IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 17, Issue 5 Ser. III (Sep. – Oct. 2021), PP 20-29
www.iosrjournals.org

Rheonomic Lagrange Spaces with Matsumoto Metric

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Abstract
In this paper we have discussed the differential geometry of rheonomic Lagrange space with Matsumoto metric.We find the coefficients of semispray, integral curve of semispray, Canonical nonlinear connection, differential equations of auto parallel curves and canonical metrical N-linear connection of rheonomic Lagrange space with Matsumoto metric.

***Keywords**: Rheonomic Lagrange space, Matsumoto metric, semispray, and autoparallel curves.*

Date of Submission: 10-10-2021 Date of Acceptance: 24-10-2021

I. Introduction

Finsler spaces endowed with $(\alpha,\ \beta)-$ metric which is studied, by a number of geometers such as Matsumoto [6, 7] and Kitayama et al [5] and several important applications of such spaces have been observed in physics and relativity theory [6]. The notion of $(\alpha,\ \beta)-$ metric was taken to a more general space called Lagrange space and then studied by several authors Miron [9], Nicolaescu [14, 15, 21] an n- dimensional Lagrange space $L^n = \left(M, L\,(x,y)\right)$is said to be endowed with$(\alpha,\ \beta)-$ metric, if Langrangian $L\,(x,y)$ is a function of $\alpha\,(x,y)$ and$\beta(x,y)$. Where $\alpha\,(x,y) = \sqrt{a_{ij}(x)y^i y^j}$is a Riemannian metric and $\beta(x,y) = b_i(x)y^i$ is a one form metric.

That Lagrange space is a pair $L^n = \left(M, L\,(x,y)\right)$ where M is a smooth manifold and $L\,(x,y)$ is a regular Lagrangian [10]. In several problems of mechanics and physics the time dependent Lagrangians play important role due to which the rheonomic Lagrangian space was introduced as the generalization of Lagrange's space. A rheonomic Lagrangian space is a pair

$RL^n = \left(M, L\,(x,y,t)\right)$ where $L\,(x,y,t)$ is a time dependent regular Langrangian. Several mathematicians Anastasiei, M., Kawaguchi, H., [2], Miron, R.,[9], [21], played important role in the development and applications of rheonomic Lagrangian space. Matsumoto metric [8], is a $(\alpha,\ \beta)-$ metric given by

$$L = \frac{\alpha^2}{\alpha-\beta} \tag{1.1}$$

An n-dimensional manifold M with Matsumoto metric is a Finsler space and we know that every Finsler space is a Lagrange space. Therefore, the Matsumoto metric can be considered as a regular Lagrangian In this paper we have discussed the differential geometry of rheonomic Lagrange space with Matsumoto metric. We find the coefficients of semispray, integral curve of semispray, Canonical nonlinear connection, differential equations of auto parallel curves and canonical metrical N-linear connection of rheonomic Lagrange space with Matsumoto metric.

II. Preliminaries

Let (TM,π,M) be the tangent bundle of an n-dimensional smooth manifold M. Let (x^i) and $(x^i,\ y^i)$ be the local coordinates on M and TM respectively. Let us consider the product manifold $E{:}\,TM\times\ \mathbb{R}$ and $(x^i,\ y^i,t)$ as local coordinates on E. A time dependent Lagrangian is a function $L{:}\,E\rightarrow\mathbb{R}$ which is smooth on$\tilde{E} = E\backslash\{(x,0,0), x\in M\}$ and continuous on its complement. This Lagrangian is said to be regular if $ranks\left(g_{ij}(x,y,t)\right) = n$, where $\left(g_{ij}(x,y,t)\right) := \frac{1}{2}\frac{\partial^2 L(x,y,t)}{\partial y^i\,\partial y^j}$ are components of a covariant symmetric tensor called the metric tensor of the Lagrangian $L(x,y,t)$. A rheonomic Lagrange space whose metric tensor g_{ij} has constant signature on $\tilde{E}$. The coordinates $(x^i,\ y^i,t)$ on E change by the following rule:

$$\tilde{x}^i = \tilde{x}^i(x^1,x^2,\ldots\ldots,x^n),\qquad \tilde{y}^i = \frac{\partial\tilde{x}^i}{\partial x^j}y^j,\ \tilde{t} = \varphi(t) \tag{2.1}$$

with

$$rank\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) = n, \qquad \varphi'(t) \neq 0. \tag{2.2}$$

In general, $\varphi(t) = at + b, a \neq 0$. A semispray S on E has the form (see [2] and [11])

$$S = y^i \frac{\partial}{\partial x^i} - \left(2G^i + G_0^i\right)\frac{\partial}{\partial y^i}, \tag{2.3}$$

where

$$2G^i = \frac{1}{2} g^{ik}\left(\frac{\partial^2 L}{\partial y^k \partial x^j} y^j - \frac{\partial L}{\partial x^k}\right), \qquad y^i = \frac{dx^i}{d\sigma}, \tag{2.4}$$

$$G_0^i = \frac{1}{2} g^{ik} \frac{\partial^2 L}{\partial y^k \partial t} \ . \tag{2.5}$$

The pair of functions$\left(G^i(x,y,t), G_0^i(x,y,t)\right)$, is the system of coefficients of the semisprey S. The semisprey S is canonical as its coefficients depend on L only. Thus, for a rheonomic Lagrange space RL^n there is a canonical semispray S with coefficients $\left(G^i(x,y,t), G_0^i(x,y,t)\right)$, of the form given by (2.4) and (2.5).

The integral of action of the Lagrangian $L(x,y,t)$along a smooth curve $c : [0,1] \to M \times \mathbb{R}$, $I(c) = \int_0^1 L\left(x, \frac{dx}{d\sigma}, \sigma\right) d\sigma$ leads, by virtue of variational calculus, to the Euler-Lagrange equations (see [1], [2]and [11]):

$$E_i(L) \equiv \frac{\partial L}{\partial x^i} - \frac{d}{d\sigma}\left(\frac{\partial L}{\partial y^i}\right) = 0 \qquad y^i = \frac{dx^i}{d\sigma} \tag{2.6}$$

For a point $u = (x,y,t) \in E$, consider the subspaces $V_u E = span\left(\frac{\partial}{\partial y^i}\ |_u\right)$ and $V_{u,0}E = span\left(\frac{\partial}{\partial t}\ |_u\right)$ of $T_u E$. Obviously,$dim\, V_u E = n$ and $dim\, V_{u,0} E = 1$. Now, the distribution

$$V \oplus V_0 : u \in E \mapsto V_u E \oplus V_{u,0} E$$

is vertical. The horizontal distribution $N : u \mapsto N_u \subset T_u E$, complementary to the above-mentioned vertical distribution, is a nonlinear connection on E. Thus, we have the following decomposition of the tangent space $T_u E$:

$$T_u E = N_u \oplus V_u \oplus V_{u,0} \tag{2.7}$$

The adapted basis of the decomposition is $\left(\frac{\delta}{\delta x^i} \equiv \delta_i, \frac{\partial}{\partial y^i} \equiv \dot{\partial}_i, \frac{\partial}{\partial y^i} \equiv \dot{\partial}_0\right)$, where

$$\delta_i = \partial_i - N_i^j \dot{\partial}_j - N_i \dot{\partial}_0, \qquad \partial_i \equiv \frac{\partial}{\partial x^i} \tag{2.8}$$

The pair $\left(N_i^j, N_j\right)$ is the system of coefficients of nonlinear connection N. For a semispray S of the rheonomic Lagrange space RL^n with coefficients $\left(G^i(x,y,t), G_0^i(x,y,t)\right)$given respectively by (2.4) and (2.5), there is a nonlinear connection N determined by only RL^n. The coefficients of N are expressed by

$$N_i^j = \frac{\partial\, G^i}{\partial y^j} \tag{2.9}$$

and

$$N_j = g_{ij} G_0^i \tag{2.10}$$

which, in view of (2.4) and (2.5), take the form

$$N_i^j = \frac{1}{4}\frac{\partial}{\partial y^j}\left[g^{ih}\left(\frac{\partial^2 L}{\partial y^h \partial x^k} y^k - \frac{\partial L}{\partial x^h}\right)\right] \tag{2.11}$$

and

$$N_j = \frac{1}{2}\frac{\partial^2 L}{\partial y^j\,\partial t} \tag{2.12}$$

The tangent vector field $\frac{dc}{d\sigma}$ of the curve

$$c: \sigma \in I \subseteq \mathbb{R} \mapsto \big(x(\sigma),\ y(\sigma),\ t(\sigma)\big) \in \tilde{E}$$

on $\tilde{E}$ is given by [1]

$$\frac{dc}{d\sigma} = \frac{dx^i}{d\sigma}\delta_i + \frac{\delta y^\alpha}{d\sigma}\dot{\partial}_\alpha\,, \qquad \alpha \in \{0, 1, \ldots\ldots, n\} \tag{2.13}$$

where

$$\frac{\delta y^\alpha}{d\sigma} = \frac{d y^\alpha}{d\sigma} + N_i^\alpha \frac{dx^i}{d\sigma}\,, \qquad N_i^0 = N_i\,.$$

The curve c is said to be horizontal if $\frac{\delta y^\alpha}{d\sigma} = 0$.

A horizontal curve c on $\tilde{E}$ for which $y^i = \frac{d x^i}{d t}$, is said to be parallel with respect to the nonlinear connection *N*.

An *N*-linear connection $D\Gamma(N) = \big(L_{jk}^i\,, C_{j\alpha}^i\big), (\alpha = 0, i)$ for a rheonomic Lagrange space is said to be a metrical *N*-linear connection if

$$g_{ij|k} = 0, \qquad g_{ij}\big|_k = 0, \qquad g_{ij}\big|_0 = 0, \tag{2.14}$$

where '|' and '$\big|$' denote respectively the $h-$ and $v-$ covariant derivatives with respect to $D\Gamma(N)$.

For given nonlinear connection *N* with coefficients given by (2.11) and (2.12), there exists a unique metrical N-linear connection with coefficients given by [1].

$$L_{jk}^i = \frac{1}{2}g^{ih}\big(\delta_j g_{hk} + \delta_k g_{jh} - \delta_h g_{jk}\big), \tag{2.15}$$

$$C_{jk}^i = \frac{1}{2}g^{ih}\big(\dot{\partial}_j g_{hk} + \dot{\partial}_k g_{jh} - \dot{\partial}_h g_{jk}\big), \tag{2.16}$$

$$C_{j0}^i = \frac{1}{2}g^{ih}\dot{\partial}_0 g_{jh} \tag{2.17}$$

In the present paper, we deal with a Lagrange space whose Langrangian L is a function of $\alpha(x, y, t)$ and $\beta\,(x, y, t)$ where $\alpha(x, y, t) = \sqrt{a_{ij}(x, t)y^i y^j}$ and $\beta\,(x, y, t) = A_i(x, t)y^i$. Let us denote this Langrangian by L. Thus

$$L(x, y, t) = \frac{\alpha^2(x,y,t)}{\alpha(x,y,t) - \beta(x,y,t)} \tag{2.18}$$

The space $\big(M, L(x, y, t)\big)$ is called a rheonomic Lagrange space with Matsumoto metric. We shall frequently use the following relations [14] for the product manifold $TM \times \mathbb{R}$:

$$\dot{\partial}_i\alpha = \alpha^{-1}y_i\,, \quad \dot{\partial}_i\dot{\partial}_j\alpha = \alpha^{-1}a_{ij}(x, t) - \alpha^{-3}y_i y_j, \quad \dot{\partial}_i\beta = A_i(x, t), \dot{\partial}_i\dot{\partial}_j\beta = 0 \tag{2.19}$$

where

$$y_i = a_{ij}(x, t)y^j\,.$$

For basic notations and terminology; we refer to the book [1].

III. Semispray and Integral Curves

For any rheonomic Lagrange space, there is a family of semisprays with coefficients G^i given by (2.4) and with arbitrary coefficients G_0^i [1] and [2]. We may consider the coefficients G_0^i of the form given by (2.5). In this section, we obtain the coefficients of canonical semispray of a rheonomic Lagrange space with Matsumoto metric, using equations (2.4) and (2.5). Nicolaescu, B. [15] obtained the coefficients G^i of a Lagrange space with Matsumoto metric as

$$2G^i(x,y) = \gamma^i_{jk}(x)y^j y^k - \lambda(x,y)F^i_j(x)y^j. \tag{3.1}$$

with

$$\lambda(x,y) = \frac{L_\beta}{L_\alpha}, \qquad F^i_j(x) = a^{ih}(x)\, F_{hj}(x), \qquad F_{hj} = \frac{1}{2}\left(\frac{\partial A_j}{\partial x^h} - \frac{\partial A_h}{\partial x^j}\right)$$

$$L_\alpha = \frac{\partial L}{\partial \alpha} = \frac{\alpha(\alpha-2\beta)}{(\alpha-\beta)^2} \qquad L_\beta = \frac{\partial L}{\partial \beta} = \frac{\alpha^2}{(\alpha-\beta)^2}$$

$$L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2} = \frac{2\beta^2}{(\alpha-\beta)^3} \qquad L_{\alpha\beta} = \frac{\partial^2 L}{\partial\alpha\partial\beta} = \frac{-2\alpha\beta}{(\alpha-\beta)^3} \tag{3.2}$$

$$L_{\beta\beta} = \frac{\partial^2 L}{\partial \beta^2} = \frac{2\alpha^2}{(\alpha-\beta)^3}$$

Equation (3.1) can be extended to get the coefficients $G^i(x,y,t)$ for a rheonomic Lagrange space with Matsumoto metric:

$$2G^i(x,y,t) = \gamma^i_{jk}(x,t)y^j y^k - \lambda(x,y,t)F^i_j(x,t)y^j \tag{3.3}$$

with

$$\lambda(x,y,t) = \frac{L_\beta}{L_\alpha}, \qquad F^i_j(x,t) = a^{ih}(x,t)F_{hj}(x,t),$$

$$F_{hj} = \frac{1}{2}\left(\frac{\partial A_j}{\partial x^h} - \frac{\partial A_h}{\partial x^j}\right), \tag{3.4}$$

Here, $F_{hj}(x,t)$ is the electromagnetic tensor of the space $L^n\big(M, L(\alpha,\beta)\big)$ and $\gamma^i_{jk}(x,t)$ are the second kind Christoffel symbols of $a_{ij}(x,t)$.

Now, differentiating (2.18) partially with respect to t, we have

$$\frac{\partial L}{\partial t} = \dot{\partial}_0\alpha\, \left[\frac{\alpha(\alpha-2\beta)}{(\alpha-\beta)^2}\right] + \frac{\alpha^2}{(\alpha-\beta)^2}\dot{\partial}_0\beta \tag{3.5}$$

Where $\quad \alpha_{.0} = \dot{\partial}_0\alpha\,, \qquad \beta_{.0} = \dot{\partial}_0\beta\,.$

Differentiating (3.5) partially with respect to y^j, we get

$$\frac{\partial^2 L}{\partial y^j \partial t} = \frac{\partial}{\partial y^j}\left[\frac{\partial L}{\partial t}\right] = \frac{\partial}{\partial y^j}\left[\dot{\partial}_0\alpha\left\{\frac{\alpha(\alpha-2\beta)}{(\alpha-\beta)^2}\right\} + \dot{\partial}_0\beta\left\{\frac{\alpha^2}{(\alpha-\beta)^2}\right\}\right]$$

$$\frac{\partial^2 L}{\partial y^j \partial t} = \frac{\partial^2\alpha}{\partial y^j\partial t}\frac{\alpha(\alpha-2\beta)}{(\alpha-\beta)^2} + \frac{\partial^2\beta}{\partial y^j\partial t}\frac{\alpha^2}{(\alpha-\beta)^2} + \alpha_{.0}\left\{\frac{\partial}{\partial\alpha}\frac{\alpha(\alpha-2\beta)}{(\alpha-\beta)^2}\frac{\partial\alpha}{\partial y^j} + \frac{\partial}{\partial\beta}\frac{\alpha(\alpha-2\beta)}{(\alpha-\beta)^2}\frac{\partial\beta}{\partial y^j}\right\} + \beta_{.0}\left\{\frac{\partial}{\partial\alpha}\frac{\alpha^2}{(\alpha-\beta)^2}\frac{\partial\alpha}{\partial y^j} + \frac{\partial}{\partial\beta}\frac{\alpha^2}{(\alpha-\beta)^2}\frac{\partial\beta}{\partial y^j}\right\}$$

which in view of (2.19), yields

$$\frac{1}{2}\frac{\partial^2 L}{\partial y^j\partial t} = \left[\frac{1}{2}\frac{(\alpha-2\beta)}{(\alpha-\beta)^2}\right]y_{j.0} + \frac{1}{2}\left[\frac{\alpha^2}{(\alpha-\beta)^2}\right]A_{j.0} + \alpha^{-2}(\alpha\,\alpha_{.0}\,y_j)\left[\frac{\beta^2}{(\alpha-\beta)^3} - \frac{1}{2}\frac{(\alpha-2\beta)}{(\alpha-\beta)^2}\right] - \frac{\beta}{(\alpha-\beta)^3}\left[\alpha\,\alpha_{.0}A_j + y_j\beta_{.0}\right] + \frac{\alpha^2}{(\alpha-\beta)^3}A_j\beta_{.0}$$

where $y_{j.0} = \dot{\partial}_0 y_j\,,\ A_{j.0} = \dot{\partial}_0 A_j$.

This gives

$$\frac{1}{2}\frac{\partial^2 L}{\partial y^j\partial t} = \rho\, y_{j.0} + \rho_1 A_{j.0} + \rho_{-2}\alpha\,\alpha_{.0}\,y_j + \rho_{-1}\big(y_j\beta_{.0} + \alpha\,\alpha_{.0}A_j\big) + \rho_0\,\beta_{.0}A_j \tag{3.6}$$

where

$$\rho = \frac{1}{2}\frac{(\alpha-2\beta)}{(\alpha-\beta)^2} \quad , \qquad \rho_0 = \frac{\alpha^2}{(\alpha-\beta)^3}$$

$$\rho_1 = \frac{1}{2} \quad \frac{\alpha^2}{(\alpha-\beta)^2} \qquad \rho_{-1} = -\frac{\beta}{(\alpha-\beta)^3} \tag{3.7}$$

$$\rho_{-2} = \frac{1}{2}\alpha^{-2}\left[\frac{2\beta^2}{(\alpha-\beta)^3} - \frac{(\alpha-2\beta)}{(\alpha-\beta)^2}\right]$$

The metric tensor g_{ij} of a Lagrange space with Matsumoto metric is given by [14]

$$g_{ij}(x,y) = \rho\, a_{ij}(x) + C_i C_j \tag{3.8}$$

where

$$C_i = q_{-1}y_i + q_0 A_i \tag{3.9}$$

and q_{-1}, q_0 satisfy

$$\rho_0 = (q_0)^2 \ , \qquad \rho_{-1} = \quad q_0 q_{-1}, \qquad \rho_{-2} = \quad (q_{-1})^2 \tag{3.10}$$

$$g_{ij}(x,y) = \rho\, a_{ij}(x) + \rho_0 A_i(x)A_j(x) + \rho_{-1}\left(y_i A_j + y_j A_i\right) + \rho_{-2} y_i y_j \tag{3.11}$$

The inverse tensor g^{ij} of g_{ij} is given by

$$g^{ij} = \frac{1}{\rho}\left[a^{ij} - \frac{1}{(\rho+c^2)}c^i c^j\right] \tag{3.12}$$

$$c^2 = a^{ij}c^i c^j \tag{3.13}$$

Equations (3.11) and (3.12) can be extended to obtain the expression for the tensor $g_{ij}$and its inverse g^{ij} for the rheonomic Lagrange space with Matsumoto metric:

$$g_{ij}(x,y,t) = \rho a_{ij}(x,t) + \rho_0 A_i(x,t)A_j(x,t) + \rho_{-1}\left(y_i A_j + y_j A_i\right) + \rho_{-2} y_i y_j \tag{3.14}$$

$$g^{ij} = \frac{1}{\rho}a^{ij}(x,t) - \frac{1}{\rho^2(1+c^2)}c^i c^j \tag{3.15}$$

where C^i satisfies conditions similar to (3.13) on $E = TM \times \mathbb{R}$.
in view of (3.6) and (2.5), we have

$$G_0^i = g^{ij}\left[\rho\, y_{j.0} + \rho_1 A_{j.0} + \rho_{-2}\alpha\, \alpha_{.0}\, y_j + \rho_{-1}\left(y_j \beta_{.0} + \alpha\, \alpha_{.0} A_j\right) + \rho_0\, \beta_{.0} A_j\right] \tag{3.16}$$

where g^{ij} is given by (3.15).
Thus, we have:

Theorem 1:*There is a semispray S of a rheonomic Lagrange space with Matsumoto metric which depends upon the Lagrange space only and whose coefficients* $\left(G^i(x,y,t), G_0^i(x,y,t)\right)$ *are given by (3.3) and (3.16).*

The integral curves of the semispray S are given by the Euler-Lagrange equations $E_i(L) = 0$, which are equivalent to [2] and [11]

$$\frac{d^2x^i}{d\sigma^2} + G^i\left(x, \frac{dx}{d\sigma}, \sigma\right) + G_0^i\left(x, \frac{dx}{d\sigma}, \sigma\right) = 0 \tag{3.17}$$

In view of (3.3) and (3.16), equation (3.17) takes the form

$$\frac{d^2x^i}{d\sigma^2} + \gamma^i_{jk}(x,\sigma)y^j y^k - \lambda(x,y,\sigma)F^i_j(x,\sigma)y^j + g^{ij}\left[\rho\, y_{j.0} + \rho_1 A_{j.0} + \rho_{-2}\alpha\, \alpha_{.0}\, y_j + \rho_{-1}\left(y_j\beta_{.0} + \alpha\, \alpha_{.0}A_j\right) + \rho_0\, \beta_{.0}A_j\right] = 0$$

$$y^j = \frac{dx^j}{d\sigma}, \qquad (3.18)$$

i.e.

$$\frac{d^2x^i}{d\sigma^2} + \gamma^i_{jk}(x,\sigma)y^j y^k = \lambda(x,y,\sigma)F^i_j(x,\sigma)y^j - g^{ij}\left[\rho\, y_{j.0} + \rho_1 A_{j.0} + \rho_{-2}\alpha\, \alpha_{.0}\, y_j + \rho_{-1}\left(y_j\beta_{.0} + \alpha\, \alpha_{.0}A_j\right) + \rho_0\, \beta_{.0}A_j\right] \qquad (3.19)$$

$$y^j = \frac{dx^j}{d\sigma}$$

Thus, we have

Theorem 2*:The integral curves of the semispray S of a rheonomic Lagrange space with Matsumoto metric are given by the second order differential equations (SODE)* (3.19).

IV. Canonical Nonlinear Connection, Autoparallel Curves

In this section, we obtain the coefficients of a canonical non-linear connection $N\left(N^i_j\,,\, N_j\right)$ for the semispray S (discussed in the preceding section) of a rheonomic Lagrange space with Matsumoto metric $L^n = \left(M, L(\alpha,\beta)\right)$. We also obtain differential equation of the autoparallel curves with respect to this nonlinear connection.

The coefficients of canonical nonlinear connection N for the semispray S are given by (2.11) and (2.12). Nicolaescu [14] obtained the following form of the coefficients N^i_j of the canonical nonlinear connection for a Lagrange space with Matsumotometric.

$$N^i_j(x,y) = \gamma^i_{jk}(x)y^k - \frac{1}{2}\lambda^k_j F^i_k(x), \qquad (4.1)$$

where $\lambda^k_j = \lambda\delta^k_j + \frac{\partial\lambda}{\partial y^j}y^k$ with λ given by (3.2).

The coefficients $N^i_j(x,y,t)$for a rheonomic Lagrange space with Matsumotometric can be written as

$$N^i_j(x,y,t) = \gamma^i_{jk}(x,t)y^k - \frac{1}{2}\lambda^k_j(x,y,t)F^i_k(x,t), \qquad (4.2)$$

where $\lambda^k_j(x,y,t) = \lambda(x,y,t)\,\delta^k_j + \frac{\partial\lambda}{\partial y^j}y^k$ with λ given by (3.4).

In view of (2.12) and (3.6), we get

$$N_j = \rho\, y_{j.0} + \rho_1 A_{j.0} + \rho_{-2}\alpha\, \alpha_{.0}\, y_j + \rho_{-1}\left(y_j\beta_{.0} + \alpha\, \alpha_{.0}A_j\right) + \rho_0\, \beta_{.0}A_j, \qquad (4.3)$$

Where ρ, ρ_1 , ρ_{-2}, ρ_{-1} and ρ_0 are given by (3.7).

***Theorem 3*:** *The coefficients of canonical nonlinear connection N produced by the semispray S of a rheonomic Lagrange space with Matsumoto metric are given by (4.2) and (4.3).*

The autoparallel curve with respect to the nonlinear connection N produced by a semispray of a rheonomic Lagrange spaces are solution curves of the following differential equations (cf. [1]):

$$\frac{d^2x^i}{dt^2} + N^i_j\left(x, \frac{dx}{dt}, t\right)\frac{dx^j}{dt} = 0\,, \qquad N_j\left(x, \frac{dx}{dt}, t\right)\frac{dx^i}{d\,\sigma} + 1 = 0 \qquad (4.4)$$

In view of (4.2) and (4.3), equations (4.4) take the form

$$\frac{d^2x^i}{dt^2} + \gamma^i_{jk}(x,t)y^j\, y^k - \frac{1}{2}\lambda^k_j(x,y,t)F^i_k(x,t)\, y^j\frac{dx^j}{d\,t} = 0\,,$$

$$\left[\rho\, y_{j.0} + \rho_1 A_{j.0} + \rho_{-2}\alpha\, \alpha_{.0}\, y_j + \rho_{-1}\left(y_j\beta_{.0} + \alpha\, \alpha_{.0}A_j\right) + \rho_0\, \beta_{.0}A_j\right]\frac{dx^i}{d\,\sigma} + 1 = 0 \qquad (4.5)$$

Thus, we have:

Theorem 4: *The autoparallel curves with respect to the nonlinear connection N produced by the semispray S of a rheonomic Lagrange space with Matsumoto metric are solution curves of the system of differential equations (4.5).*

V. Canonical Metrical N-linear Connection

In this section we deal with the canonical metrical N-linear connection $C\Gamma(N) = \left(L^i_{jk}\,, C^i_{jk}\,, C^i_{j0}\right)$of a rheonomic Lagrange space with Matsumoto metric and obtain its coefficients.

If we partially differentiate the quantities appearing in (3.7) with respect to x^j , y^j and *t*, we respectively find the following sets of quantities:

$$\partial_j\rho = \frac{1}{2}\rho_{-2}\xi_j + \rho_{-1}\zeta_j\,, \qquad \partial_j\rho_0 = \frac{1}{2}\mu_{-1}\xi_j + \mu_0\zeta_j\,,$$

$$\partial_j\rho_{-1} = \frac{1}{2}\mu_{-2}\xi_j + \mu_{-1}\zeta_j\,, \qquad \partial_j\rho_{-2} = \frac{1}{2}\mu_{-3}\xi_j + \mu_{-2}\zeta_j\,, \tag{5.1}$$

$$\dot\partial_j\rho = \rho_{-2}y_j + \rho_{-1}A_j\,, \qquad \dot\partial_j\rho_0 = \mu_{-1}y_j + \mu_0 A_j\,,$$

$$\dot\partial_j\rho_{-1} = \mu_{-2}y_j + \mu_{-1}A_j\,, \qquad \dot\partial_j\rho_{-2} = \mu_{-3}\,y_j + \mu_{-2}A_j\,, \tag{5.2}$$

And

$$\dot\partial_0\rho = \alpha\,\alpha_{.0}\rho_{-2} + \beta_{.0}\rho_{-1}\,, \qquad \dot\partial_0\rho_0 = \alpha\,\alpha_{.0}\mu_{-1} + \mu_0\beta_{.0}$$

$$\dot\partial_0\rho_{-1} = \alpha\,\alpha_{.0}\,\mu_{-2} + \mu_{-1}\beta_{.0}\,, \qquad \dot\partial_0\rho_{-2} = \mu_{-3}\,\alpha\,\alpha_{.0} + \mu_{-2}\beta_{.0} \tag{5.3}$$

where $\xi_j = \partial_j a_{rs}y^r y^s,\ \zeta_j = \partial_j A_r y^r,\quad \mu_0 = \frac{1}{2}L_{\beta\beta\beta} = \left[\frac{3\,\alpha^2}{(\alpha-\beta)^4}\right],$

$$\mu_{-1} = \frac{1}{2}\alpha^{-1}L_{\alpha\beta\beta} = \frac{-(\alpha+2\beta)}{(\alpha-\beta)^4}\,, \qquad \mu_{-2} = \frac{1}{2}\alpha^{-2}\left(L_{\alpha\alpha\beta} - \alpha^{-1}L_{\alpha\beta}\right) = \alpha^{-1}\left\{\frac{3\beta}{(\alpha-\beta)^4}\right\}$$

$$\mu_{-3} = \frac{1}{2}\alpha^{-3}(L_{\alpha\alpha\alpha} - 3\alpha^{-1}L_{\alpha\alpha} + 3\alpha^{-2}L_{\alpha}) = \frac{3}{2}\frac{\alpha^{-3}(\alpha^2+\beta^2-4\alpha\beta)}{(\alpha-\beta)^4} \tag{5.4}$$

Where

$$L_{\beta\beta\beta} = \frac{6\,\alpha^2}{(\alpha-\beta)^4}\,, \quad L_{\alpha\beta\beta} = \frac{-2\alpha\,(\alpha+2\beta)}{(\alpha-\beta)^4}\,, \qquad L_{\alpha\alpha\beta} = -\frac{2\beta}{(\alpha-\beta)^3} + \frac{6\alpha\beta}{(\alpha-\beta)^4}$$

$$L_{\alpha\alpha\alpha} = -\frac{6\,\beta^2}{(\alpha-\beta)^4}$$

In view of (3.14), we have

$$2\,C_{jhk} = \dot\partial_j g_{hk} = a_{hk}\dot\partial_j\,\rho + A_hA_k\dot\partial_j\rho_0 + (\dot\partial_j\rho_{-1})\mathfrak{S}_{(hk)}\{y_hA_k\} + \rho_{-1}\mathfrak{S}_{(hk)}\{a_{hj}A_k\} + (\dot\partial_j\rho_{-2})y_hy_k + \rho_{-2}\mathfrak{S}_{(hk)}\{a_{hj}y_k\}, \tag{5.5}$$

where $\mathfrak{S}_{(hk)}$ denotes the interchange of indices h and k and addition. Using (5.2) and (5.5), we get

$$2\,C_{jhk} = a_{hk}\left(\rho_{-2}y_j + \rho_{-1}A_j\right) + A_hA_k\left(\mu_{-1}y_j + \mu_0A_j\right) + \left(\mu_{-2}y_j + \mu_{-1}A_j\right)\mathfrak{S}_{(hk)}\{y_hA_k\} + \rho_{-1}\mathfrak{S}_{(hk)}\{a_{hj}A_k\}$$
$$+\left(\mu_{-3}y_j + \mu_{-2}A_j\right)y_hy_k + \rho_{-2}\mathfrak{S}_{(hk)}\{a_{hj}y_k\}, \tag{5.6}$$

Applying (5.6) in (2.16), we obtain

$$C^i_{jk} = \frac{1}{2}g^{ih}[a_{hk}\left(\rho_{-2}y_j + \rho_{-1}A_j\right) + A_hA_k\left(\mu_{-1}y_j + \mu_0A_j\right) + \left(\mu_{-2}y_j + \mu_{-1}A_j\right)\mathfrak{S}_{(hk)}\{y_hA_k\}$$
$$+\rho_{-1}\mathfrak{S}_{(hk)}\{a_{hj}A_k\} + \left(\mu_{-3}y_j + \mu_{-2}A_j\right)y_hy_k + \rho_{-2}\mathfrak{S}_{(hk)}\{a_{hj}y_k\}] \tag{5.7}$$

Differentiating (3.14) partially with respect to t, we have

$$2\, C_{jh0} = \dot{\partial}_0 g_{jh} = a_{jh}\dot{\partial}_0 \rho + \rho\, a_{jh.0} + A_j A_h \dot{\partial}_0 \rho_0 + \rho_0 \mathfrak{S}_{(hk)}\{A_{j.o} A_h\} + \left(\dot{\partial}_0 \rho_{-1}\right)\mathfrak{S}_{(jh)}\{y_j A_h\}$$
$$+\rho_{-1}\mathfrak{S}_{(jh)}\{y_{j.0} A_h + y_j A_{h.0}\} + \left(\dot{\partial}_0 \rho_{-2}\right) y_j y_h + \rho_{-2}\mathfrak{S}_{(jh)}\{y_{j.0} y_h\},$$

Which, in view of (5.3) becomes

$$2\, C_{jh0} = a_{jh}(\rho_{-2}\, \alpha\, \alpha_{.0} \ + \rho_{-1}\, \beta_{.0}) + \rho\, a_{jh.0} + A_j A_h(\mu_{-1}\, \alpha\, \alpha_{.0} \ + \mu_0\, \beta_{.0}) + \rho_0 \mathfrak{S}_{(jh)}\{A_{j.o} A_h\}$$
$$+ (\mu_{-2}\, \alpha\, \alpha_{.0} \ + \mu_{-1}\, \beta_{.0})\mathfrak{S}_{(jh)}\{y_j A_h\} + \rho_{-1}\mathfrak{S}_{(jh)}\{y_{j.o} A_h + y_j A_{h.0}\}$$
$$+ (\mu_{-3}\, \alpha\, \alpha_{.0} \ + \mu_{-2}\, \beta_{.0}) y_j y_h + \rho_{-2}\mathfrak{S}_{(jh)}\{y_{j.0} y_h\}, \tag{5.8}$$

Using (5.8) in (2.17), we get

$$C^i_{j0} = \frac{1}{2} g^{ih}[a_{jh}(\rho_{-2}\, \alpha\, \alpha_{.0} \ + \rho_{-1}\, \beta_{.0}) + \rho\, a_{jh.0} + A_j A_h(\mu_{-1}\, \alpha\, \alpha_{.0} \ + \mu_0\, \beta_{.0}) + \rho_0 \mathfrak{S}_{(jh)}\{A_{j.o} A_h\}$$
$$+ (\mu_{-2}\, \alpha\, \alpha_{.0} \ + \mu_{-1}\, \beta_{.0})\mathfrak{S}_{(jh)}\{y_j A_h\} + \rho_{-1}\mathfrak{S}_{(jh)}\{y_{j.o} A_h + y_j A_{h.0}\}$$
$$+ (\mu_{-3}\, \alpha\, \alpha_{.0} \ + \mu_{-2}\, \beta_{.0}) y_j y_h + \rho_{-2}\mathfrak{S}_{(jh)}\{y_{j.0} y_h\}], \tag{5.9}$$

Differentiating (3.14) partially with respect to x^j, we have

$$\partial_j g_{hk} = X_{hk}\xi_j + Y_{hk}\zeta_j + \rho\, \partial_j a_{hk} + \rho_0 \mathfrak{S}_{(hk)}\{A_k \partial_j\, A_h\} + \rho_{-1}\mathfrak{S}_{(hk)}\{y_h \partial_j A_k + A_k \partial_j\, y_h\}$$
$$+\rho_{-2}\mathfrak{S}_{(hk)}\{y_h\, \partial_j y_k\} \tag{5.10}$$

where

$$X_{hk} = \frac{1}{2}\left(\rho_{-2}\, a_{hk} + \mu_{-1} A_h A_k + \mu_{-2}\mathfrak{S}_{(hk)}\{y_h A_k\} + \mu_{-3} y_h y_k\right),$$

$$Y_{hk} = \rho_{-1}\, a_{hk} + \mu_0 A_h A_k + \mu_{-1}\mathfrak{S}_{(hk)}\{y_h A_k\} + \mu_{-2} y_h y_k$$

Now from (2.8), we have $\delta_j g_{hk} = \left(\partial_j - N^r_j \dot{\partial}_r - N_j \dot{\partial}_0\right) g_{hk}$, which in view of (5.6), (5.8) and (5.10) yields

$$\delta_j g_{hk} = X_{hk}\xi_j + Y_{hk}\zeta_j + \rho\, \partial_j a_{hk} + \rho_0 \mathfrak{S}_{(hk)}\{A_k \partial_j\, A_h\} + \rho_{-1}\mathfrak{S}_{(hk)}\{y_h \partial_j A_k + A_k \partial_j\, y_h\}$$
$$+\rho_{-2}\mathfrak{S}_{(hk)}\{y_h\, \partial_j y_k\} - 2\, N^r_j C_{rhk} - 2\, N_j C_{hk0} \tag{5.11}$$

Similarly, we have

$$\delta_k g_{jh} = X_{jh}\xi_k + Y_{jh}\zeta_k + \rho\, \partial_k a_{jh} + \rho_0 \mathfrak{S}_{(jh)}\{A_j \partial_k\, A_h\} + \rho_{-1}\mathfrak{S}_{(jh)}\{y_j \partial_k A_h + A_h \partial_k\, y_j\}$$
$$+\rho_{-2}\mathfrak{S}_{(jh)}\{y_j\, \partial_k y_h\} - 2\, N^r_k C_{rjh} - 2\, N_k C_{jh0} \tag{5.12}$$

and

$$\delta_h g_{jk} = X_{jk}\xi_h + Y_{jk}\zeta_h + \rho\, \partial_h a_{jk} + \rho_0 \mathfrak{S}_{(jk)}\{A_j \partial_h\, A_k\} + \rho_{-1}\mathfrak{S}_{(jk)}\{y_j \partial_h A_k + A_k \partial_h\, y_j\}$$
$$+\rho_{-2}\mathfrak{S}_{(jk)}\{y_j\, \partial_h y_k\} - 2\, N^r_h C_{rjk} - 2\, N_k C_{jk0} \tag{5.13}$$

Using (5.11)-(5.13) in (2.15), we have

$$L^i_{jk} = \rho\, \gamma^i_{jk} - \mathfrak{S}_{(jk)}\{N^m_j C^i_{mk} + N_j C^i_{k0}\} + N^{im} C_{mjk} - N^i C_{jk0} + \frac{1}{2} g^{ih} [X_{hk}\xi_j + X_{jh}\xi_k - X_{jk}\xi_h + Y_{hk}\zeta_j$$
$$+ Y_{jh}\zeta_k - Y_{jk}\zeta_h$$
$$+\rho_0\big(A_h \mathfrak{S}_{(jk)}\{\partial_j A_k\} + 2\,\mathfrak{S}_{(jk)}\{A_k F_{jh}\}\big) + \rho_{-1}(y_h \mathfrak{S}_{(jk)}\{\partial_j A_k\} + A_h \mathfrak{S}_{(jk)}\{\partial_j y_k\} + 2\,\mathfrak{S}_{(jk)}\{y_k F_{jh} + A_k K_{jh}\})$$
$$+\rho_{-2}\left(y_h \mathfrak{S}_{(jk)}\{\partial_j\, y_k\} + 2\,\mathfrak{S}_{(jk)}\{y_j K_{kh}\}\right)] \tag{5.14}$$

where

$$N^{im} = g^{ih} N^m_h, \quad N^i = g^{ih} N_h\,, \qquad K_{kh} = \tfrac{1}{2}(\partial_k y_h - \partial_h y_k).$$

Thus, we have

***Theorem 5*:** *For a rheonomic Lagrange space with Matsumoto metric, endowed with a nonlinear connection whose coefficients are given by (4.2) and (4.3), there is a unique canonical metrical n-linear connection $C\Gamma(N) = \left(L^i_{jk}\,, C^i_{jk}\,, C^i_{j0}\right)$ with the coefficients given by (5.7), (5.9) and (5.14).*

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Neetu Singh. "Rheonomic Lagrange Spaces with Matsumoto Metric." *IOSR Journal of Mathematics (IOSR-JM)*, 17(5), (2021): pp. 20-29.