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Sum of Generalized Tribonacci Sequence: The Sum Formulas of $\sum_{k=0}^{n} x^k W_k$ via Generating Functions

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Abstract. *In this paper, we present the sum formula of generalized Tribonacci numbers via generating functions.*
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1. Introduction

The generalized (r, s, t) sequence (or generalized Tribonacci sequence or generalized 3-step Fibonacci sequence)

$$\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$$

(or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \tag{1.1}$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r, s, t are real numbers. This sequence has been studied by many authors, see for example [1,2,3,4,5,6,7,9,10,12,13]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

LEMMA 1. *Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized (r,s,t) sequence (the generalized Tribonacci sequence) $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2}{1 - rx - sx^2 - tx^3}. \tag{1.2}$$

We define three special cases of the generalized (r,s,t) sequence $\{W_n\}$. (r,s,t) sequence$\{G_n\}_{n\geq 0}$, Lucas (r,s,t) sequence $\{H_n\}_{n\geq 0}$ and modified (r,s,t) sequence $\{E_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$G_{n+3} = rG_{n+2} + sG_{n+1} + tG_n, \qquad G_0 = 0, G_1 = 1, G_2 = r, \tag{1.3}$$

$$H_{n+3} = rH_{n+2} + sH_{n+1} + tH_n, \qquad H_0 = 3, H_1 = r, H_2 = 2s + r^2, \tag{1.4}$$

$$E_{n+3} = rE_{n+2} + sE_{n+1} + tE_n, \qquad E_0 = 1, E_1 = r - 1, E_2 = -r + s + r^2. \tag{1.5}$$

The sequences $\{\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{s}{t}G_{-(n-1)} - \frac{r}{t}G_{-(n-2)} + \frac{1}{t}G_{-(n-3)}, \\ H_{-n} &= -\frac{s}{t}H_{-(n-1)} - \frac{r}{t}H_{-(n-2)} + \frac{1}{t}H_{-(n-3)}, \\ E_{-n} &= -\frac{s}{t}E_{-(n-1)} - \frac{r}{t}E_{-(n-2)} + \frac{1}{t}E_{-(n-3)} \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.3)-(1.5) hold for all integers n.

Some special cases of (r,s,t) sequence $\{G_n(0,1,r;r,s,t)\}_{n\geq 0}$ and Lucas (r,s,t) sequence $\{H_n(3,r,2s+r^2;r,s,t)\}_{n\geq 0}$ are as follows:

(1) $G_n(0,1,1;1,1,1) = T_n$, Tribonacci sequence,
(2) $H_n(3,1,3;1,1,1) = K_n$, Tribonacci-Lucas sequence,
(3) $G_n(0,1,2;2,1,1) = P_n$, third order Pell sequence,
(4) $H_n(3,2,6;2,1,1) = Q_n$, third order Pell-Lucas sequence,
(5) $G_n(0,1,0;0,1,1) = U_n$, adjusted Padovan sequence,
(6) $H_n(3,0,2;0,1,1) = E_n$, Perrin (Padovan-Lucas) sequence,
(7) $G_n(0,1,0;0,2,1) = M_n$, adjusted Pell-Padovan sequence
(8) $H_n(3,0,4;0,2,1) = B_n$, third order Lucas-Pell sequence,
(9) $G_n(0,1,0;0,1,2) = K_n$, adjusted Jacobsthal-Padovan sequence,
(10) $H_n(3,0,2;0,1,2) = L_n$, Jacobsthal-Perrin (-Lucas) sequence,
(11) $G_n(0,1,1;1,0,1) = N_n$, Narayana sequence,
(12) $H_n(3,1,1;1,0,1) = U_n$, Narayana-Lucas sequence,
(13) $G_n(0,1,1;1,1,2) = J_n$, third order Jacobsthal sequence,
(14) $H_n(3,1,3;1,1,2) = j_n$, modified third order Jacobsthal-Lucas sequence.

Lemma 1 gives the following results as particular examples (generating functions of (r,s,t),Lucas (r,s,t) and modified (r,s,t) numbers).

COROLLARY 2. *Generating functions of (r,s,t),Lucas (r,s,t) and modified (r,s,t) numbers are*

$$\begin{aligned}\sum_{n=0}^{\infty} G_n x^n &= \frac{x}{1-rx-sx^2-tx^3},\\ \sum_{n=0}^{\infty} H_n x^n &= \frac{3-2rx-sx^2}{1-rx-sx^2-tx^3},\\ \sum_{n=0}^{\infty} E_n x^n &= \frac{1-x}{1-rx-sx^2-tx^3},\end{aligned}$$

respectively.

The generalized Fibonacci sequence (or generalized (r,s)-sequence or Horadam sequence or 2-step Fibonacci sequence) $\{V_n(V_0,V_1;r,s)\}_{n\geq 0}$ (or shortly $\{V_n\}_{n\geq 0}$) is defined as follows:

$$V_n = rV_{n-1} + sV_{n-2}, \qquad V_0 = d, V_1 = e, \quad n \geq 2 \tag{1.6}$$

where V_0, V_1 are arbitrary complex (or real) numbers and r,s are real numbers. Now we define two special cases of the sequence $\{V_n\}$. (r,s) sequence $\{X_n(0,1;r,s)\}_{n\geq 0}$ and Lucas (r,s) sequence $\{Y_n(2,r;r,s)\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$X_{n+2} = rX_{n+1} + sX_n, \qquad X_0 = 0, X_1 = 1, \tag{1.7}$$

$$Y_{n+2} = rY_{n+1} + sY_n, \qquad Y_0 = 2, Y_1 = r. \tag{1.8}$$

Let

$$K_n = \sum_{k=0}^{n} V_k.$$

In a quite recent preprint, Prodinger [8] proved the following Theorem via generating functions.

THEOREM 3. *(Prodinger) For $n \geq 0$, we have*

$$K_n = \frac{V_0+V_1-V_0r}{1-r-s} - \frac{(2V_0s+V_1r+2V_1s-V_0rs)}{2(1-r-s)}X_n - \frac{V_1+V_0s}{2(1-r-s)}Y_n.$$

Let

$$M_n = \sum_{k=0}^{n} x^k V_k.$$

In [11], Theorem *3* was generalized as follows:

THEOREM 4. *Let x be a nonzero complex (or real) number.*

(a): *If $1-rx-sx^2 \neq 0$ then*

$$\begin{aligned} M_n &= \frac{V_0+x(V_1-rV_0)}{1-rx-sx^2} - \frac{2V_1sx^2+2V_0sx+V_1rx-V_0rsx^2}{2(1-rx-sx^2)}x^nX_n - \frac{V_1x+V_0sx^2}{2(1-rx-sx^2)}x^nY_n \\ &= \frac{\Lambda(x)}{2(1-rx-sx^2)} \end{aligned} \tag{1.9}$$

where

$$\Lambda(x) = 2(V_0 + (V_1 - rV_0)x) - (s(2V_1 - rV_0)x + (rV_1 + 2sV_0))x^{n+1}X_n - (V_1 + sxV_0)x^{n+1}Y_n.$$

(b): *If* $1 - rx - sx^2 = u(x-a)(x-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $x = a$ *or* $x = b$, *then*

$$M_n = \frac{\Lambda_1(x)}{-2(r + 2sx)}$$

where

$\Lambda_1(x) = 2(V_1 - rV_0) + (-(r + nr + 4sx + 2nsx)V_1 + s(-2n + 2rx + nrx - 2)V_0)x^n X_n - x^n((n+1)V_1 + sx(n+2)V_0)Y_n.$

(c): *If* $1 - rx - sx^2 = u(x-c)^2 = 0$ *for some* $u, c \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $x = c$, *then*

$$M_n = \frac{\Lambda_2(x)}{4s}$$

where

$\Lambda_2(x) = (n+1)((nr + 4sx + 2nsx)V_1 + s(2n - 2rx - nrx)V_0)x^{n-1}X_n + (n+1)(nx^{n-1}V_1 + sx^n(n+2)V_0)Y_n.$

In the next section, we extend the results of Theorem 4 to the generalized Tribonacci numbers.

2. Main Result: The Sum Formula $\sum_{k=0}^{n} x^k W_k$ via Generating Functions

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} x^n W_n z^n$ of the sequence $\{x^n W_n\}$.

LEMMA 5. *Suppose that* $f_{x^n W_n}(z) = \sum\limits_{n=0}^{\infty} x^n W_n z^n$ *is the ordinary generating function of the sequence* $\{x^n W_n\}_{n \geq 0}$. *Then,* $\sum\limits_{n=0}^{\infty} x^n W_n z^n$ *is given by*

$$\sum_{n=0}^{\infty} x^n W_n z^n = \frac{W_0 + x(W_1 - rW_0)z + x^2(W_2 - rW_1 - sW_0)z^2}{1 - rxz - sx^2z^2 - tx^3z^3} \tag{2.1}$$

Proof. Note that

$$x^n W_n = x^n(rW_{n-1} + sW_{n-2} + tW_{n-3}).$$

Using the definition of generalized Tribonacci numbers, and substracting $rxz\sum_{n=0}^{\infty}x^nW_nz^n$, $sx^2z^2\sum_{n=0}^{\infty}x^nW_nz^n$ and $tx^3z^3\sum_{n=0}^{\infty}x^nW_nz^n$ from $\sum_{n=0}^{\infty}x^nW_nz^n$ we obtain

$$
\begin{aligned}
& (1-rxz-sx^2z^2-tx^3z^3)\sum_{n=0}^{\infty}x^nW_nz^n \\
= & \sum_{n=0}^{\infty}x^nW_nx^n - rxz\sum_{n=0}^{\infty}x^nW_nz^n - sx^2z^2\sum_{n=0}^{\infty}x^nW_nz^n - tx^3z^3\sum_{n=0}^{\infty}x^nW_nz^n \\
= & \sum_{n=0}^{\infty}x^nW_nz^n - r\sum_{n=0}^{\infty}x^{n+1}W_nz^{n+1} - s\sum_{n=0}^{\infty}x^{n+2}W_nz^{n+2} - tx^3\sum_{n=0}^{\infty}x^{n+3}W_nz^{n+3} \\
= & \sum_{n=0}^{\infty}x^nW_nz^n - r\sum_{n=1}^{\infty}x^nW_{n-1}z^n - s\sum_{n=2}^{\infty}x^nW_{n-2}z^n - t\sum_{n=3}^{\infty}x^nW_{n-3}z^n \\
= & (W_0+xW_1z+x^2W_2z^2) - r(xW_0z+x^2W_1z^2) - sx^2W_0z^2 \\
& +\sum_{n=3}^{\infty}x^n(W_n-rW_{n-1}-sW_{n-2}-tW_{n-3})z^n \\
= & W_0+xW_1z+x^2W_2z^2-rxW_0z-rx^2W_1z^2-sx^3W_0z^2 \\
= & W_0+x(W_1-rW_0)z+x^2(W_2-rW_1-sW_0)z^2.
\end{aligned}
$$

Rearranging the above equation, we obtain (2.1). □

Lemma 5 gives the following results as particular examples.

COROLLARY 6. *Generating functions* $\sum_{n=0}^{\infty}x^nG_nz^n$, $\sum_{n=0}^{\infty}x^nH_nz^n$ *and* $\sum_{n=0}^{\infty}x^nE_nz^n$ *are*

$$
\begin{aligned}
\sum_{n=0}^{\infty}x^nG_nz^n &= \frac{xz}{1-rxz-sx^2z^2-tx^3z^3}, \\
\sum_{n=0}^{\infty}x^nH_nz^n &= \frac{3-2rxz-sx^2z^2}{1-rxz-sx^2z^2-tx^3z^3}, \\
\sum_{n=0}^{\infty}x^nE_nz^n &= \frac{1-xz}{1-rxz-sx^2z^2-tx^3z^3},
\end{aligned}
$$

respectively.

Let

$$S_n=\sum_{k=0}^{n}x^kW_k.$$

The following theorem presents some sum formulas of generalized Tribonacci numbers with positive subscripts.

THEOREM 7. *Let x be a nonzero complex (or real) number.*

(a): *If* $1-rx-sx^2-tx^3\neq 0$ *then*

$$S_n=\frac{\Theta(x)}{t(1-rx-sx^2-tx^3)} \tag{2.2}$$

and

$$S_n = \frac{\Psi(x)}{s(1 - rx - sx^2 - tx^3)} \tag{2.3}$$

where

$\Theta(x) = t(x^2W_2 - x(rx-1)W_1 - (sx^2+rx-1)W_0) + (-tx^3W_2 + tx^2(rx-1)W_1 + tx(sx^2+rx-1)W_0)x^nG_{n+2} + (tx^2(rx-1)W_2 - tx(rx-1)^2W_1 + tx(r - r^2x - tx^2 - rsx^2)W_0)x^nG_{n+1} + (tx(sx^2+rx-1)W_2 + tx(r - r^2x - tx^2 - rsx^2)W_1 + tx(rtx^2 - s^2x^2 - tx - rsx + s)W_0)x^nG_n$

and

$\Psi(x) = s(x^2W_2 - x(rx-1)W_1 - (sx^2+rx-1)W_0) - (x^2(s+3tx)W_2 - x(s+3tx)(rx-1)W_1 - tx(2sx^2+3rx-3)W_0)x^nG_{n+1} + (x(2rtx^2+rsx-s)W_2 - x(stx^2+2r^2tx^2+s^2x+r^2sx-2rtx-rs)W_1 + tx\left(-rsx^2-2r^2x-sx+2r\right)W_0)x^nG_n - (-tx^3W_2 + tx^2(rx-1)W_1 + tx(sx^2+rx-1)W_0)x^nH_n.$

(b): *If* $1 - rx - sx^2 - tx^3 = u(x-a)(x-b)(x-c) = 0$ *for some* $u, a, b, c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $x = a$ *or* $x = b$ *or* $x = c$ *then*

$$S_n = \frac{\Theta_1(x)}{-t\left(3tx^2 + 2sx + r\right)}$$

where

$\Theta_1(x) = t(2xW_2 - (2rx-1)W_1 - (r+2sx)W_0) + t(-x^2(n+3)W_2 + x(3rx+nrx-n-2)W_1 + (nsx^2+3sx^2+2rx+nrx-n-1)W_0)x^nG_{n+2} - t(x(n-3rx-nrx+2)W_2 + (rx-1)(3rx+nrx-n-1)W_1 + (2r^2x+3tx^2+3rsx^2+ntx^2+nrsx^2+nr^2x-r-nr)W_0)x^nG_{n+1}$
$-t((-3sx^2-nsx^2-2rx-nrx+n+1)W_2 + (ntx^2+3rsx^2+nrsx^2+3tx^2+2r^2x+nr^2x-r-nr)W_1 + (3s^2x^2-nrtx^2-3rtx^2+ns^2x^2+2tx+ntx+2rsx+nrsx-s-ns)W_0)x^nG_n.$

(c): *If* $1 - rx - sx^2 - tx^3 = u(x-a)^2(x-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $x = a$ *or* $x = b$ *then for* $x = a$ *we get*

$$S_n = \frac{\Theta_2(x)}{-t\left(2s + 6tx\right)}$$

where

$\Theta_2(x) = 2t(W_2 - rW_1 - sW_0) + t(-x^2(n+3)(n+2)W_2 + x(n+2)(3rx+nrx-n-1)W_1 + (5nsx^2+n^2sx^2+6sx^2+n^2rx+3nrx+2rx-n^2-n)W_0)x^{n-1}G_{n+2} - t(x(n+2)(-3rx-nrx+n+1)W_2 + (6r^2x^2+n^2r^2x^2+5nr^2x^2-4rx-2n^2rx-6nrx+n^2+n)W_1 + (5ntx^2+6rsx^2+n^2tx^2+5nrsx^2+n^2rsx^2+6tx^2+3nr^2x+n^2r^2x+2r^2x-nr-n^2r)W_0)x^{n-1}G_{n+1} - t((-6sx^2-5nsx^2-n^2sx^2-3nrx-n^2rx-2rx+n^2+n)W_2 + (n^2tx^2+5nrsx^2+n^2rsx^2+6tx^2+5ntx^2+6rsx^2+2r^2x+3nr^2x+n^2r^2x-nr-n^2r)W_1 + (6s^2x^2+n^2s^2x^2-6rtx^2-5nrtx^2-n^2rtx^2+5ns^2x^2+3ntx+2rsx+n^2rsx+3nrsx+n^2tx+2tx-n^2s-ns)W_0)x^{n-1}G_n$

and for $x = b$ *we get*

$$S_n = \frac{\Theta_3(x)}{-t\left(3tx^2 + 2sx + r\right)}$$

where

$\Theta_3(x) = t(2xW_2 - (2rx-1)W_1 - (r+2sx)W_0) + t(-x^2(n+3)W_2 + x(3rx+nrx-n-2)W_1 + (nsx^2+3sx^2+2rx+nrx-n-1)W_0)x^nG_{n+2} - t(x(n-3rx-nrx+2)W_2 + (rx-1)(3rx+nrx-n-1)W_1 + (2r^2x+3tx^2+3rsx^2+ntx^2+nrsx^2+nr^2x-r-nr)W_0)x^nG_{n+1}$
$-t((-3sx^2-nsx^2-2rx-nrx+n+1)W_2 + (ntx^2+3rsx^2+nrsx^2+3tx^2+2r^2x+nr^2x-r-nr)W_1 + (3s^2x^2-nrtx^2-3rtx^2+ns^2x^2+2tx+ntx+2rsx+nrsx-s-ns)W_0)x^nG_n.$

(d): *If* $1-rx-sx^2-tx^3 = u(x-a)^3 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $x = a$, *then*

$$S_n = \frac{\Theta_4(x)}{-6t^2}$$

where

$\Theta_4(x) = t(n+1)(-x^2(n+3)(n+2)W_2 + x(n+2)(3rx+nrx-n)W_1 + (6sx^2+5nsx^2+n^2sx^2+n^2rx+2nrx-n^2+n)W_0)x^{n-2}G_{n+2}$
$+t(n+1)(x(n+2)(3rx+nrx-n)W_2 - (6r^2x^2+n^2r^2x^2+5nr^2x^2-2n^2rx-4nrx+n^2-n)W_1 + (-6tx^2-n^2tx^2-5nrsx^2-n^2rsx^2-5ntx^2-6rsx^2-n^2r^2x-2nr^2x+n^2r-nr)W_0)x^{n-2}G_{n+1}$
$+t(n+1)((5nsx^2+n^2sx^2+6sx^2+n^2rx+2nrx-n^2+n)W_2 + (-n^2tx^2-5nrsx^2-n^2rsx^2-6tx^2-5ntx^2-6rsx^2-2nr^2x-n^2r^2x+n^2r-nr)W_1 + (5nrtx^2+n^2rtx^2-6s^2x^2-n^2s^2x^2+6rtx^2-5ns^2x^2-2ntx-n^2rsx-2nrsx-n^2tx+n^2s-ns)W_0)x^{n-2}G_n.$

Proof.

(a): Note that using generating functions, we get

$$\begin{aligned} S(z) &= \sum_{n=0}^{\infty} S_n z^n = \frac{1}{1-z}\frac{W_0 + x(W_1 - rW_0)z + x^2(W_2 - rW_1 - sW_0)z^2}{1-rxz-sx^2z^2-tx^3z^3} \\ &= \frac{A}{1-z} + B\frac{xz}{1-rxz-sx^2z^2-tx^3z^3} + C\frac{3-2rxz-sx^2z^2}{1-rxz-sx^2z^2-tx^3z^3} + D\frac{1-xz}{1-rxz-sx^2z^2-tx^3z^3} \\ &= A\sum_{n=0}^{\infty} z^n + B\sum_{n=0}^{\infty} x^nG_nz^n + C\sum_{n=0}^{\infty} x^nH_nz^n + D\sum_{n=0}^{\infty} x^nE_nz^n \\ &= \sum_{n=0}^{\infty}\left(A + Bx^nG_n + Cx^nH_n + Dx^nE_n\right)z^n \end{aligned}$$

where

$$\begin{aligned} A &= \frac{x^2W_2 - x(rx-1)W_1 - (sx^2+rx-1)W_0}{1-rx-sx^2-tx^3}, \\ B &= -\frac{\begin{array}{c} x(3tx^2-2rtx^2+sx-rsx+s)W_2 \\ +x(-3rtx^2+stx^2+2r^2tx^2+s^2x+3tx+r^2sx-rsx-2rtx+s-rs)W_1 \\ +tx(rsx^2-2sx^2+2r^2x-3rx+sx+3-2r)W_0 \end{array}}{s(1-rx-sx^2-tx^3)}, \\ C &= -\frac{-tx^3W_2 + tx^2(rx-1)W_1 + tx(sx^2+rx-1)W_0}{s(1-rx-sx^2-tx^3)}, \\ D &= -\frac{x^2(s+3tx)W_2 - x(s+3tx)(rx-1)W_1 - tx(2sx^2+3rx-3)W_0}{s(1-rx-sx^2-tx^3)}, \end{aligned}$$

i.e.,

$$\sum_{n=0}^{\infty} S_n z^n = \sum_{n=0}^{\infty} \left(A + Bx^n G_n + Cx^n H_n + Dx^n E_n\right) z^n.$$

Comparing on both sides, we obtain

$$\sum_{k=0}^{n} x^k W_k = S_n = \sum_{n=0}^{\infty} \left(A + Bx^n G_n + Cx^n H_n + Dx^n E_n\right).$$

Since

$$E_n = G_{n+1} - G_n$$

we get

$$\sum_{k=0}^{n} x^k W_k = A + Dx^n G_{n+1} + (B - D)x^n G_n + Cx^n H_n$$

The last formula can be written as (2.3).

Note that using the identity

$$tH_n = -sG_{n+2} + (3t + rs)G_{n+1} + (-2rt + s^2)G_n$$

we obtain (2.2).

(b): We use (2.2). For $x = a$ or $x = b$ or $x = c$, the right hand side of the above sum formula (2.2) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) by using

$$\begin{aligned} \sum_{k=0}^{n} a^k W_k &= \left.\frac{\frac{d}{dx}\Theta(x)}{\frac{d}{dx}(t(1 - rx - sx^2 - tx^3))}\right|_{x=a}, \\ \sum_{k=0}^{n} b^k W_k &= \left.\frac{\frac{d}{dx}\Theta(x)}{\frac{d}{dx}(t(1 - rx - sx^2 - tx^3))}\right|_{x=b}, \\ \sum_{k=0}^{n} c^k W_k &= \left.\frac{\frac{d}{dx}\Theta(x)}{\frac{d}{dx}(t(1 - rx - sx^2 - tx^3))}\right|_{x=c}. \end{aligned}$$

(c): We use (2.2). For $x = a$ and $x = b$, the right hand side of the above sum formula (2.2) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (c) by using

$$\sum_{k=0}^{n} a^k W_k = \left.\frac{\frac{d^2}{dx^2}\Theta(x)}{\frac{d^2}{dx^2}(t(1 - rx - sx^2 - tx^3))}\right|_{x=a}$$

and

$$\sum_{k=0}^{n} b^k W_k = \left.\frac{\frac{d}{dx}\Theta(x)}{\frac{d}{dx}(t(1 - rx - sx^2 - tx^3))}\right|_{x=b}.$$

(d): We use (2.2). For $x = a$, the right hand side of the above sum formula (2.2) is an indeterminate form. Now, we can use L'Hospital rule (three times). Then we get (d) by using

$$\sum_{k=0}^{n} a^k W_k = \left.\frac{\frac{d^3}{dx^3}\Theta(x)}{\frac{d^3}{dx^3}(t(1 - rx - sx^2 - tx^3))}\right|_{x=a}. \square$$

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