(0) - n\lambda \sqrt{\frac{2}{\pi}} \{cosnx.V\}_0^{\infty} - \int_0^{\infty} - sinnxnV\, dx$$

$$= 0 - n\lambda \sqrt{\frac{2}{\pi}}\{0-1\} - n^2\lambda \sqrt{\frac{2}{\pi}} \int_0^{\infty} V\, sinnx\, dx$$

$$= n\lambda \sqrt{\frac{2}{\pi}} - n^2\,\lambda V_s(n,t) \qquad \text{...(6)}$$

Taking Laplace transform of equation (6)

$L\left\{D_t^{\alpha,\beta} V_s(n,t); s\right\} = L\left\{n\lambda \sqrt{\frac{2}{\pi}} - n^2\,\lambda V_s(n,t)\right\}$

Using equation (2), we get

$s^{\alpha}\widetilde{V}_s(n,s) - s^{\beta(\alpha-1) I_{0+}^{(1-\alpha)(1-\beta)}} \widetilde{V}_s(0+) = n\lambda \sqrt{\frac{2}{\pi}} \frac{1}{s} - n^2\,\lambda \widetilde{V}_s(n,s)$

On simplification (5) it gives

$\widetilde{V}_s(n,s) = n\lambda \sqrt{\frac{2}{\pi}} \left[\frac{1}{s(s^{\alpha}+n^2\lambda)}\right]$ ...(7)

On considering inverse Laplace transform of equation (7)

$L^{-1}\left[\widetilde{V}_s(n,s)\right] = n\lambda \sqrt{\frac{2}{\pi}} L^{-1}\left[\left[\frac{1}{s(s^{\alpha}+n^2\lambda)}\right]\right]$

We get,

$$V_s(n,t) = n\lambda\sqrt{\frac{2}{\pi}}t^\alpha\, E_{\alpha,\alpha+1}(-n^2\lambda t^\alpha) \qquad \ldots(8)$$

Now, taking inverse Fourier sine transform of equation (8),

$$V(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty V_s(n,t)\; sinnx\; dn$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty n\lambda\sqrt{\frac{2}{\pi}}t^\alpha\, E_{\alpha,\alpha+1}(-n^2\lambda t^\alpha)\; sinnx\; dn$$

$$= \frac{2}{\pi}t^\alpha\lambda\int_0^\infty n\sqrt{\frac{2}{\pi}}\, E_{\alpha,\alpha+1}(-n^2\lambda t^\alpha) sinnx\; dn$$

The integral on the right-hand side is equal to W(.) which is defined in equation (5). This completes the proof of theorem-1.

Corollary: Let $\alpha = 1$, then the solution of the heat equation $D_t^{\alpha,\beta}V(x,t) = \lambda\, V''(x), (\alpha > 0,\ \beta > 0\ ,\ \lambda > 0, x \in R)$ holds the following solution

$$V(x,t) = erfc\left(\frac{x}{2\sqrt{\lambda t^\alpha}}\right)$$

Where $erfc(.)$ Is the complementary error function given in the equation (3)

Heat flux has significant influence on the heat release rate per unit area per unit time on infinite solid.

Theorem-3.2 Let $D_t^{\alpha,\beta}V(x,t) = \lambda V''(x), (\alpha > 0, \beta > 0, \lambda > 0, x \in R)$ with boundary conditions $V(x,t=0) = 0\ , V(x=0,t) = 1\ , x \in R,\ \lim_{x\to 0} V(x,t) = 0$, and

$$V(x,t) = 1 - \frac{\frac{x}{\sqrt{\lambda t^\alpha}}}{\Gamma\left(\frac{-\alpha}{2}+1\right)1!} + \frac{\left(\frac{x}{\sqrt{\lambda t^\alpha}}\right)^2}{\Gamma(-\alpha+1)2!} - \cdots = W\left(-{}^{\alpha}/_{2}\,, 1; \frac{-x}{\sqrt{\lambda t^\alpha}}\right)$$

where W(.) is Wright function. The heat flux at a certain point $x = 0$ at the surface of solid is

$$\Phi_q = \frac{C(T_s - T_0)}{\sqrt{\lambda t^\alpha}\,\Gamma\left(1-\frac{\alpha}{2}\right)}; c > 0, \alpha > 0, \lambda > 0 \qquad \ldots(9)$$

Proof: We know that the heat flux at the surface of the solid is (5)

$$\Phi_q = -C\left[\frac{\partial T}{\partial X}\right]$$

By using the result of theorem- 1

$$\Phi_q = -C\frac{\partial}{\partial X}\left\{T_0 + (T_s - T_0)\, W\left(-{}^{\alpha}/_{2}\,, 1; \frac{-x}{\sqrt{\lambda t^\alpha}}\right)\right\}$$

The function $W\left(-{}^{\alpha}/_{2}\,, 1; \frac{-x}{\sqrt{\lambda t^\alpha}}\right)$ is defined in the equation (1)

$$\Phi_q = -C\left\{0 + (T_s - T_0)\frac{\partial}{\partial X}\left(\sum_{m=0}^\infty \frac{\left(\frac{-x}{\sqrt{\lambda t^\alpha}}\right)^m}{\Gamma\left(\frac{-\alpha m}{2}+1\right)m!}\right)\right\}$$

On expending the function in right hand side

$$\Phi_q = -C(T_s - T_0) \times \frac{\partial}{\partial X}\left\{1 - \frac{\frac{x}{\sqrt{\lambda t^\alpha}}}{\Gamma\left(\frac{-\alpha}{2}+1\right)1!} + \frac{\left(\frac{x}{\sqrt{\lambda t^\alpha}}\right)^2}{\Gamma(-\alpha+1)2!} - \cdots\right\}$$

On differentiating partially with respect to t, we have

$$\Phi_q = -C(T_s - T_0)\left\{0 - \frac{\frac{1}{\sqrt{\lambda t^\alpha}}}{\Gamma\left(\frac{-\alpha}{2}+1\right)1!} + \frac{2\frac{x}{\sqrt{\lambda t^\alpha}}}{\Gamma(-\alpha+1)2!} - \ \ldots\right\}$$

On the surface of solid by considering,$x = 0$, we get

$$\Phi_q = -C(T_s - T_0)\left(-\frac{\frac{1}{\sqrt{\lambda t^\alpha}}}{\Gamma\left(\frac{-\alpha}{2}+1\right).1}\right)$$

Finally, we get the desired result (9)

Special Case: At the time $t = 0$, the curve between heat flux and time has no critical (maximum/minimum) point. When time is zero the heat flux on the surface of solid is not defined.

IV. NUMERICAL ASSESSMENT

4.1: Evaluate the surface heat flux for 12 cm thick concrete firewall which has a black silicone paint surface. The wall is come close to black body at $900\ K$. It will take 2 minutes for the surface to reach $600\ K$. The initial temperature of the wall is $400\ K$.

Solution: Here we have given

$$T_s = 600\ K, T_0 = 400\ K$$

The required concrete properties are

$$C = 1.4\ W/m\ K, \qquad \lambda = 0.75 \times 10^{-6}\ m^2/s$$

and $t = 2\ min = 120\ seconds$

For particular $\alpha = \frac{1}{8}$, using the result of theorem 2, heat flux is evaluated as follow

$$\Phi_q = \frac{1.4\times(600-400)}{\sqrt{0.75\times10^{-6}\times(120)^{1/8}\Gamma 1-\left(\frac{1}{.8\times2}\right)}};\ \alpha = \frac{1}{8},$$

Or $$\Phi_q = \frac{1.4\times(200)}{\sqrt{0.75\times10^{-6}\times(120)^{\frac{1}{8}}\Gamma\left(\frac{15}{16}\right)}}$$

Or $$\Phi_q = 230447.13658\ W/m^2$$

It is observed that during heat transfer, heat flux is directly proportional to the difference of temperature between solid, liquid, or gaseous media. Under the process of conduction, the heat flux vector is directly proportional to and usually parallel to the temperature gradient vector. The heat flux formation due to radiation is a flux of electromagnetic radiation. In contrast to convection and heat conduction, it may occur without any intervening medium.

Now in second part of this paper we show that how heat flux behaves with time t.

V. RESULTS

On differentiating the obtained result of theorem -2 we have observed the nature of heat flux with respect to time is inversely and in decreasing in nature as show in following steps

$$\frac{d\Phi_q}{dt} = \frac{C(T_S-T_0)}{\Gamma\left(1-\frac{\alpha}{2}\right)}\frac{d\left[(\lambda t^{\alpha})^{-\frac{1}{2}}\right]}{dt}$$

$$= \frac{C(T_S-T_0)}{\Gamma\left(1-\frac{\alpha}{2}\right)\sqrt{\lambda}}\frac{dt^{-\frac{\alpha}{2}}}{dt}$$

$$= \frac{C(T_S-T_0)}{\Gamma\left(1-\frac{\alpha}{2}\right)\sqrt{\lambda}}\left[\frac{-\alpha}{2}t^{-\frac{\alpha}{2}-1}\right]$$

$$= \frac{C(T_S-T_0)}{\Gamma\left(1-\frac{\alpha}{2}\right)\sqrt{\lambda}}\left[\frac{-\alpha}{2}t^{-\left(\frac{\alpha}{2}+1\right)}\right]$$

$$\Rightarrow \frac{d\Phi_q}{dt} = \frac{C(T_S-T_0)}{\Gamma\left(1-\frac{\alpha}{2}\right)\sqrt{\lambda}t^{\left(\frac{\alpha}{2}+1\right)}}\left(\frac{-\alpha}{2}\right)$$

So here we have studied, that the accelerate heat flux behaves negatively in direction with time. When we consider $\alpha = \frac{1}{8}$, time, $t = 2min$

$\Phi_q = -120.02189941$

If we take $\Phi_q = f(t)$, with conditions $|\Phi_q| > 1$and $|\Phi_q{}'| > 1$then such scheme can be adopted for the numerical results through numerical analysis.

VI. CONCLUSIONS

In this paper, we have investigated the constitutive relationship model in the form of generalizing the phenomena of heat flow through infinite solid medium. We have obtained the solution of fractional partial differential heat equation in the form of well-known Wright $W(\alpha,\beta;z)$ function by using inverse Laplace transform, Fourier sine transform and their fractional derivatives with the help of boundary value conditions. The solution obtained also in term of complementary error function for $\alpha = 1$.The performance of the complete procedure is exemplified with example. In the last section we have focused on accelerate heat flux which is to be found negative in nature for $0 < \alpha < 1$.The adaption and extension of this proposed model is possible in numerical approximation, pure chemistry, thermodynamics etc.

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