IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 17, Issue 5 Ser. III (Sep. – Oct. 2021), PP 14-19
www.iosrjournals.org

Application for sequence of compact metric spaces U_n and X_n

Ahmed M.H.Yahya[1], Elsiddeg Ali [1,2]
1. *Shendi University Faculty of Science and Technology, Department of Mathematics, Sudan*
2. *Taif University ,Turabah University College , Department of Mathematics , Kingdom of Saudi Arabia,*

Abstract
in this present study, we discuss and generalize the proof that, if the sequence of Banach spaces X_n contains some regular convex sequence of subspaces at a given geometrical position, then the $C(U_n, X_n)$ distances (U_n, X_n) for all continuous functions with value X_n specified on the sequence of combined metric spaces U_n have exactly the same isomorphism classes Such as $C(U_n)$ spaces . As a consequence, we show that if $1 < p < \infty$ and $\varepsilon > 0$ then for every infinite sequence of countable compact metric spaces U_{n+1} , U_{n+2}, U_{n+3} and U_{n+4} are equivalent:
(a) $C(U_{n+1}, l_p) \oplus C(U_{n+2}, l_{p+\varepsilon})$ is isomorphic to $C(U_{n+3}, l_p) \oplus C(U_{n+4}, l_{p+\varepsilon})$.
(b) $C(U_{n+1})$ is isomorphic to $C(U_{n+3})$ and $C(U_{n+2})$ is isomorphic to $C(U_{n+4})$.
Keywords: *Isomorphic classification of $C(U_n, X_n)$spaces Bessaga–Pełczyn´ ski's and Milutin's theorems on separable $C(U_n)$spaces*

Date of Submission: 10-10-2021 Date of Acceptance: 24-10-2021

I. Introduction

Let X_n besequence of a Banach spaces and U_n sequence ofa compact Hausdorff spaces. By $C(U_n, X_n)$we denote the sequence ofBanach spaces of all continuousX_n-valued functions defined on $U_n$and equipped with the supremum norm. This spaces will be denoted by $C(U_n)$in the case$X_n = \mathbb{R}$. We write $X_n \sim X_{n+1}$when thesequence ofBanach spaces $X_n$and $X_{n+1}$are isomorphic and $X_{n+1} \hookrightarrow X_n$when $X_n$contains a copy of , thatis, a subspace isomorphic to X_{n+1} The interval of real numbers $[0, 1]$ will be denoted by I. By $[0, \alpha]$ we denote the interval ofordinals $\{(\gamma+\varepsilon), 0 \le (\gamma+\varepsilon) \le \alpha\}$ endowed with the order topology (See[13]). We will follow [6] for the related notation and terminology onBanach spaces.(See[2,9]).

Theorem 1.1. Let $\omega \le \alpha < \omega_1, \varepsilon > 0$ be ordinals andU_n an infinitesequence of compact metric spaces. Then
(a) $C([0, \alpha]) \sim C([0, \alpha+\varepsilon]) \Leftrightarrow (\alpha+\varepsilon) < \alpha^{\omega}$.
(b) $C(I) \sim C(U_n) \Leftrightarrow U_n$ is uncountable.

We recall that a well-known theorem of Mazurkiewicz and Sierpin´ ski [7] states that every infinite sequence ofcountable compactmetric spaces is homeomorphic to an interval $[0, \alpha]$ with $\omega \le \alpha < \omega_1$.(See[12]).

We consider here the problem of determining geometric conditions on the subspaces of a sequence ofBanach spaces$X_n$which would allowgeneralizations of Theorem 1.1 to the spaces of the form $C(U_n, X_n)$. In order to state our main result (Theorem 1.3) it isconvenient to introduce:

Definition 1.2. We say that sequence ofa subspace $H_n$of sequence ofa Banach space $X_n$is a maximal factor of $X_n$whenever $X_n$is the direct sum of $H_n$and some subspace $X_{n+1}$of $X_n$such that every finite sum $(X_{n+1})^n$of X_{n+1}contains no copy of H_n.

Recall also thatsequence of a Banach space $H_n$is said to be uniformly convex [3] if for each $\varepsilon, 0 < \varepsilon \le 2$, there corresponds a $\delta(\varepsilon)$such that for every h_n and h_{n+1} in $H_n$we have

$$\|h_n\| = \|h_{n+1}\| = 1 \; and \; \|h_n - h_{n+1}\| \ge \varepsilon \Rightarrow \|h_n + h_{n+1}\| \le 2(1-\delta(\varepsilon)).$$

Thus the principal purpose of the present work is to prove:

Theorem 1.3. Suppose that $1 < p < \infty$ and $H_n$is a uniformly convex space containing a copy of l_p . Then for very Banach spacesX_{n+1}containing no copy of l_p and infinite compact metric spaces U_{n+1} and U_{n+2}

$$C(U_{n+1}, X_{n+1} \oplus H_n) \sim C(U_{n+2}, X_{n+1} \oplus H_n) \Leftrightarrow C(U_{n+1}) \sim C(U_{n+2}).$$

We use Theorem 1.3. to provide the isomorphic classification ofthe following spaces.
(a) $C(U_{n+1}, l_p) \oplus C(U_{n+2}, l_{(p+\varepsilon)})$, where $1 < p < \infty, \varepsilon > 0$ and U_{n+1} and U_{n+2} are arbitrary infinite countable compact metric spaces(Theorem 3.1).
(b) $C(I, X_n) \oplus C(U_n, l_p(\Gamma))$, where $X_n$is an arbitrary separable Banach space, $1 < p < \infty, \Gamma$is an uncountable set and $U_n$isarbitrary infinite countable compact metric space.

II. Preliminary results

Before proving Theorem 3.1, we establish some auxiliary results. From now on following [2] the $C([0,\alpha],X_n)$spaces willbe denoted by $(X_n)^\alpha$. We set $(X_n)_0^\alpha = \{f_n \in (X_n)^\alpha : f_n(\alpha) = 0\}$. In what follows, we will often make use without explicit mentionof [2] which states that $(X_n)^\alpha$is isomorphic to $(X_n)_0^\alpha$whenever $\alpha \geq \omega$.

Proposition 2.1. Let H_n be a uniformly convex Banach space and X_{n+1} a Banach space containing no copy of H_n. Suppose that

$$H_n^{(\gamma+\varepsilon)^\omega} \hookrightarrow X_{n+1} \oplus H_n^{(\gamma+\varepsilon)} ,$$

for some $\omega \leq \gamma + \varepsilon < \omega_1$, $\varepsilon > 0$. Then there exists $\omega \leq \gamma$such that

$$H_n^{(\gamma+\varepsilon)} \hookrightarrow X_{n+1} \oplus H_n^{(\gamma)} .$$

Proof. First of all notice that since $H_n$is a uniformly convex space, it follows from a Pisier theorem, see [11] that$H_n$admits an equivalent norm which will be denoted by $\|.\|$ such that there exist $\delta > 0$ and $p \in \mathbb{R}, 2 < p < \infty$ in such a waythat if $b \in \mathbb{R}_+$ and $h_n, h_{n+1} \in H_n$with $\|h_n\| \geq 1$ and $\|h_{n+1}\| \geq b$, then

$$\|h_n + h_{n+1}\| \geq \sqrt[p]{1+\delta b} \quad or \quad \|h_n - h_{n+1}\| \geq \sqrt[p]{1+\delta b}.$$

So, given $h_n, h_{n+1}, \dots, h_{n+m} \in H_n$, with $\|h_n\| = 1, \|h_{n+i}\| \geq \sqrt[p]{b}, i = 2,3,\dots,m$, there exists $c_i \in \mathbb{R}, |c_i| = 1, i = 1,2,\dots,m, n = 1,2,3,\dots$ such that

$$\left\| \sum_{n=1}^{\infty} \sum_{i=1}^{m} c_i h_{n+i} \right\| \geq \sqrt[p]{1+(m-1)\delta b}. \quad (1)$$

We now assume that

$$H_n^{(\gamma+\varepsilon)} \not\hookrightarrow X_{n+1} \oplus (H_n)_0^\gamma , \quad \forall\, 0 < \varepsilon,$$

and argue to a contradiction. By hypothesis there exist two bounded linear operators $\Pi_n : H_n^{(\gamma+\varepsilon)^\omega} \to X_{n+1}$ *and* $\Pi_{n+1} : H_n^{(\gamma+\varepsilon)^\omega} \to (H_n)_0^{(\gamma+\varepsilon)}$and $a \in \mathbb{R}_+$ such that for every $f_n \in H_n^{(\gamma+\varepsilon)^\omega}$we have
$a\|f_n\| \leq sup(\|\Pi_n(f_n)\|, \|\Pi_{n+1}(f_n)\|) \leq \|f_n\|$.(2)
Fix $m \in N$ such that

$$a\sqrt[p]{1+(m-1)\delta} > 1,$$

and $\varepsilon > 0$ such that
$1 + \varepsilon < a\sqrt[p]{1+(m-1)\delta}$. (3)
Denote for every $\eta \in [0,(\gamma+\varepsilon))$,

$$\Delta_\eta^1 = [(\gamma+\varepsilon)^m \eta + 1, (\gamma+\varepsilon)^m(\eta+1)],$$

and

$$(H_n)_m = \{f_n \in H_n^{(\gamma+\varepsilon)^m} : \forall \eta \in [0,(\gamma+\varepsilon)), f_n \textit{ is constant in } \Delta_\eta^1 \textit{ and } f_n(\gamma) = 0, \forall \gamma \in [(\gamma+\varepsilon)^{m+1}, (\gamma+\varepsilon)^m]\}.$$

It is easy to check that $(H_n)_m$is isometric to $H_n^{(\gamma+\varepsilon)}$. Since X_{n+1}contains no copy of H_n, the restriction of Π_n to $(H_n)_m$is not anisomorphism onto its image. Therefore there exists $f_n \in (H_n)_m$with

$$\|f_n\| = 1 \textit{ and } \|\Pi_n(f_n)\| \leq \varepsilon/2.$$

Let $0 \leq \eta_n < (\gamma+\varepsilon)$be such that there exists $h_n \in H_n$with

$$\|h_n\| = 1 \textit{ and } f_n(\gamma) = h_n, \forall \gamma \in \Delta_{\eta_1}^1 .$$

Since $\Pi_{n+1}(f_n) \in (H_n)_0^{(\gamma+\varepsilon)}$, it follows that there exists $\gamma_1 < (\gamma+\varepsilon)$such that

$$\|\Pi_{n+1}(f_n(\gamma))\| \leq \frac{\varepsilon}{2}, \qquad \forall \gamma \in [\gamma_1 + 1, (\gamma+\varepsilon)).$$

Denote for every $\eta \in [0,(\gamma+\varepsilon))$,

$$\Delta_\eta^2 = [(\gamma+\varepsilon)^m \eta_n + (\gamma+\varepsilon)^{m-1}\eta + 1, (\gamma+\varepsilon)^m \eta_n + (\gamma+\varepsilon)^{m-1}(\eta+1)],$$

and

$$(H_n)_{m-1} = \{f_n \in H^{(\gamma+\varepsilon)^m} : \forall \eta \in [0,(\gamma+\varepsilon)), f_n \textit{ is constant in } \Delta_\eta^2 \textit{ and } f_n(\gamma) = 0, \forall \gamma \notin [(\gamma+\varepsilon)^m \eta_n, (\gamma+\varepsilon)^m(\eta_n+1)]\}.$$

Again it is easy to see that $(H_n)_{m-1}$ is isometric to $H_n^{(\gamma+\varepsilon)}$. Let P_{γ_1} be the canonical projection from $H_n^{(\gamma+\varepsilon)}$onto $H_n^{\gamma_1}$. Since byhypothesis

$$H_n^{(\gamma+\varepsilon)} \not\hookrightarrow X_{n+1} \oplus H_n^{\gamma_1} ,$$

it follows that the restriction of the bounded linear operator$\Pi_n + P_{\gamma_1}\Pi_{n+1}$defined by

$$(\Pi_n + P_{\gamma_1}\Pi_{n+1})(g) = (\Pi_n(g), P_{\gamma_1}\Pi_{n+1}(g))$$

to $(H_n)_{m-1}$cannot be an isomorphism onto the image. Therefore there exists $f_{n+1} \in (H_n)_{m-1}$ such that

$$\|f_{n+1}\| = 1, \qquad \|\Pi_n(f_{n+1})\| < \frac{\varepsilon}{2^2} \ and \ \|\Pi_{n+1}(f_{n+1})\| \leq \varepsilon/2^2, \qquad \forall\gamma \in [0,\gamma_1].$$

Pick $\gamma_2 \in [\gamma_1 + 1, (\gamma+\varepsilon))$ such that

$$\|\Pi_2(f_{n+1})\| < \varepsilon/2^2, \qquad \forall\gamma \in [\gamma_2 + 1, (\gamma+\varepsilon)).$$

Let $0 \leq \eta_{n+1} < (\gamma+\varepsilon)$be such that there exists $h_{n+1} \in H_n$with $\|h_{n+1}\| = 1 \ and \ f(\gamma) = h_{n+1}, \forall\gamma \in \Delta^2_{\eta_{n+1}}$.

Repeating this procedure $m$times we can find ordinals $\eta_n, \eta_{n+1}, \ldots, \eta_{n+m}$, functions $f_n, f_{n+1}, \ldots, f_{n+m}$and elements $h_n, h_{n+1}, \ldots, h_{n+m} \in H_n$such that:

(i) $\|\Pi_n(f_{n+i})\| < \frac{\varepsilon}{2^i}$, $1 \leq i \leq m$.

(ii) $\|\Pi_{n+1}(f_{n+i})(\gamma)\| < \frac{\varepsilon}{2^i}$, $\forall\gamma \in [0,\gamma_{i-1}] \ and \ 2 \leq i \leq m-1$.

(iii) $\|\Pi_{n+2}(f_{n+i})(\gamma)\| < \frac{\varepsilon}{2^i}$, $\forall\gamma \in [\gamma_i, (\gamma+\varepsilon)] \ and \ 2 \leq i \leq m$

(iv) $\|h_{n+i}\| = 1, 1 \leq i \leq m$.

(v)$f_{n+i}(t) = h_{n+i}, \forall\gamma \in \Delta^i_{\eta_i} \ and \ 1 \leq i \leq m$, where

$$\Delta^i_{\eta_i} = \left[(\gamma+\varepsilon)^m\eta_1 + (\gamma+\varepsilon)^{m-1}\eta_2 + \cdots + (\gamma+\varepsilon)^{m-(i-1)}\eta_i, (\gamma+\varepsilon)^m\eta_1 + (\gamma+\varepsilon)^{m-1}\eta_2 + \cdots + (\gamma+\varepsilon)^{m-(i-1)}(\eta_i - 1)\right].$$

According to (1) there exists $c_i \in \mathbb{R}$with $|c_i| = 1, i = 1,2,\ldots,m$, such that

$$\left\|\sum_{i=1}^m c_i h_i\right\| \geq \sqrt[p]{1+(m-1)\delta}.$$

Let $f_n = \sum_{m=1}^{\infty}\sum_{i=1}^m f_i$. Hence it is clear that the following hold

(vi) $\|f_n\| \geq \sqrt[p]{1+(m-1)\delta}$.

(vii) $\|\Pi_{n+1}(f_n)\| \leq \varepsilon$.

(viii) $\|\Pi_{n+2}(f_n)\| \leq 1+\varepsilon$

Therefore by (2) we conclude that

$$a\sqrt[p]{1+(m-1)\delta} \leq 1+\varepsilon,$$

a contradiction with (3) and the proof is complete. □

Lemma 2.2. Let H_n and X_{n+1} be Banach spaces such that every bounded linear operator from c_0 to H is compact andX_{n+1} contains no copyof H_n. Then

$$H_0^\omega \nrightarrow X_{n+1} \oplus H_n.$$

Proof. Let $(H_n)_j = \{f \in H_0^\omega : f(m) = 0, \forall m < j\}$ and $P_2 : X_{n+1} \oplus H_n \to H_n$be the canonical projection. Let Tbe a bounded linearoperator from $(H_n)_0^\omega$ to $X_{n+1} \oplus H_n$. Since every bounded linear operator from $c_0$to $H_n$is compact, it follows that

$$P_2 T|_{H_n} \xrightarrow{n\to\infty} 0.$$

Putting $Q = I - P_2$ we deduce that

$$\left\|T - Q\,T|_{(H_n)_j}\right\| \xrightarrow{j\to\infty} 0.$$

Therefore if $T$was one-to-one and with closed image, then we would have that for $j$large enough $Q\,T|_{(H_n)_j}$would be anisomorphism onto its image, a contradiction because X_{n+1}contains no copy of H_n.□

Proposition 2.3. Let H_n be a uniformly convex Banach space and X_{n+1} a Banach space containing no copy of H_n. Then for every $\omega \leq (\gamma+\varepsilon) < \omega_1$

$$H_n^{(\gamma+\varepsilon)^\omega} \nrightarrow X_{n+1} \oplus H_n^{(\gamma+\varepsilon)} \quad .$$

Proof. Let Abe the set of ordinals $\omega \leq (\gamma+\varepsilon) < \omega_1$ satisfying the following condition:

$H_n^{(\gamma+\varepsilon)^\omega} \hookrightarrow X_{n+1} \oplus H_n^{(\gamma+\varepsilon)}$ for some$X_{n+1}$with $H_n \nrightarrow X_{n+1}$.

Suppose that $A \neq \emptyset$ and let $(\gamma+\varepsilon)_1$ be the minimum of A. So we infer that

$H_n^{(\gamma+\varepsilon)_1^\omega} \hookrightarrow (X_{n+1})_1 \oplus H_n^{(\gamma+\varepsilon)}$,(4)

for some Banach space $(X_{n+1})_1$ containing no copy ofH_n.

On the other hand, a well-known Milman–Pettis theorem states that every uniformly convex space is reflexive [8]or [10]. Therefore $(H_n)_j$contains no copy of c_0, for every $1 \leq j < \omega$. Hence every bounded linear operator from c_0 to $(H_n)_j$is compact, see for instance [1]. Thus, if $(X_{n+1})_2$ is an arbitrary Banach space containing no copy of H, then byLemma 2.2 and [2] we deduce that

$$H_n^\omega \sim H_n^{j\omega} \sim \left(H_n^j\right)^\omega \sim \left(H_n^j\right)_0^\omega \nrightarrow (X_{n+1})_2 \oplus H_n^j,$$

for every $1 \leq j < \omega$. So, by Proposition 2.1 ,we infer that

$$H_n^{\omega^\omega} \nrightarrow (X_{n+1})_2 \oplus H_n^\omega.$$

This means that $\omega < (\gamma+\varepsilon)_1$.

Next observe that

$H_n^{(\gamma+\varepsilon)_1} \nrightarrow (X_{n+1})_1 \oplus H_n^{(\gamma+\varepsilon)}$, $\forall(\gamma+\varepsilon) < (\gamma+\varepsilon)_1$.(5)

Indeed, otherwise, there exists an ordinal $(\gamma+\varepsilon)_0 < (\gamma+\varepsilon)_1$ such that

$H_n^{(\gamma+\varepsilon)_1} \hookrightarrow (X_{n+1})_1 \oplus H_n^{(\gamma+\varepsilon)_0}$. (6)

Hence by the minimality of $(\gamma+\varepsilon)_1$ we deduce that

$H_n^{(\gamma+\varepsilon)_0^\omega} \nrightarrow (X_{n+1})_1 \oplus H_n^{(\gamma+\varepsilon)_0}$,(7)

Thus (6) and (7) together imply that

$H_n^{(\gamma+\varepsilon)_0^\omega} \nrightarrow H_n^{(\gamma+\varepsilon)_1}$.(8)

Now we distinguish two cases:

Case 1. $(\gamma+\varepsilon)_0^\omega < (\gamma+\varepsilon)_1^\omega$. Then by [2] we see that

$$R^{(\gamma+\varepsilon)_0^\omega} \hookrightarrow R^{(\gamma+\varepsilon)_1}.$$

Consequently,

$$H_n^{(\gamma+\varepsilon)_0^\omega} \hookrightarrow H_n^{(\gamma+\varepsilon)_1},$$

which is absurd by (8).

Case 2. $(\gamma+\varepsilon)_1^\omega \le (\gamma+\varepsilon)_0^\omega$. In this case, since $(\gamma+\varepsilon)_0^\omega \le (\gamma+\varepsilon)_1^\omega$, it follows that $(\gamma+\varepsilon)_0^\omega = (\gamma+\varepsilon)_1^\omega$. That is, $(\gamma+\varepsilon)_0 < (\gamma+\varepsilon)_1 < (\gamma+\varepsilon)_1^\omega = (\gamma+\varepsilon)_0^\omega$. Then again by [2] we conclude that

$$R^{(\gamma+\varepsilon)_0} \sim R^{(\gamma+\varepsilon)_1},$$

and hence

$H_n^{(\gamma+\varepsilon)_0} \sim H_n^{(\gamma+\varepsilon)_1}$.

Then we can rewrite (7) as follows

$$H_n^{(\gamma+\varepsilon)_1^\omega} \nrightarrow (X_{n+1})_1 \oplus H_n^{(\gamma+\varepsilon)_1},$$

again a contradiction with (4).

So (5) holds and therefore by Proposition 2.1 applied in (5) we conclude that

$$H_n^{(\gamma+\varepsilon)_1^\omega} \nrightarrow (X_{n+1})_1 \oplus H_n^{(\gamma+\varepsilon)_1},$$

this contradicts (4). Thus $A = \emptyset$ and the proposition is proved. □

Theorem 2.4. Let X_n be a Banach space containing some maximal factor H_n which is uniformly convex, $\omega \le \alpha < \omega_1$, $\varepsilon > 0$ ordinals and$U_n$an infinite compact metric space. Then

(a) $C([0,\alpha],X_n) \sim C([0,\alpha+\varepsilon],X_n) \Leftrightarrow 1+\varepsilon < \alpha^{\omega-1}$.

(b) $C(I,X_n) \sim C(U_n,X_n) \Leftrightarrow U_n$is uncountable.

Proof. By Bessaga and Pełczyn´ ski's theorem and Milutin's theorem the conditions areof course sufficient. Let us show that they are also necessary.Since $H_n$is a maximal factor of $X_n$there exists a subspace $X_{n+1}$of $X_n$such that

$X_n = X_{n+1} \oplus H_n$ *and* $H_n \nrightarrow (X_{n+1})^j, \forall 1 \le j < \omega$.(9)

(a) Suppose then that $\omega \le \alpha < \omega_1, \varepsilon > 0$and

$$(X_n)^\alpha \sim (X_n)^{\alpha+\varepsilon}.$$

Assume the contrary that $\alpha^{\omega-1} \le \varepsilon+1$. Thus we have

$H_n^{\alpha^\omega} \hookrightarrow (X_n)^{\alpha^\omega} \hookrightarrow (X_n)^{\alpha+\varepsilon} \sim (X_n)^\alpha \sim (X_{n+1})^\alpha \oplus H_n^\alpha$. (10)

As we noticed in the proof of Proposition 2.3, $H_n$contains no copy of $c_0$and since (9) holds, it follows from [4] That

$$H_n \nrightarrow (X_{n+1})^\alpha.$$

Hence according to Proposition 2.3, (10) cannot hold. So we are done.

(b) Finally suppose that

$$C(I,X_n) \sim C(U_n,X_n),$$

for some infinite compact metric space U_n. We assume that $U_n$is countable and derive a contradiction. So by Mazurkiewiczand Sierpin´ ski's theorem there exists an infinite countable ordinal αsuch that $U_n$is homeomorphic to $[0,\alpha]$. Consequently

$$H_n^{\alpha^\omega} \hookrightarrow (X_n)^{\alpha^\omega} \hookrightarrow C(I,X_n) \sim (X_n)^\alpha.$$

Hence proceeding as in (10) we obtain the required contradiction. Thus Theorem 2.4. is proved.

III. Applications

In this section we devoted to providing some applications of Theorem 1.3 together with Theorem 1.1. Let us begin recalling if $U_n$is a compact Hausdorff space, the derivative of a subset $A$of $U_n$is defined to be the set of limits points of A. Thus a transfinite inductive sequence is defined as follows: $A^{(0)} = A, A^{(1)}$is the derivative of A, in general suppose $A^{(\alpha)}$has been defined for all ordinals $0 < \varepsilon, if\ \alpha + \varepsilon = \gamma + 1$, then $A^{(\alpha+\varepsilon)}$is the derivative of $A^{(\gamma)}$, otherwise $A^{(\alpha+\varepsilon)} = \bigcap_{0<\varepsilon} A^{(\alpha)}$.

Theorem 3.1 Let $1 < p < \infty, \varepsilon > 0$. Then for every infinite countable compact metric spaces $U_{n+1}, U_{n+2}, U_{n+3}$ and U_{n+4}

$$C(U_{n+1}, l_p) \oplus C(U_{n+2}, l_{p+\varepsilon}) \sim C(U_{n+3}, l_p) \oplus C(U_{n+4}, l_{p+\varepsilon}) \Leftrightarrow C(U_{n+1}) \sim C(U_{n+3})\ and\ C(U_{n+2}) \sim C(U_{n+4}).$$

Proof. Let us show the non-trivial implication. So suppose that

$C(U_{n+1}, l_p) \oplus C(U_{n+2}, l_{p+\varepsilon}) \sim C(U_{n+3}, l_p) \oplus C(U_{n+4}, l_{p+\varepsilon})$.(11)

Observe that Theorem 1.1(a) means that the spaces $C([0, \omega^{\omega^{\gamma}}])$for $0 \le \gamma < \omega_1$ are a complete set of representatives of theisomorphism classes of $C(U_n)$where $U_n$is countably infinite, compact metric space. Then, let $\delta_1, \delta_2, \delta_3$ and δ_4 be ordinalssuch that for every $1 \le i \le 4$

$$C(U_{n+i}) \sim C\left(\left[0, \omega^{\omega^{\delta_i}}\right]\right).$$

Fix $\omega < (\alpha + \varepsilon) < \omega_1$ satisfying $(\alpha + \varepsilon) > \delta_i$for every $1 \le i \le 4$. Now, for every $1 \le i \le 4$ consider the following compact metric spaces

$$F_i = \left[0, \omega^{\omega^{\delta_i}}\right] \times \left[0, \omega^{\omega^{(\alpha+\varepsilon)}}\right].$$

Since that $[0, \omega^{\lambda}](\lambda) = \{\omega^{\lambda}\}$ for every ordinal λ [5], it follows that

$$(F_i)^{(\omega^{(\alpha+\varepsilon)})} = \left[0, \omega^{\omega^{\delta_i}}\right] \times \left[0, \omega^{\omega^{(\alpha+\varepsilon)}}\right],$$

and therefore

$$(F_i)^{\left(\omega^{(\alpha+\varepsilon)} + \omega^{\delta_i}\right)} = \left((F_i)^{(\omega^{(\alpha+\varepsilon)})}\right)^{\omega^{\delta_i}} = \left\{\left(\omega^{\omega^{(\alpha+\varepsilon)}}, \omega^{\omega^{\delta_i}}\right)\right\}.$$

Hence by [5]$F_i$is homeomorphic to $[0, \omega^{\left(\omega^{(\alpha+\varepsilon)} + \omega^{\delta_i}\right)}]$,for every $1 \le i \le 4$. Set $U_{n+5} = [0, \omega^{\omega^{(\alpha+\varepsilon)}}]$. So again by Theorem 1.1(a) we have

$$C\left([0, \omega^{\left(\omega^{(\alpha+\varepsilon)} + \omega^{\delta_i}\right)}]\right) \sim C\left([0, \omega^{\omega^{(\alpha+\varepsilon)}}]\right).$$

Hence

$$C(U_{n+i} \times U_{n+5}) \sim C(F_i) \sim C\left(\left[0, \omega^{\left(\omega^{(\alpha+\varepsilon)} + \omega^{\delta_i}\right)}\right]\right) \sim C\left(\left[0, \omega^{\omega^{(\alpha+\varepsilon)}}\right]\right) \sim C(U_{n+5}), \qquad (12)$$

and

$C(U_{n+5}) \oplus C(U_{n+i}) \sim C\left(\left[0, \omega^{\omega^{(\alpha+\varepsilon)}}\right] \oplus \left[0, \omega^{\omega^{\delta_i}}\right]\right) \sim C\left(\left[0, \omega^{\left(\omega^{(\alpha+\varepsilon)} + \omega^{\delta_i}\right)}\right]\right) \sim C(U_{n+5})$.(13)

On the other hand, by (11) we deduce

$C(U_{n+5}, l_p) \oplus C(U_{n+1}, l_p) \oplus C(U_{n+2}, l_{p+\varepsilon}) \sim C(U_{n+5}, l_p) \oplus C(U_{n+3}, l_p) \oplus C(U_{n+4}, l_{p+\varepsilon})$.(14)

Thus by (13) and (14), we infer

$$C(U_{n+5}, l_p) \oplus C(U_{n+2}, l_{p+\varepsilon}) \sim C(U_{n+5}, l_p) \oplus C(U_{n+4}, l_{p+\varepsilon}).$$

But in view of (12) we conclude

$$C(U_{n+2} \times U_{n+5}, l_p) \oplus C(U_{n+2}, l_{p+\varepsilon}) \sim C(U_{n+4} \times U_{n+5}, l_p) \oplus C(U_{n+4}, l_{p+\varepsilon}),$$

that is

$$C\big(U_{n+2}, C(U_{n+5}, l_p)\big) \oplus C(U_{n+2}, l_{p+\varepsilon}) \sim C\big(U_{n+4}, C(U_{n+5}, l_p)\big) \oplus C(U_{n+4}, l_{p+\varepsilon}).$$

Consequently,

$$C\big(U_{n+2}, C(U_{n+5}, l_p) \oplus l_{p+\varepsilon}\big) \sim C\big(U_{n+4}, C(U_{n+5}, l_p) \oplus l_{p+\varepsilon}\big).$$

Furthermore, by [2] and [4] we know that $C(U_{n+5}, l_p)$contains no copy of $l_{p+\varepsilon}$. Hence by Theorem 1.3, we concludethat $C(U_{n+2})$is isomorphic to $C(U_{n+4})$. Analogously we prove that $C(U_{n+1})$is isomorphic to $C(U_{n+3})$.□

Theorem 3.2. Let U_n be an infinite countable compact space and $1 < p < \infty$. Then for every infinite countable compact metric spacesU_{n+1} and U_{n+2}

$$C(U_n) \oplus C(U_{n+1}, l_p) \sim C(U_n) \oplus C(U_{n+2}, l_p) \Leftrightarrow C(U_{n+1}) \sim C(U_{n+2}).$$

Theorem 3.3. Let X_n be a separable Banach space, $1 < p < \infty$and Γ an uncountable set. Then for every infinite compact metric spacesU_{n+1} and U_{n+2}

$$C(I, X_n) \oplus C(U_{n+1}, l_p(\Gamma)) \sim C(I, X_n) \oplus C(U_{n+2}, l_p(\Gamma)) \Leftrightarrow C(U_{n+1}) \sim C(U_{n+2}).$$

Proof. Observe that by the Milutin theorem

$$C(I, X_n) \sim C(U_n \times I, X_n) \sim C(U_n, C(I, X_n)),$$

for every infinite compact metric space U_n. Therefore

$$C(I, X_n) \oplus C\left(U_n, l_p(\Gamma)\right) \sim C\left(U_n, C(I, X_n) \oplus l_p(\Gamma)\right).$$

Thus by Theorem 1.3 we are done.□

References

[1]. E.M. Bator, Unconditionally converging and compact operators on c_0, Rocky Mountain J. Math. 22 (2) (1992) 417–422.
[2]. C. Bessaga, A. Pełczyn´ ski, Spaces of continuous functions, IV, Studia Math. XIX (1960) 53–61.
[3]. J.A. Clarkson, Uniformly convex spaces, Trans. Amer. Math. Soc. 40 (3) (1936) 396–414.
[4]. E.M. Galego, On subspaces and quotients of Banach spaces $C(K, X)$, Monatsh. Math. 136 (2) (2002) 87–97.
[5]. H.E. Lacey, The Isometric Theory of Classical Banach Spaces, Grundlehren Math. Wiss., vol. 208, Springer-Verlag, New York–Heidelberg, 1974.
[6]. W.B. Johnson, J. Lindenstrauss, Basic concepts in the geometry of Banach spaces, in: Handbook of the Geometry of Banach Spaces, North-Holland Publishing Co., Amsterdam, 2001, pp. 1–84.
[7]. S. Mazurkiewicz, W. Sierpin´ ski, Contribution a la topologie des ensembles denombrables, Fund. Math. 1 (1920) 17–27.
[8]. V.D. Milman, Geometric theory of Banach spaces, I: Theory of basic and minimal systems, Russian Math. Surveys 25 (1970) 111–170.
[9]. A.A. Milutin, Isomorphisms of spaces of continuous functions on compacts of power continuum, TieoriaFunc. (Kharkov) 2 (1966) 150–156 (in Russian).
[10]. B.J. Pettis, A proof that every uniformly convex space is reflexive, Duke Math. J. 5 (1939) 249–253.
[11]. H.P. Rosenthal, Some recent discoveries in the isomorphic theory of Banach spaces, Bull. Amer. Math. Soc. 5 (1978) 803–831.
[12]. H.P. Rosenthal, The Banach space $C(K)$, in: Handbook of the Geometry of Banach Spaces, North-Holland Publishing Co., Amsterdam, 2001, pp. 1547–1602.
[13]. Elói Medina Galego. The $C(K, X)$ spaces for compact metric spaces K and X with a uniformly convex maximal factor. J. Math. Anal. Appl. 384 (2011) 357–365.

Ahmed M.H.Yahya. "Application for sequence of compact metric spaces U_nand X_n." *IOSR Journal of Mathematics (IOSR-JM)*, 17(5), (2021): pp. 14-19.