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Proof of Generalized Continuum Hypothesis

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Abstract:
Background*: The continuum hypothesis was originally proposed by Georg Cantor. The continuum hypothesis has remained a prominent conjecture, mainly because the mathematical tools of truth and provability have been developed much later. The continuum hypothesis has been shown to be independent of the axiom of choice.*
Materials and Methods*: Besides proof and truth, the objects to which the mathematical statements apply need to be well-formulated. Inductive or recursive constructibility of the mathematical objects is a well-acknowledged prerequisite, for a mathematical theory to be consistent and verifiable. By recursion, only well-founded recursion is intended to be of opportuneness, in the construction of proofs and the mathematical objects to which the theorems refer.*
Results*: For inductively constructible sets, the continuum hypothesis as well as its generalization can be proved.*
Conclusion: *Well-founded recursion and inductive constructability are prerequisites for a mathematical theory to become free from paradoxes.*
Key Word*: Set Theory; Mathematical Induction; Well-founded Recursion;*

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I. Introduction

The continuum hypothesis was originally formulated by Georg Cantor. Since the time of Cantor's publication of set theory, the continuum hypothesis has remained a prominent mathematical conjecture, mainly because the mathematical tools of truth and provability have been developed much later. The continuum hypothesis has been shown to be independent of the axiom of choice.

Besides proof and truth, the objects to which the mathematical statements apply need to be well-formulated. Inductive or recursive constructability of the mathematical objects is a well-acknowledged prerequisite, for a mathematical theory to be consistent and verifiable. By recursion, only well-founded recursion is intended to be of opportuneness, in the construction of proofs and the mathematical objects to which the theorems refer.

Cantor's set theory lays out the basic operations on sets. There are a few basic sets, including the empty set, using which more complex can be constructed. However, there is no provision in Cantor's set theory, to conclude that any set is constructible only using these operations. This extra provision must be supplemented in the proofs about any general set that may occur as a properly constructible set. Though it is assumed to be so, it is not always explicitly stated, meaning that it is taken for granted that a set is somehow properly constructed, when one is referring to a set, in general. Otherwise, the prover may run into numerous paradoxes.

In formal logic, such a clause as being discussed in the preceding paragraph may appear like the phrase, "… the smallest collection, closed under the operations …". The danger here is the immediate self-referential question, as to whether the class of sets is a set by itself. It is also possible that this was the intuition behind the formulation of Zorn's maximality principle.

In this paper, inductive constructibility of sets is emphasized. Well-founded recursion can be converted into inductive construction.

For inductively constructible sets, the continuum hypothesis as well as its generalization can be proved. Well-founded recursion and inductive constructability are prerequisites for a mathematical theory to become free from paradoxes.

II. Material And Methods

Let $\mathbb{N}$ be the set of positive integers, and let $\mathbb{R}$ the set of real numbers. Any reference to recursion must be understood to be well-founded. For constructible sets, the following set operations are defined:

(1) <u>Union $\bigcup$</u>: for an inductively or recursively constructible index set I, and inductively or recursively constructible sets, A_i, $i \in I$, the set union is $\bigcup_{i \in I} A_i$,

(2) <u>Intersection $\bigcap$</u>: for an inductively or recursively constructible index set I, and inductively or recursively constructible sets, A_i, $i \in I$, the set intersection is $\bigcap_{i \in I} A_i$,

(3) Cartesian Product $\times$ or Π : for an inductively or recursively constructible index set I, and inductively or recursively constructible sets, A_i, $i \in I$, the set intersection is $\prod_{i \in I} A_i$,

(4) Power Set $\mathcal{P}$: for an inductively or recursively constructible set A, the set of all its subsets, denoted by $\mathcal{P}(A)$, is the power set of A, and

(5) Set Difference $A - B$ and Symmetric Difference $A \Delta B$: for two inductively or recursively defined sets A and B, $A - B$ is the collection of elements in A that are not in B, and $A \Delta B$ is the union of the two sets $A - B$ and $B - A$, i.e., $A \Delta B = A - B \cup B - A$.

The set complementation by itself is undefined. In item (4), the subsets of a set are not attempted to be defined further. If a single subset is required, it may have to be constructible appropriately. But, since the power set is taken as a whole, when referring to the subsets of a set, no further constructability conditions are implied in (4). The power set $\mathcal{P}(A)$ is also written as 2^A, as a subset can be identified with its binary valued indicator function defined on the set A.

The complementation of a set is undefined, unless a constructible superset is explicitly or implicitly specified, owing to the discovery of paradoxes originating from the notion of the set of all sets. Instead, for two inductively or recursively defined sets A and B, the set differences, $A - B$ and $B - A$, as well as the symmetric difference, $A \Delta B = B \Delta A$, are defined.

In view of the fact that unbounded recursion is mainly responsible for the unreasonableness of the paradoxes discovered, recursive or inductive constructability is specifically included as a prerequisite. In this paper, whenever a reference to a set is made, it is implicitly assumed that the set is inductively or recursively constructible. From the standpoint of formal logic, except for some few sets that are taken to be basic and subject to no further definability conditions, each set may need to be definable recursively or inductively, by means of functions, that are recursively or inductively constructible.

Two sets A and B may be compared with respect to a partial order $\preccurlyeq$, defined as follows: if there exists a one-to-one mapping from A into B or there exists a surjective mapping from B onto A, then $A \preccurlyeq B$. If $A \preccurlyeq B$, then the cardinality of A is no larger than the cardinality of B.

Proposition 1 *There is no surjective mapping from a finite set A onto the set of natural numbers $\mathbb{N}$, and there is no one-to-one mapping from the set of natural numbers $\mathbb{N}$ into a finite set A,* i.e., *the partial relation $\mathbb{N} \preccurlyeq A$ never holds, for a finite set A.*

Proof Both statements can be proved by induction the number of elements of A.

□

For an infinite set A, for some strictly proper subset $B \subseteq A$, it may hold that $A \preccurlyeq B$.

Proposition 2 (Cantor) *For any set A, there is no surjective mapping from A onto its power set $\mathcal{P}(A)$, and there is no one-to-one mapping from the power set $\mathcal{P}(A)$ into A,* i.e., *the partial relation $\mathcal{P}(A) \preccurlyeq A$ never holds, for any set A.*

Proof Cantor introduced the diagonal argument in the proof this proposition. □

Assumption 1 *There is no set B of cardinality strictly larger than that of every finite set, but strictly smaller than that of the set of natural numbers $\mathbb{N}$,* i.e., *if the partial relation $A \preccurlyeq B$ holds, for every finite set A, then the partial relation $\mathbb{N} \preccurlyeq B$ holds, too.*

Proposition 2 $\mathbb{R} \preccurlyeq \mathcal{P}(\mathbb{N}) = 2^{\mathbb{N}} \preccurlyeq \mathbb{R}$ *and* $\mathbb{N}^k \preccurlyeq \mathbb{N}$, *for every* $k \in \mathbb{N}$.

Proof Known to be standard results.

□

Proposition 3 *Among the sets inductively constructed so far, at any point of inductive construction, let A be an infinite set such that the cardinality of A is at least that of every other set constructed so far. Then, the set of strictly larger cardinality constructible in the next induction step is 2^A.*

Proof The set $\bigcup_{i \in I} A_i$, where $A_i \preccurlyeq A$, for $i \in I$, can be identified with $I \times A \preccurlyeq A^2$, whenever $I \preccurlyeq A$. If $A \preccurlyeq \mathbb{N}$ and A is infinite, then $A^2 \preccurlyeq A$. Now, after constructing sets of cardinality equal to that of $\mathbb{N}$, by a finite number of steps of induction, starting from the basic sets, any larger set can be constructed only by taking $2^{\mathbb{N}}$, which completes the proof. It is also true that $\mathbb{R} \preccurlyeq 2^{\mathbb{N}} \preccurlyeq \mathbb{R}$, and so $\mathbb{R}^{\mathbb{N}} \preccurlyeq 2^{\mathbb{N} \times \mathbb{N}} \preccurlyeq 2^{\mathbb{N}} \preccurlyeq \mathbb{R}$.

Otherwise, if $A \preccurlyeq 2^B$ and $2^B \preccurlyeq A$, for a constructible infinite set B, then $A^2 \preccurlyeq 2^{2 \times B} \preccurlyeq 2^B \preccurlyeq A$. If $B \preccurlyeq \mathbb{N}$ and B is infinite, then $2 \times B \preccurlyeq B$, and if B is of a larger cardinality, it must be inductively proved that the partial relation $2 \times B \preccurlyeq B$ holds. Therefore, the set $\bigcup_{i \in I} A_i$ cannot be of larger cardinality, for an infinite set A, with $A_i \preccurlyeq A$, for $i \in I$, and also $I \preccurlyeq A$. Since $2^A \preccurlyeq A^A = \prod_{i \in A} A_i$, where A_i is a copy of A, for every $i \in A$, the contention becomes obvious for the recursion to be well-founded. □

Inductively, it can be shown that $A^A \preccurlyeq \mathcal{P}(A)$, for every infinite set A.

III. Result

The main results of the paper are summarized in this section.

Theorem 1 (Special Continuum Hypothesis) For an inductively constructible set A, if the partial relation $\mathbb{N} \preccurlyeq A$ holds, but the partial relation $A \preccurlyeq \mathbb{N}$ cannot hold, then $\mathbb{R} \preccurlyeq A$.

Proof Since $\mathbb{N} \preccurlyeq A$, but it is not true that $A \preccurlyeq \mathbb{N}$, it may be assumed that, for any $k \in \mathbb{N}$, it is not true that $A \preccurlyeq \mathbb{N}^k$. The least cardinality of any inductively constructible set by rising $\mathbb{N}$ to an exponent set B, *i.e.*, by taking $\mathbb{N}^B$, for some inductively constructible set B, is when $\mathbb{N} \preccurlyeq B$, by Assumption 1 of the preceding section. But since $2^{\mathbb{N}} \preccurlyeq \mathbb{N}^{\mathbb{N}}$, the contention follows. □

Theorem 2 (Generalized Continuum Hypothesis) For two inductively constructible sets A and B, such that B is infinite, if the partial relation $A \preccurlyeq B$ holds, but the partial relation $B \preccurlyeq A$ cannot hold, then $\mathcal{P}(A) \preccurlyeq B$.

Proof Inductive proof is constructed, with the proof of preceding theorem as the induction basis step. □

IV. Discussion

Well-foundedness of recursion is an indispensable prerequisite for constructability. Well-founded recursion can be shown to be some form of induction. For inductive constructability, the number of basic operations starting from basic sets is required to be finite.

V. Conclusion

The continuum hypothesis and its generalization can be proved for inductively constructible sets.

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