IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 18, Issue 1 Ser. II (Jan. – Feb. 2022), PP 24-30
www.iosrjournals.org

Even Sum Property of $J_n, B(3,n), TB_n, P_m(+)\ \overline{K_n}$ and $(\overline{K_n} \cup P_3) + 2K_1$

P. P. Andharia[1], V. J. Kaneria[2]
[1]*(Department of Applied Sciences and Humanity, Shri S. N. Patel Institute of Technology, Bardoli, Gujarat, India)*
[2]*(Department of Mathematics, Saurashtra University, Rajkot, Gujarat, India)*

Abstract:
For a graph $G = (V, E)$, an injective vertex labelling function $f: V(G) \to \{0, \pm2, \pm4, \dots, \pm2|E(G)|\}$ is said to be an even sum labeling of a graph G if the induced edge labeling map $f^*: E(G) \to \{2, 4, \dots, 2|E(G)|\}$ defined by $f^*(u_i u_j) = f(u_i) + f(u_j)$, $\forall u_i u_j \in E(G)$ is bijection. A graph which admits even sum labeling is called an even sum graph. In this article we prove that the Jewel graph J_n, triangular book graph $B(3,n)$, triangular book graph with bookmark TB_n, a graph $P_m(+)\ \overline{K_n}$ and a graph $(\overline{K_n} \cup P_3) + 2K_1$ are even sum graphs.
***Key Word**: Jewel graph; Triangular book graph; Even sum labeling; Even sum graph.*

Date of Submission: 08-01-2022 Date of Acceptance: 22-01-2022

I. Introduction

In this article we use a word 'graph' for an undirected simple and finite graph, V(G) for vertex set of a graph G and E(G) for edge set of G. The order of a graph or number of vertices in a graph is denoted by p and the size or number of edges is denoted by q. The terms not defined here are used in the sense of Harary[1]. For the bibliographic references on graph labeling we refer to Gallian[2]. The idea of odd sum labeling was first given by Arockiaraj and Mahalakshmi[3]. The odd sum labeling of various types of graph are found in different investigations[4,5,6]. Monika and Murugan[7] introduce the concept of odd-even sum labeling. Some general results on odd-even sum labeling of graphs are presented by Kaneria and Andharia[8]. Andharia and Kaneria[9] introduce the new concept of labeling called even sum labeling. Kaneria and Andharia[10,11] have presented the even sum labeling of various graphs. In this article we have presented some more graphs with even sum labeling property.

Following is a brief summary of definitions which are useful for the present investigations.

Definition 1: An injection $f: V(G) \to \{0, \pm2, \pm4, \dots, \pm2|E(G)|\}$ is said to be an even sum labeling of a graph G if the induced edge labeling map $f^*: E(G) \to \{2, 4, \dots, 2|E(G)|\}$ defined by $f^*(u_i u_j) = f(u_i) + f(u_j)$, $\forall u_i u_j \in E(G)$ is bijection. A graph which admits even sum labeling is called an even sum graph.

Definition 2: The Jewel graph J_n is a graph with $V(J_n) = \{u_i, v_j : i = 0, 1, 2, 3; j = 1, 2, \dots, n\}$ and the edge set $E(J_n) = \{u_0u_1, u_0u_2, u_0u_3, u_1u_3, u_2u_3, u_1v_i, u_2v_i : 1 \le i \le n\}$.

Definition 3: An n copies of cycle C_3 sharing a common edge is known as a triangular Book graph with n-pages. It is denoted by $B(3,n)$. The common edge is referred as spine of the book.

Definition 4: A triangular book graph with bookmark is a triangular book graph $B(3,n)$ alongwith a pendant edge attached at any one of the end vertices of the spine. It is denoted by TB_n.

II. Main Results

Theorem 1: Every Jewel Graph is even sum graph.

Proof: Consider a Jewel graph J_n with $V(J_n) = \{u_0, u_1, u_2, u_3, v_1, v_2, \dots, v_n\}$ and $E(J_n) = \{u_0 u_1, u_0u_2, u_0 u_3, u_1u_3, u_2u_3\} \cup \{u_1v_i, u_2v_i : i = 1, 2, \dots, n\}$ as shown in Figure 1.

Clearly the order of a Jewel graph J_n is $n + 4$ and size is $2n + 5$.

Define $f: V(J_n) \to \{0, \pm2, \pm4, \dots, \pm(4n + 10)\}$ as follows:

$f(u_i) = 2i$, $i = 0,1,2,3$ and

$f(v_i) = 4i + 6$, $\forall i = 1, 2, \dots, n$.

Note that f is injective as

$f(V(J_n)) = \{0, 2, 4, 6, 10, 14, \dots, 4n + 6\}$ and $|V(J_n)| = n + 4 = |f(V(J_n))|$.

Further, its edge induced function

$f^*: E(J_n) \to \{2, 4, \dots, 4n + 10\}$ is bijective.

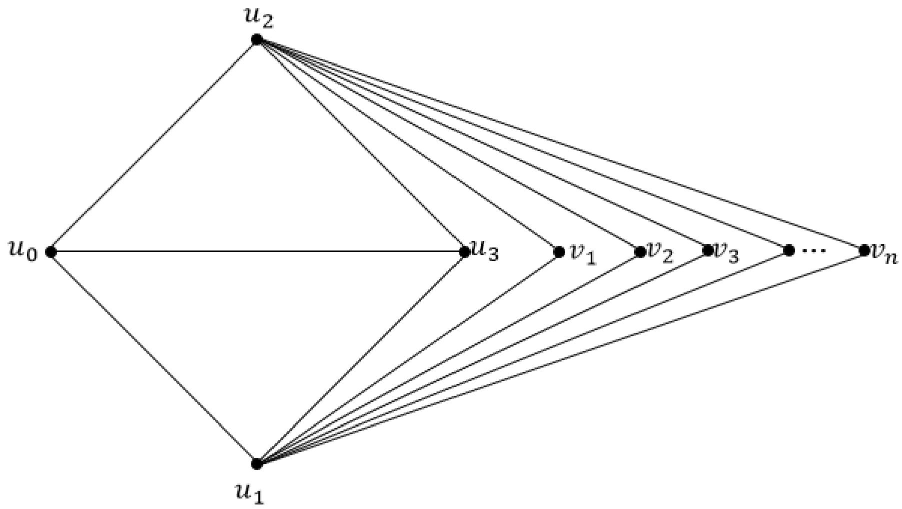

Figure – 1: Ordinary labeling of a Jewel graph J_n

Because $f^*(E(J_n)) = \{2, 4, \dots, 4n + 10\}$,
$|E(J_n)| = 2n + 5 = |f^*(E(J_n))|$ and
$f^*(u_0u_i) = f(u_0) + f(u_i) = f(u_i) = 0 + 2i = 2i,\ \ i = 1, 2, 3;$
$f^*(u_iu_3) = f(u_i) + f(u_3) = f(u_i) + 6 = 2i + 6,\ \ i = 1, 2;$
$f^*(u_1v_i) = f(u_1) + f(v_i) = 2 + 4i + 6 = 4i + 8,\ \ i = 1, 2, \dots\dots, n;$
$f^*(u_2v_i) = f(u_2) + f(v_i) = 4 + 4i + 6 = 4i + 10,\ \ i = 1, 2, \dots\dots, n.$
Hence the Jewel graph J_n is even sum graph.

Illustration 1: Even sum labeling of J_5 is shown in Figure 2.

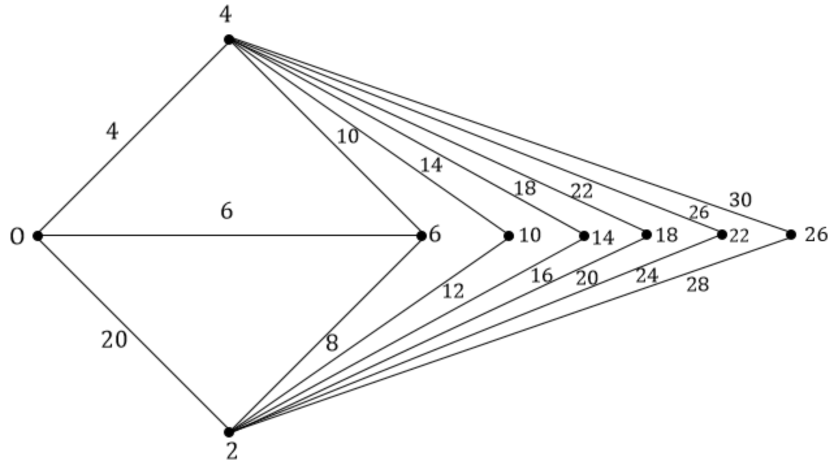

Figure – 2: Jewel graph J_5 with its even sum labeling

Theorem 2: Every triangular book graph is even sum graph.
Proof: Consider a triangular book graph $B(3, n)$ with its ordinary vertex labeling as shown in Figure 3.
Here, $V(B(3, n)) = \{u_0, u_1, v_i : 1 \le i \le n\}$ and $E(B(3, n)) = \{e_i : 1 \le i \le 2n + 1\}$, where $e_1 = u_0u_1$, $e_{2i} = u_0v_i$ and $e_{2i+1} = u_1v_i, \forall i = 1,2, \cdots, n$.
Clearly, the order and size of any triangular book graph $B(3, n)$ is $p = n + 2$ and $q = 2n + 1$ respectively.
Now define $f: V(B(3, n)) \rightarrow \{0, \pm 2, \pm 4, \dots, \pm(4n + 2)\}$ as below:
$f(u_0) = 0$; $f(u_1) = 2$ and $f(v_i) = 4i,\ \forall i = 1,2, \dots, n$.

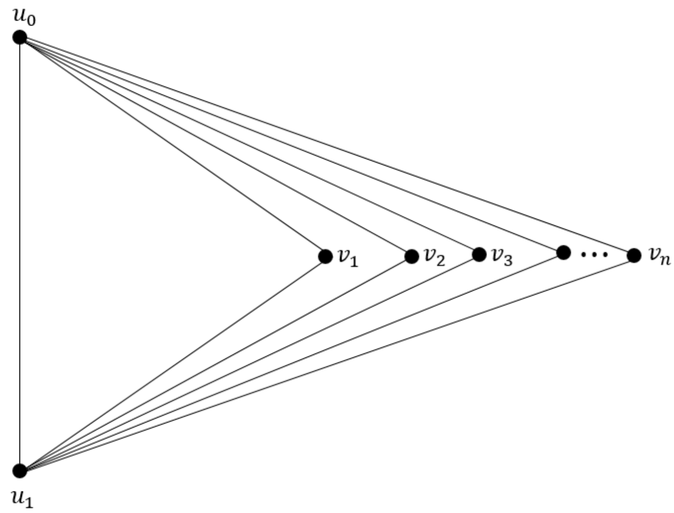

Figure – 3: $B(3, n)$ and its ordinary labeling

Note that f is an injective map as
$f(V(B(3,n))) = \{0, 2, 4, 8, 12, \cdots, 4n\}$ and $|V(B(3,n))| = n + 2 = |f(V(B(3,n)))|$.
Further, its edge induced function
$f^*: E(B(3,n)) \to \{2, 4, \ldots, 4n + 2\}$ is bijective.
Because $f^*(E(B(3,n))) = \{2, 4, 6, \ldots, 4n + 2\}$,
$|E(B(3,n))| = 2n + 1 = \left|f^*\left(E(B(3,n))\right)\right|$ and

$$f^*(e_1) = f^*(u_0u_1) = f(u_0) + f(u_1) = 0 + 2 = 2;$$
$$f^*(e_{2i}) = f^*(u_0v_i) = f(u_0) + f(v_i) = f(v_i) = 4i;$$
$$f^*(e_{2i+1}) = f^*(u_1v_i) = f(u_1) + f(v_i) = 2 + f(v_i) = 4i + 2, \forall i = 1, 2, \cdots, n.$$

Thus, f is even sum labeling for the triangular book graph $B(3, n)$ and hence every triangular book graph is even sum graph.

Illustration 2: Even sum labeling of a triangular book graph $B(3, 5)$ is shown in Figure 4.

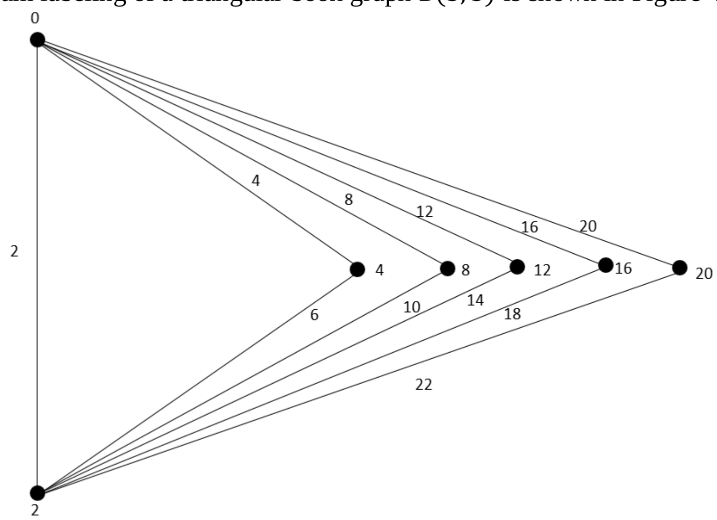

Figure – 4: Even sum labeling of $B(3, 5)$

Theorem 3: The triangular book graph with bookmark TB_n is even sum graph.
Proof: Consider a triangular book graph with bookmark TB_n with its vertex labeling as shown in Figure 5.

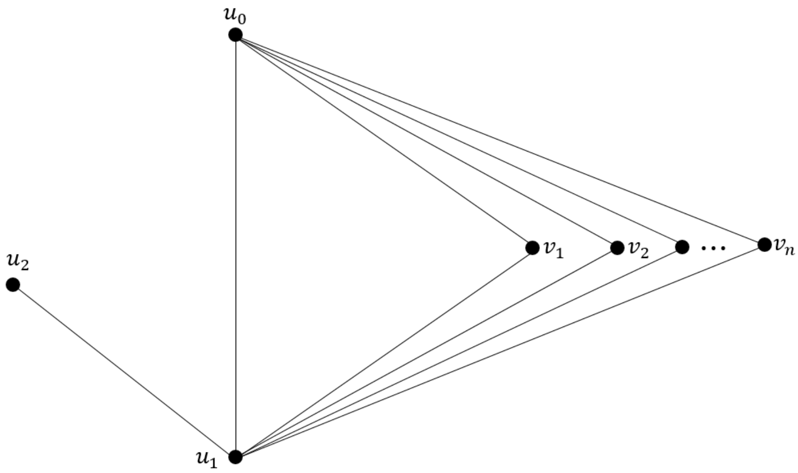

Figure – 5: Ordinary labeling of $\mathrm{TB_n}$

Here, $V(TB_n) = \{u_0, u_1, u_2, v_1, v_2, \dots, v_n\}$ and $E(TB_n) = \{e_0, e_1, \cdots, e_{2n+1}\}$, where $e_0 = u_1u_2$, $e_1 = u_0u_1, e_{2i} = u_0v_i$ and $e_{2i+1} = u_1v_i, \forall i = 1,2,\cdots,n$.
It is clear that, $|V(TB_n)| = n + 3$ and $|E(TB_n)| = 2n + 2$.
Now define $f: V(TB_n) \to \{0, \pm 2, \pm 4, \dots, \pm(4n + 2)\}$ as
$f(u_0) = 0,\ f(u_1) = 2,\ f(u_2) = 4n + 2$ and $f(v_i) = 4i,\ \forall i = 1,2,\dots,n$.
Note that f is an injective map as
$f(V(TB_n)) = \{0, 2, 4, 8, 12, \cdots, 4n, 4n + 2\}$ and
$|V(TB_n)| = n + 3 = |\{0, 2, 4, 8, 12, \cdots, 4n, 4n + 2\}| = |f(V(TB_n))|$.
Further, its edge induced function
$f^*: E(TB_n) \to \{2, 4,6,8, \dots, 4n + 4\}$ is bijective.
Because $f^*(E(TB_n)) = \{2, 4,6,8, \dots, 4n + 4\}$,
$|E(TB_n)| = 2n + 2 = |f^*(E(TB_n))|$ and

$f^*(e_0) = f^*(u_1u_2) = f(u_1) + f(u_2) = 2 + 4n + 2 = 4n + 4;$
$f^*(e_1) = f^*(u_0u_1) = f(u_0) + f(u_1) = 0 + 2 = 2;$
$f^*(e_{2i}) = f^*(u_0v_i) = f(u_0) + f(v_i) = 0 + 4i = 4i;$
$f^*(e_{2i+1}) = f^*(u_1v_i) = f(u_1) + f(v_i) = 2 + 4i = 4i + 2, \forall i = 1, 2, \cdots, n.$

Thus, f is even sum labeling for the triangular book with book mark graph TB_n and hence TB_n is even sum graph.

Illustration 3: Even sum labeling of a triangular book graph with book mark TB_4 is shown in Figure 6.

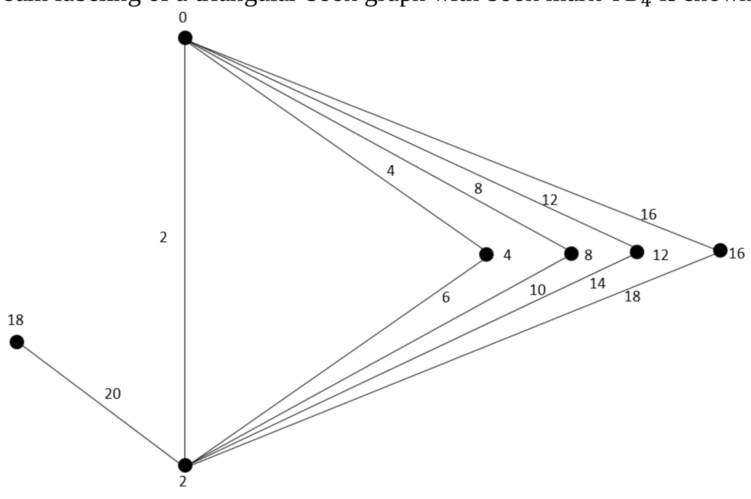

Figure – 6: TB_4 with its even sum labeling

Theorem 4: The graph $P_m(+)\ \overline{K_n}$ is even sum graph when m is odd and $n = \frac{m-1}{2}$.
Proof: Let $G = P_m(+)\ \overline{K_n}$ where m is odd and $n = \frac{m-1}{2}$.

As shown in Figure 7, the vertices of G are u_i; $1 \leq i \leq m$ and v_j; $1 \leq j \leq n$ and edges are $u_i u_{i+1}$, $u_1 v_j$, $u_m v_j$ where $1 \leq i \leq m-1, 1 \leq j \leq n$.

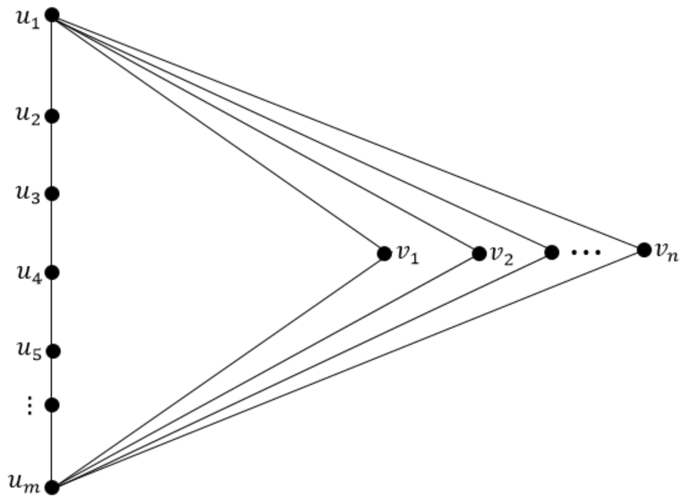

Figure – 7: Vertex labeling of $P_m(+)\overline{K_n}$

Clearly, G has order $m + n$ and size 4n.
We define $f: V(G) \rightarrow \{0, \pm 2, \pm 4, \ldots, \pm 2q\}$ as
$f(u_i) = \begin{cases} 1 - i, & \text{if i is odd} \\ 8n - i + 2, & \text{if i is even} \end{cases}$ and
$f(v_j) = 2(n + j), \ \forall j = 1,2, \ldots, n$.
The above labeling pattern give rise even sum labeling to the graph G. Hence the graph $P_m(+)\overline{K_n}$ is even sum graph when m is odd and $n = \frac{m-1}{2}$.

Illustration 4: Even sum labeling of a graph $P_7(+)\overline{K_3}$ is shown in Figure 8.

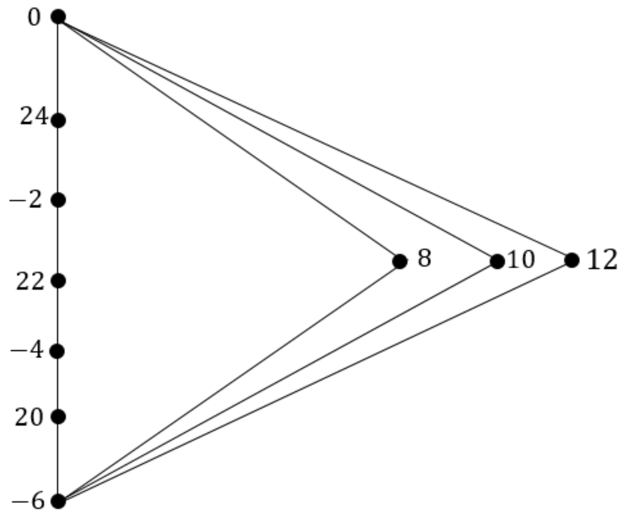

Figure – 8: $P_7(+)\overline{K_3}$ and its even sum labeling

Theorem 5: The graph $(\overline{K_n} \cup P_3) + 2K_1$ admits even sum labeling.
Proof: Let $G = (\overline{K_n} \cup P_3) + 2K_1$.

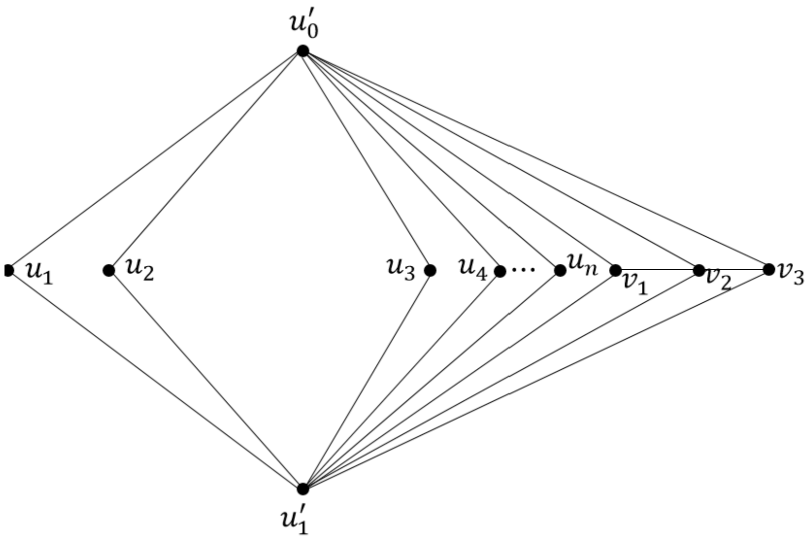

Figure – 9: Ordinary labeling of $(\overline{K_n} \cup P_3) + 2K_1$

As shown in Figure 9, the vertices of G are $u_i, v_1, v_2, v_3, u_0', u_1'$ where $i = 1, 2, \dots, n$ and edges are $u_0'u_i, u_1'u_i, u_0'v_j, u_1'v_j, v_1v_2, v_2v_3$ where $i - 1,2, \dots, n; j = 1, 2, 3$.
Clearly, G has order $n + 5$ and size $2n + 8$.
Now, define a map $f: V(G) \rightarrow \{0, \pm 2, \pm 4, \dots, \pm(4n + 16)\}$ as
$f(u_i) = -2i, \forall\, i = 1, 2, \dots, n;$
$f(v_1) = 4;\ f(v_2) = 0;\ f(v_3) = 2;$
$f(u_0') = 4(n + 3);\ f(u_1') = 2(n + 3).$
Note that f is an injective map as
$f\big(V(G)\big) = \{-2n, -2(n - 1), \dots\dots, -4, -2, 0, 2, 4, 2n + 6, 4n + 12\}$ and
$|V(G)| = n + 5 = |\{-2n, -2(n - 1), \dots\dots, -4, -2, 0, 2, 4, 2n + 6, 4n + 12\}| = \big|f\big(V(G)\big)\big|.$
Further, its edge induced function
$f^*: E(G) \rightarrow \{2, 4, 6, 8, \dots, 4n + 16\}$ is bijective.
Because $f^*\big(E(G)\big) = \{2, 4, 6, 8, \dots, 4n + 16\},$
$|E(G)| = 2n + 8 = \big|f^*\big(E(G)\big)\big|$ and

$$f^*(u'_0u_i) = f(u'_0) + f(u_i) = 4(n + 3) - 2i = 4n + 12 - 2i,\ \forall i = 1, 2, \cdots, n$$
$$= \{4n + 10, 4n + 8, \dots\dots, 2n + 14, 2n + 12\};$$
$$f^*(u'_1u_i) = f(u'_1) + f(u_i) = 2(n + 3) - 2i = 2n + 6 - 2i,\ \forall i = 1, 2, \cdots, n$$
$$= \{2n + 4, 2n + 2, \dots\dots, 8, 6\};$$
$$f^*(u'_0v_1) = f(u'_0) + f(v_1) = 4(n + 3) + 4 = 4n + 16;$$
$$f^*(u'_0v_2) = f(u'_0) + f(v_2) = 4(n + 3) + 0 = 4n + 12;$$
$$f^*(u'_0v_3) = f(u'_0) + f(v_3) = 4(n + 3) + 2 = 4n + 14;$$
$$f^*(u'_1v_1) = f(u'_1) + f(v_1) = 2(n + 3) + 4 = 2n + 10;$$
$$f^*(u'_1v_2) = f(u'_1) + f(v_2) = 2(n + 3) + 0 = 2n + 6;$$
$$f^*(u'_1v_3) = f(u'_1) + f(v_3) = 2(n + 3) + 2 = 2n + 8;$$
$$f^*(v_1v_2) = f(v_1) + f(v_2) = 4 + 0 = 4 \text{ and}$$
$$f^*(v_2v_3) = f(v_2) + f(v_3) = 0 + 2 = 2.$$

Thus, f is even sum labelling for the graph $G = (\overline{K_n} \cup P_3) + 2K_1$ and hence $(\overline{K_n} \cup P_3) + 2K_1$ is an even sum graph.

Illustration 5: Even sum labeling of a graph $(\overline{K_5} \cup P_3) + 2K_1$ is shown in Figure 10.

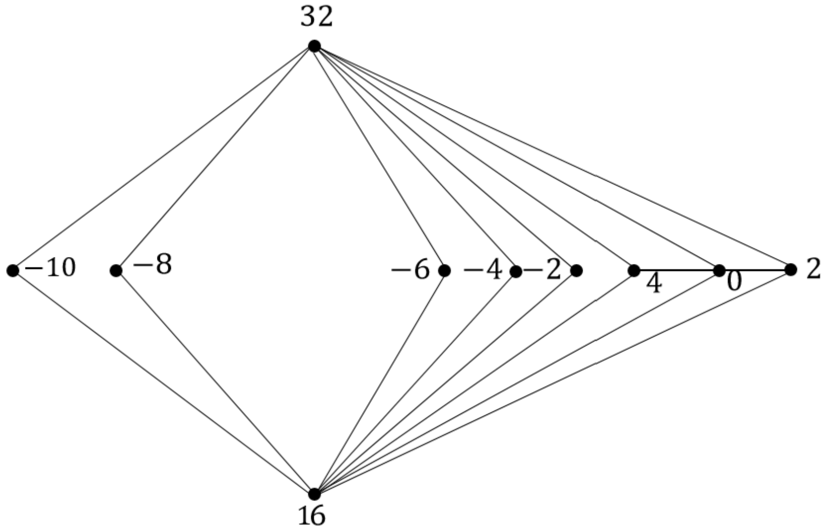

Figure – 10: Even sum labeling of $(\overline{K_5} \cup P_3) + 2K_1$

III. Conclusion

In this paper, we have discussed even sum labeling property of Jewel graph, triangular book graph, triangular book graph with book mark, the graph $P_m(+)\ \overline{K_n}$ and the graph $(\overline{K_n} \cup P_3) + 2K_1$.

References

[1]. Haray F. Graph theory. Narosa Publishing House, New Delhi. 2001.
[2]. Gallian JA. A dynamic survey of graph labeling. The Electronic Journal of Combinatorics. 2021; 16: # DS6.
[3]. Arockiaraj S, Mahalakshmi P. On odd sum graphs. International Journal of Mathematical Combinatorics. 2013; 4: 58–77.
[4]. Arockiaraj S, Mahalakshmi P, Namasivayam P. Odd sum labeling of some subdivision graphs. Kragujevac Journal of Mathematics. 2014; 38(1): 203–222.
[5]. Gopi R, Irudaya Mary A. Odd sum labeling of some more graphs. International Journal of Engineering Science, Advanced Computing and Bio-Technology. 2016; 7(4): 95–103.
[6]. Gopi R. Odd sum labeling of tree related graphs. International Journal of Mathematics and its Applications. 2016; 4(4): 11–16.
[7]. Monika K, Murugan K. Odd-even sum labeling of some graphs. International Journal of Mathematics and Soft Computing. 2017; 7(1): 57–63.
[8]. Kaneria VJ, Andharia PP. General results on odd-even sum labeling of graphs. International Journal of Mathematics And its Applications. 2017; 3–B: 185–187.
[9]. Andharia PP, Kaneria VJ. Even sum labeling of some graphs. International Journal of Computer & Mathematical Sciences. 2018; 7(5): 199–202.
[10]. Kaneria VJ, Andharia PP. Some results on even sum labeling of graphs. Journal of the Calcutta Mathematical Society. 2019; 15(2): 129–138.
[11]. Kaneria VJ, Andharia PP. Further results on even sum labeling of graphs. Global Journal of Pure and Applied Mathematics. 2020; 16(2): 155–166.

P. P. Andharia, et. al. " Even Sum Property of $J_n, B(3,n), TB_n, P_m(+)\ \overline{K_n}$ and $(\overline{K_n} \cup P_3) + 2K_1$." *IOSR Journal of Mathematics (IOSR-JM)*, 18(1), (2022): pp. 24-30.