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Numerical Solution for Stochastic Mixed Nonlinear Shrödinger Equation

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In this paper, we numerically solve the one-dimensional stochastic nonlinear Schrödinger equation in the case of an additive white noise and with mixed concave convex, sub-super nonlinearities. The aim is to investigate their influence on the well-known deterministic solutions: stationary states and blowing-up solutions.

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I. Introduction

In this work, we are interested to the study of the one-dimensional stochastic nonlinear Schrödinger (NLS) equation with both a subcritical and a supercritical power nonlinearities in the presence of an additive noise. The deterministic equation occurs as a basic model in many areas of physics, hydrodynamics, plasma physics, nonlinear optics, molecular biology. It describes the propagation of waves in media with both nonlinear and dispersive responses. It is an idealized model and does not take into account many aspects such as in-homogeneities, high order terms, thermal fluctuations, external forces which may be modeled as random excitations (see [?, ?, ?, ?, ?]). Propagation in random media may also be considered. The resulting re-scaled equation is a random perturbation of the dynamical system of the following form

$$\begin{cases} i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+|u|^{p-1}u+\alpha|u|^{q-1}u=\varepsilon f(u), \quad L_1<x<L_2,\ t>t_0, u(t_0,x)=u_0(x), \quad \frac{\partial u}{\partial x}(t_0,x)=\varphi(x), \quad L_1\le x\le L_2 \frac{\partial u}{\partial x}(t,L_1)=\frac{\partial u}{\partial x}(t,L_2)=0 \quad t\ge t_0. \end{cases} \quad (1.1)$$

where $u=u(t,x)$, φ $t\geq 0,$ $x\in\mathbb{R}$; is a complex-valued function and φ is $\mathcal{C}^2$ $\alpha\geq 0,$ L_1 and L_2 are real parameters. The term $f(u)$ includes the stochastic contribution. For an additive noise, $f(u)=\dot{\chi}$ is real-valued, Gaussian, white in time and either white or correlated in space. In this case, the noise does not depend on the solution. The size of the noise is controlled by the parameter $\varepsilon>0$.

Here, we are particularly interested in the influence of a noise acting as a potential on the behavior of resolution. Such noise has been considered in [7] : there the paths of the noise are smooth functions and the nonlinearity is subcritical. In the case of a white noise, considered here, this type of model has been introduced in the context of crystals (see [?, ?] and also [?, ?] for other models). It is expected that such noise has a strong influence on the solutions which blow-up. It may delay or even prevent the formation of a singularity. In [1], some numerical simulations tend to show that this is the case for a very irregular noise: for a space-time white noise. However, in the supercritical case and for a noise which is correlated in space but non degenerate, it has been observed that, on the contrary, any solution seems to blow-up in a finite time. Recall that in the deterministic case, only a restricted class of solutions blow-up. Our aim is to prove rigorously such a behavior.

The case of an additive noise has been considered in [?, ?] and it has been proved that for any initial data, blow-up occurs in the sense that, for arbitrary $t>0$, the probability that the solution blows up before the time t is strictly positive. Thus, the noise strongly influences this blow-up phenomenon. In the present paper, result is in perfect agreement with the numerical simulations. The argument is based on three ingredient: first, we generalize the deterministic argument to prove that blow-up occurs for some initial data: this is based on a stochastic version of the variance identity (see [?, ?]). Then, we use the fact that the NLS equation is controllable by a forcing term. Thus, any initial data can be transformed into a state which yields a singular solution. Finally, since the noise is non-degenerate and the solution depends continuously on the path of the noise, we can argue that, with positive probability, the noise will be close to the control so that blow-up will happen afterward.

The paper is organized as follows: In Section 2, we introduce the discrete scheme and transform problem (1.1) into a linear algebraic system.In Section 3, the analysis of local truncation error and stability are shown by introducing Fourier development replacements in order to apply the Von Neumann techniques. The scheme is proved to be uniquely solvable. We study the convergence and we analyze the method for consistency and stability. The Von Neumann method consists in testing the impact of the proposed scheme on an isolated Fourier mode. Some numerical examples are exposed in order to validate the scheme.

II. The stochastic Schrödinger equation

The considered noises have their paths in $\mathcal{H}^1$. In order to state precisely the equation considered in this work, we introduce the probability space $(\Omega,\ \mathcal{F},\ \mathbb{P})$, endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ and a sequence $(\beta_i)_{i\in\mathbb{N}}$ of independent real valued Brownian motions on $\mathbb{R}^+$, associated to the filtration $(\mathcal{F}_t)_{t\geq 0}$. We are interested to the NLS equations with additive noise. We consider the space of complex (resp. real) valued square integrable functions on $\mathbb{R}^n$ endowed with an Hilbertian basis $(e_i)_{i\in\mathbb{N}}$ and we are given a bounded linear operator ϕ on this space. The process

$$W(t,x,w) = \sum_{i=0}^{\infty} \beta_i(t,w)\phi e_i(x),$$

defined for $t \geq 0$, $x \in \mathbb{R}^n$, and $w \in \Omega$, is then a Wiener process on the space of complex valued square integrable functions on $\mathbb{R}^n$, with covariance operator $\phi\phi^\star$. We then set

$$\dot{\chi} = \frac{\partial W}{\partial t}.$$

Note that if ϕ is defined through a kernel $\mathcal{K}$, which means that for any square integrable function u,

$$\phi u(x) = \int_{\mathbb{R}^n} \mathcal{K}(x,y)u(y)\ dy.$$

As a consequence, the correlation function of the noise is given by

$$E\left(\frac{\partial W}{\partial t}(x,t),\ \frac{\partial W}{\partial t}(y,s)\right) = c(x,y)\delta_{t-s},$$

with

$$c(x,\ y) = \int_{\mathbb{R}^n} \mathcal{K}(x,z)\mathcal{K}(y,z)\ dz.$$

This kind of noise is always uncorrelated -or white- in time. Particularly, if W is a cylindrical Wiener process, i.e. if $\phi = Id$, the noise is also white in space and the spatial correlation $c(x,\ y)$ is the Dirac mass δ_{x-y}. Moreover, a correlation described by the Dirac mass $\delta_{x-y}\delta_{t-s}$ indicates a white noise both in time and space.

Taking in account these notations, we then rewrite the system (1.1) in the form

$$\begin{cases} i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{dx^2} + |u|^{p-1}u + \alpha|u|^{q-1}u = \varepsilon\dot{\chi} \quad , \quad L_1 < x < L_2,\ t > t_0, u(t_0,x) = u_0(x) \quad , \quad L_1 \leq x \leq L_2 \frac{\partial u}{\partial x}(t,L_1) = \frac{\partial u}{\partial x}(t,L_2) = 0 \quad , \quad t \geq t_0. \end{cases} \quad (2.1)$$

2.1 Finite difference scheme

Consider a time step $l = \Delta t$ and denote

$$t_k = t_0 + kl, \qquad k \in \mathbb{N}.$$

Now, fix some integer N and consider, a space step

$$h = \Delta x = \frac{L_2 - L_1}{N+1}.$$

We subdivide the interval $[L_1,\ L_2]$ into subintervals $[x_m,\ x_{m+1}]$ where

$$x_m = L_0 + mh \qquad for \qquad m = 0, \ldots,\ N+1.$$

Consider also positive parameters $\lambda_1,\ \lambda_2,\ \lambda_3$ and $\mu_1,\ \mu_2,\ \mu_3$ such that $\lambda_1 + \lambda_2 + \lambda_3 = \mu_1 + \mu_2 + \mu_3 = 1$

Denote by u_m^k the approximation of $u(t_k,\ x_m)$ and U_m^k the numerical solution. We introduce the following notations

$$\partial_m^k U = \frac{U_m^{k+1} - U_m^{k-1}}{2l} \qquad and \qquad \Delta_k^m U = \frac{U_{m+1}^k - 2U_m^k + U_{m-1}^k}{h^2},$$

$$\frac{\partial U_m^k}{\partial t} = \lambda_1 \partial_{m-1}^k U + \lambda_2 \partial_m^k U + \lambda_3 \partial_{m+1}^k U,$$

$$\frac{\partial U_m^k}{\partial x} = \frac{U_{m+1}^k - U_{m-1}^k}{2h}$$

$$\frac{\partial^2 U_m^k}{\partial x^2} = \mu_1\ \Delta_{k+1}^m\ U + \mu_2\ \Delta_k^m\ U + \mu_3\ \Delta_{k-1}^m\ U.$$

Let's denote

$$g(u) = |u|^{p-1}u + \alpha|u|^{q-1}u.$$

We then discretize problem (1.1) as follows

$$i\frac{\partial U_m^k}{\partial t} + \frac{\partial^2 U_m^k}{\partial x^2} + g(U_m^k) = \varepsilon f_m^{k+\frac{1}{2}} \qquad (2.2)$$

where

$$f_m^{k+\frac{1}{2}} = \frac{1}{\sqrt{\Delta t \Delta x}}\ \chi_m^{k+\frac{1}{2}}.$$

In this case of an additive noise, $f_m^{k+\frac{1}{2}}$ should be an approximation of

$$\frac{1}{\Delta x\Delta t}\int_{(m-\frac{1}{2})\Delta x}^{(m+\frac{1}{2})\Delta x}\int_{t_k}^{t_{k+1}}\dot{\chi}ds\ \ dx=\frac{1}{\Delta x\Delta t}\int_{(m-\frac{1}{2})\Delta x}^{(m+\frac{1}{2})\Delta x}\sum_{i\in\mathbb{N}}e_i(x)d\beta_i(s)dx$$
$$=\frac{1}{\Delta x\Delta t}\sum_{i\in\mathbb{N}}\left(\int_{(m-\frac{1}{2})\Delta x}^{(m+\frac{1}{2})\Delta x}e_i(x)dx\right)(\beta_i(t_{k+1})-\beta_i(t_k))$$

If the orthonormal basis $(e_i)_{i\in\mathbb{N}}$ of $\mathbb{L}^2(L_1,\ L_2)$ is chosen in such a way that

$$e_m=\frac{1}{\sqrt{\Delta x}}\mathbf{1}_{[(m-\frac{1}{2})\Delta x,(m+\frac{1}{2})\Delta x)};\quad m=1,\ \cdots,\ N$$

$$e_0=\frac{1}{\sqrt{\frac{\Delta x}{2}}}\mathbf{1}_{[0,\frac{1}{2}\Delta x]},$$

$$e_{N+1}=\frac{1}{\sqrt{\frac{\Delta x}{2}}}\mathbf{1}_{[(N+\frac{1}{2})\Delta x,(N+1)\Delta x]},$$

then, by orthogonality, we obtain

$$\int_{(m-\frac{1}{2})\Delta x}^{(m+\frac{1}{2})\Delta x}e_i(x)dx=0\qquad for\ m=1,\cdots,N,\quad and\quad i\in\mathbb{N},\ i\neq m.$$

Furthermore,

$$f_m^{k+\frac{1}{2}}=\frac{1}{\Delta t\sqrt{\Delta x}}(\beta_m(t_{k+1})-\beta_m(t_k)),\quad m=1,\ \cdots,\ N$$

$$f_0^{k+\frac{1}{2}}=\frac{1}{\Delta t\sqrt{\frac{\Delta x}{2}}}(\beta_0(t_{k+1})-\beta_0(t_k)),$$

$$f_{N+1}^{k+\frac{1}{2}}=\frac{1}{\Delta t\sqrt{\frac{\Delta x}{2}}}(\beta_{N+1}(t_{k+1})-\beta_{N+1}(t_k)).$$

Since the random variables defined by

$$\frac{\beta_m(t_{k+1})-\beta_m(t_k)}{\sqrt{\Delta t}}$$

are independent random with normal law $N(0,\ 1)$, we choose also the sequence

$$(\chi_m^{k+\frac{1}{2}})_{k\geq 0,\ \ m=1,\dots,N+1}$$

to be a sequence of independent random variables with normal law $N(0,\ 1)$.

Note that from $f_m^{k+\frac{1}{2}}$, we can see that the numerical noise has the form $\phi_{num}dW$, where ϕ_{num} is the orthogonal projector on the space spanned by $<e_0,\cdots,e_{N+1}>$ and the associated correlation function is given by

$$C_{num}=\begin{cases}\frac{1}{\Delta x}, & if\quad x,\ y\in[(m-\frac{1}{2})\ \Delta x,\ (m+\frac{1}{2})\ \Delta x],\\ \frac{2}{\Delta x}, & if\quad x,\ y\in[0,\ \frac{1}{2}\ \Delta x]\ or\ x,\ y\in[(N+\frac{1}{2})\ \Delta x,\ (N+1)\ \Delta x],\\ 0 & otherwise.\end{cases}$$

This confirms that we have an approximation of a space-time white noise. Indeed, it is easily seen that C_{num} in an approximation of δ_{x-y}. However, it also shows that the numerical noise has length scales always larger than Δx and in that sense, it is not white.

The numerical problem is considered under the initial data

$$U_m^0=u(t_0,\ x_m)=u_0(x_m),\quad 0\leq m\leq N+1,$$

$$U_m^1=U_m^0+il(u_0''(x_m)+g(u_0(x_m))),$$

and the boundary conditions

$$U_0^k=U_1^k\quad and\quad U_{N+1}^k=U_N^k,\quad\forall k.$$

Let $\nu\in[0,\ 1]$ and consider the approximation

$$g(U_m^k)\approx\tilde{g}(\nu U_m^k+(1-\nu)U_m^{k-1}),$$

where

$$\tilde{g} = \max_m \left| \frac{g(U_m^0)}{U_m^0} \right|.$$

Next, denote

$a_1 = 2\mu_1\sigma + i\lambda_1,$ $\qquad$ $b_1 = b_3 = -2\mu_2\sigma,$

$a_2 = -4\mu_1\sigma + i\lambda_2,$ $\qquad$ $b_2 = 4\mu_2\sigma - 2\tilde{g}\nu l,$

$a_3 = 2\mu_1\sigma + i\lambda_3,$ $\qquad$ $c_1 = -2\mu_3\sigma + i\lambda_1,$

$c_2 = 4\mu_3\sigma - 2\tilde{g}(1-\nu)l + i\lambda_2,$ $\qquad$ $c_3 = -2\mu_3\sigma + i\lambda_3,$

where $\sigma = \frac{l}{h^2}$. We then obtain

$$\begin{aligned} a_1 U_{m-1}^{k+1} + a_2 U_m^{k+1} + a_3 U_{m+1}^{k+1} &= b_1 U_{m-1}^k + b_2 U_m^k + b_3 U_{m+1}^k + c_1 U_{m-1}^{k-1} + c_2 U_m^{k-1} \\ &+ c_3 U_{m+1}^{k-1} + \varepsilon f_m^{k+\frac{1}{2}} \end{aligned} \tag{2.3}$$

with the boundary conditions

$$\begin{aligned} (a_1 + a_2) U_1^{k+1} + a_3 U_2^{k+1} &= (b_1 + b_2) U_1^k + b_3 U_2^k + (c_1 + c_2) U_1^{k-1} \\ &+ c_3 U_2^{k-1} + \varepsilon f_0^{k+\frac{1}{2}} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} a_1 U_{N-1}^{k+1} + (a_2 + a_3) U_N^{k+1} &= b_1 U_{N-1}^k + (b_2 + b_3) U_N^k + c_1 U_{N-1}^{k-1} \\ &+ (c_2 + c_3) U_N^{k-1} + \varepsilon f_{N+1}^{k+\frac{1}{2}}. \end{aligned} \tag{2.5}$$

2.2 Solvability of the difference scheme

In the matrix form, (2.3)-(2.5) becomes

$$AU^{k+1} = BU^k + CU^{k-1} + F, \qquad k \geq 1 \tag{2.6}$$

where F is the white noise vector and $A,\ B$ and C are the matrix defined as follows

$$A = \begin{pmatrix} a_1 + a_2 & a_3 & 0 & \cdots & \cdots & 0 \\ a_1 & a_2 & a_3 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_1 & a_2 & a_3 \\ 0 & \cdots & \cdots & 0 & a_1 & a_2 + a_3 \end{pmatrix}$$

$$B = \begin{pmatrix} b_1 + b_2 & b_3 & 0 & \cdots & \cdots & 0 \\ b_1 & b_2 & b_3 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & b_1 & b_2 & b_3 \\ 0 & \cdots & \cdots & 0 & b_1 & b_2 + b_3 \end{pmatrix}$$

$$C = \begin{pmatrix} c_1 + c_2 & c_3 & 0 & \cdots & \cdots & 0 \\ c_1 & c_2 & c_3 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & c_1 & c_2 & c_3 \\ 0 & \cdots & \cdots & 0 & c_1 & c_2 + c_3 \end{pmatrix}$$

$$F = \begin{pmatrix} f_0^{k+\frac{1}{2}} \\ \vdots \\ f_m^{k+\frac{1}{2}} \\ \vdots \\ f_{N+1}^{k+\frac{1}{2}} \end{pmatrix}$$

In order to prove the solvability of the difference scheme (2.3)-(2.5), we have to calculate the determinant of the matrix A. This is based on techniques developed in [?, ?, ?, ?] and treating the invertibility of a general tri-diagonal matrix. We recall the basic result in what follows

Lemma 2.1 *Consider the following real matrix* A

$$A = \begin{pmatrix} d_1 & a_1 & 0 & \dots & \dots & 0b_2 & d_2 & a_2 & \dots & \dots \\ 00 & b_3 & d_3 & a_3 & \dots & 0\,\vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots\,0 & \dots & \dots & b_{n-1} & d_{n-1} & a_{n-1}0 & 0 & \dots & 0 & b_n \\ d_n & & & & & & & & & \end{pmatrix}$$

and define the real vector

$$\alpha = (\alpha_0, \alpha_1, \dots \alpha_n)$$

as follows

$$\alpha_i = \{1 \quad if \quad i = 0 d_1 \quad if \quad i = 1 d_i \alpha_{i-1} - b_i a_{i-1} \alpha_{i-2} \quad if \quad i = 2,3, \dots, n.$$

Then, there holds

$$det(A) = \alpha_n.$$

Now, we are able to state the main result of this section.

Theorem 2.2 *The difference scheme (2.3)-(2.5) is uniquely solvable.*

Proof. Denote by $Det_{N+2}(A)$ the determinant of the matrix A. Then, from Lemma 2.1, we have the following recursive equation

$$Det_{N+2}(A) - (a_2 + a_3) Det_{N+1}(A) + a_1 a_3 Det_N(A) = 0.$$

We now divide the proof into three cases.

Case 1: $\mu_1 = 0,\ \lambda_2 = \frac{(1-\lambda_1)(5\lambda_1 - 1)}{4\lambda_1}$ and $\lambda_1 \in]0,\ 1[$.

Standard computations yield

$$Det_{N+2}(A) = (N+3)(\lambda_1 + \lambda_2)(1 - \lambda_1)^{(N+1)} (\frac{i}{2})^{(N+2)} \neq 0.$$

Case 2: $\mu_1 = 0,\ \lambda_2 \neq \frac{(1-\lambda_1)(5\lambda_1 - 1)}{4\lambda_1}$ and $\lambda_1 \in]0,\ 1[$.

If we denote by

$$\delta^2 = (\lambda_1 - 1)(5\lambda_1 - 1) + 4\lambda_1 \lambda_2,$$

$$X_1 = i\ \frac{\lambda_1 + \lambda_2 + \delta}{2},$$

$$X_2 = i\ \frac{\lambda_1 + \lambda_2 - \delta}{2},$$

$$C_1 = -\ \frac{\lambda_1 + \lambda_2}{\delta} + \frac{i}{\delta} \frac{X_2[\lambda_1 - (\lambda_1 + \lambda_2)(1 + \lambda_2)]}{\lambda_1(1 - \lambda_1 - \lambda_2)},$$

$$C_2 = \frac{\lambda_1 + \lambda_2}{\delta} - \frac{i}{\delta}\ \frac{X_1[\lambda_1 - (\lambda_1 + \lambda_2)(1 + \lambda_2)]}{\lambda_1(1 - \lambda_1 - \lambda_2)}.$$

Then, also by standard computations, we obtain

$$Det_{N+2}(A) = C_1 X_1^{N+2} + C_2 X_2^{N+2} \neq 0,$$

Case 3: $\mu_1 \neq 0$.

Let's consider the complex values

$$\delta = \sqrt{(a_2 + a_3)^2 - 4a_1 a_3}, \quad \zeta = \frac{(a_2 + a_3) + \delta}{2} \quad and \quad \xi = \frac{(a_2 + a_3) - \delta}{2}.$$

Then, we have

$$Det_{N+2}(A) = C_1 \zeta^{N+2} + C_2 \xi^{N+2} \neq 0.$$

It follows from the cases above that the system (2.6) is uniquely solvable.

2.3 Consistency, Stability And Convergence of the Discrete method

The principal part of the local truncation error of the method arising from the scheme (2.2) is given by

$$\begin{aligned}\mathcal{L}(t,\ x) &= ih\frac{\partial^2 u}{\partial t\partial x}(\lambda_3-\lambda_1)+l(\mu_1-\mu_3)\left(\frac{2}{h^2}\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^2\partial t}\right)\\ &+l^2\left(\frac{1}{h^2}(\mu_1+\mu_3)\frac{\partial^2 u}{\partial t^2}+\frac{i}{6}\frac{\partial^3 u}{\partial t^3}\right)+h^2\left(\frac{i}{2}(\lambda_1+\lambda_3)\frac{\partial^3 u}{\partial x^2\partial t}+\frac{1}{24}\frac{\partial^4 u}{\partial x^4}\right) \qquad (2.10)\\ &+o(l^2+h^2)\end{aligned}$$

It follows, particularly that, taking $l=o(h^3)$, it becomes that $\mathcal{L}(t,\ x)$ tends to 0 as l and h tend to 0. This means obviously that the method is consistent.

Our good is to optimize numerically (2.10), (relativity to the parameters λ_i and μ_i, for $i=1,2,3$) in order to obtain the minimal possible error.

We remark that, in the particular case when

$$\begin{cases}\lambda_1=\lambda_3=\lambda & \in\ ]0,\ 1[\\ \lambda_2=1-2\lambda & \in\ ]0,\ 1[\end{cases} \qquad and \qquad \begin{cases}\mu_1=\mu_3=\mu & \in\ ]0,\ 1[\\ \mu_2=1-2\mu & \in\ ]0,\ 1[\end{cases}$$

we obtain

$$\mathcal{L}(t,\ x)=l^2\left(\frac{2\mu}{h^2}\frac{\partial^2 u}{\partial^2 t}+\frac{i}{6}\frac{\partial^3 u}{\partial t^3}\right)+h^2\left(i\lambda\frac{\partial^3 u}{\partial x^2\partial t}+\frac{1}{24}\frac{\partial^4 u}{\partial x^4}\right)+o(l^2+h^2).$$

This particular case makes that the scheme is consistent just when taking the condition $l=o(h^2)$, and not necessarily $l=o(h^2)$ as in the general case. This is due to the fact that the second term of the summation in the right hand of (2.10), which was in the origin of the last condition $(l=o(h^2))$, is simplified as $\mu_1=\mu_3$.

We now proceed by proving the stability of the method by applying the Lyapunov criterion. A linear system $\mathcal{L}(x_{n+1},\ x_n,\ x_{n-1},\ \dots)=0$ is stable in the sense of Lyapunov if for any bounded initial solution x_0 the solution x_n remains bounded for all $n\geq 0$. Here, we will precisely prove the following result.

Lemma 2.3 $\mathcal{P}_n$: *The solution U^n is bounded independently of n whenever the initial solution U^0 is bounded.*

We will proceed by recurrence on n. Assume firstly that $\| U^0 \|\leq\eta$ for some η positive. Using the system (2.6), we obtain

$$\| AU^{k+1} \|\leq\| B \|\| U^k \| +\| C \|\| U^{k-1} \| +\| F \|, \qquad k\geq 1 \quad (2.11)$$

Next, recall that, for $l=o(h^{s+2})$ small enough, $s>0$, we have

$$a_1=2\mu_1 h^s+i\lambda_1\to i\lambda_1, \qquad b_1=b_3=-2\mu_2 h^s\to 0,$$

$$a_2=-4\mu_1 h^s+i\lambda_2\to i\lambda_2, \qquad b_2=4\mu_2 h^s-2\tilde{g}\nu l\to 0,$$

$$a_3=2\mu_1 h^s+i\lambda_3\to i\lambda_3, \qquad c_1=-2\mu_3 h^s+i\lambda_1\to i\lambda_1,$$

$$c_2=4\mu_3 h^s-2\tilde{g}(1-\nu)l+i\lambda_2\to i\lambda_2, \qquad c_3=-2\mu_3 h^s+i\lambda_3\to i\lambda_3,$$

As a consequence, for h small enough,

$$\| B \|\leq \max(|2b_1+b_2|,|b_1+b_2+b_3|+|2b_3+b_2|)\leq\frac{1}{2}, \quad (2.12)$$

and

$$\| C \|\leq \max(|2c_1+c_2|,|c_1+c_2+c_3|,|2c_3+c_2|)\leq \max(|2\lambda_1+\lambda_2|,1,|2\lambda_3+\lambda_2|)\leq 3, \quad (2.13)$$

The operator A converges to the operator A_0 as h goes towards 0 and $l=o(h^{s+2})$ with $s>0$. where

$$A_0 = i\begin{pmatrix} \lambda_1+\lambda_2 & \lambda_3 & 0 & \cdots & \cdots & 0 \\ \lambda_1 & \lambda_2 & \lambda_3 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & \cdots & \cdots & 0 & \lambda_1 & \lambda_2+\lambda_3 \end{pmatrix}$$

Next, observing

$$\| A - A_0 \| \leq Ch^s. \qquad (2.14)$$

Lemma 2.4 *It exists C' such as $\| A_0 U \| \geq C' \| U \|$, $\forall \quad U$,*

$Proof$. Denote by $Det_{N+2}(A_0)$ the determinant of the matrix A_0. Then,let a_1, a_2, and a_3 are continuous and $h, l \to 0$ observing that

$$Det_{N+2}(A_0) - i(\lambda_2+\lambda_3)Det_{N+1}(A_0) - \lambda_1\lambda_3 Det_N(A_0) = 0.$$

Recall here that we have already proved in (2.2) for $\mu_1 = 0$ (Case 1 and Case 2) that the $Det_{N+2}(A_0) \neq 0$, which means that A_0 is invertible.

The following lemma deduced from (2.14).

Lemma 2.5 *For h small enough, it holds for all U that*

$$\frac{\|A_0\|}{2} \| U \| \leq (\| A_0 \| - Ch^s) \| U \| \leq \| A \| \leq (\| A_0 \| + C(\alpha)h^s) \| U \| \leq \frac{3\|A_0\|}{2} \| U \|.$$

-17pt Indeed, recall that equation (2.14) affirms that $\| A - A_0 \| \leq Ch^s$ for some constant $C > 0$. Consequently, for any U we get

$$(\| A_0 \| - Ch^s) \| U \| \leq \| A \| \leq (\| A_0 \| + C(\alpha)h^s) \| U \|.$$

For $h \leq \frac{\|A_0\|}{2(C)^{1/s}}$, we obtain

$$\frac{\|A_0\|}{2} \leq (\| A_0 \| - Ch^s) < (\| A_0 \| + Ch^s) \leq \frac{3\|A_0\|}{2}$$

and thus Lemma 2.5. As a result, (2.19) yields that

$$\frac{\|A_0\|}{2} \| U^{n+1} \| \leq \frac{1}{2} \| U^n \| + 3 \| U^{n-1} \| + \| F \|. \qquad (2.15)$$

For $n = 0$, this implies that

$$\| U^1 \| \leq \frac{1}{\|A_0\|} \| U^0 \| + \frac{6}{\|A_0\|} \| U^{-1} \| + \frac{2}{\|A_0\|} \| F \|. \qquad (2.16)$$

Using the discrete initial condition

$$U^0 = U^{-1} + l\varphi.$$

Here we identify the function φ to the matrix whom coefficients are $\varphi_m = \varphi(x_m)$. We obtain

$$\| U^{-1} \| \leq \| U^0 \| + l \| \varphi \|. (2.17)$$

Hence, equation (2.18) yields that

$$\| U^1 \| \leq \frac{7}{\|A_0\|} \| U^0 \| + \frac{6l\|\varphi\|}{\|A_0\|} + \frac{2\|F\|}{\|A_0\|}. (2.18)$$

Now, the Lyapunov criterion for stability states exactly that

$$\forall\ \varepsilon > 0, \exists\ \eta > 0 \quad s.t; \ \| U^0 \| \leq \eta \ \Rightarrow \ \| U^n \| \leq \varepsilon,\ \forall\, n \geq 0. \quad (2.19)$$

For $n = 1$ and $\| U^1 \| \leq \varepsilon$, we seek an $\eta > 0$ for which $\| U^0 \| \leq \eta$. Indeed, using (2.18), this means that, it suffices to find η such that

$$\frac{7}{\|A_0\|}\eta + \frac{6l\|\varphi\|}{\|A_0\|} + \frac{2\|F\|}{\|A_0\|} - \varepsilon < 0. \qquad (2.20)$$

For $\eta_1 = \frac{\|A_0\|\varepsilon - 6l\|\varphi\| - 2\|F\|}{7}$,

Consequently, choosing $\eta \in]0, \eta_1[$ we obtain (2.20). Finally, (2.18) yields that $\| U^1 \| \leq \varepsilon$.

Assume now that the U^k is bounded for $k = 1,2,\ldots,n$ (by ε_1) whenever U^0 is bounded by η and let $\varepsilon > 0$. We shall prove that it is possible to choose η satisfying $\| U^{n+1} \| \leq \varepsilon$. Indeed, from (2.15), we have

$$\| U^{n+1} \| \leq \frac{1}{\|A_0\|}(7\varepsilon_1 + 2 \| F \|). \qquad (2.21)$$

So, one seeks, ε_1 for which $\frac{1}{\|A_0\|}(7\varepsilon_1 + 2 \| F \|) - \varepsilon \leq 0$

For $\varepsilon_1 = \frac{\|A_0\|\varepsilon - 2\|F\|}{7}$,then $\| U^{n+1} \| \leq \varepsilon$ whenever $\| U^k \| \leq \varepsilon_1$, $k = 1,2,\ldots,n$.

Next, it holds from the recurrence hypothesis for ε_1, that there exists $\eta > 0$ for which $\| U^0 \| \leq \eta$ implies that $\| U^k \| \leq \varepsilon_1$, for $k = 1,2,\ldots,n$, which by the next induces that $\| U^{n+1} \| \leq \varepsilon$.

Lemma 2.6 *As the numerical scheme is consistent and stable, it is then convergent.*

This lemma is a consequence of the well known Lax-Richtmyer equivalence theorem, which states that for consistent numerical approximations, stability and convergence are equivalent. Recall here that we have already proved in (2.10) that the used scheme is consistent. Next, Lemma 2.3, Lemma 2.5 and equation (2.19) yields the stability of the scheme. Consequently, the Lax equivalence Theorem guarantees the convergence. So as Lemma 2.6.

III. Numerical implementations

We want to investigate the noise effects on stationary solutions in different cases. As mentioned in the introduction, stationary waves play an important role in physics and the effect of white noise on propagation in not well-known. Noise effects on solitary waves have already been studied for the NLS equation and for Korteweg-de-Vries equation (see [?, ?, ?, ?]). Let us recall that there exist deterministic solutions of the following form ([?])

$$u(x,\ t) = \sqrt{\frac{2a}{q_s}} \exp(i(\frac{1}{2}cx - \theta t + \varphi)) sech(\sqrt{a}(x - ct) + \phi),$$

where $a,\ q_s,\ \theta = \frac{c^2}{4} - a,\ \varphi$ and ϕ are some appropriate constants. For fixed t, this function decays exponentially as $|x| \to \infty$. It is a soliton-type disturbance which travels with speed c and with a governed amplitude.

In the first one, subject of Figure 1, the time and the space partial derivative parameters are fixed to the particular case where

$$\lambda_1 = \lambda_3 = \lambda = \frac{1}{5} \qquad and \qquad \mu_1 = \mu_3 = \mu = \frac{1}{3}.$$

In the second case, which is more general then the first one, we take different values for the parameters λ_i and μ_i $(1 \le i \le 3)$, precisely,

$$\begin{cases} \lambda_1 & = 0.29 \\ \lambda_2 & = 0.38 \\ \lambda_3 & = 0.33 \end{cases} \qquad and \qquad \begin{cases} \mu_1 & = 0.25 \\ \mu_2 & = 0.45 \\ \mu_3 & = 0.3 \end{cases}$$

This last example is expressed numerically in Figures 2, 3 and 4.

In both cases, the presented simulations of the equation (2.1) are given considering an additive noise. In the numerical scheme (2.2), the computations are done for $-80 \le x \le 100$ with a space step $h = 1$ and for $0 \le t \le 10$ with a time step $l = 0.01$. We fix also the soliton parameters $a = 0.01,\ q_s = 1,\ c = 0.1$ and the phase parameters $\phi = \varphi = 0$. Finally, we consider the parameters of the nonlinearity $q = 0.73,\ p = 1.5$ and $\nu = 0.5$.

For small amplitudes of the noise, corresponding to small values of the parameter ε, we can see that the solitary wave is not strongly perturbed and the noise does not prevent its propagation. This is clearly expressed in the particular case where $\lambda_1 = \lambda_3 = \lambda = \frac{1}{5}$ and $\mu_1 = \mu_3 = \mu = \frac{1}{3}$ by Figure 1 (e, f and g). It is also confirmed, in the more general case where $\lambda_1 \ne \lambda_3$ and $\mu_1 \ne \mu_3$, by Figure 3 (g, h, and i). However, going to Figure 1 (c, d and e), we can remark that, as the noise level becomes higher, the wave is progressively destroyed. We can remark the same behavior in the general case, especially in Figure 2 (c, d, e and f). For the deterministic case corresponding to $\varepsilon = 0$, and physically interpreted by the absence of noise, the solution of the problem is given, in the first case by Figure 1-h and in the second one by Figure 1-j. It represents the stationary wave.

Now, in both cases, taking the amplitude of the noise $\varepsilon \ge 0.0003$, it is clearly seen that the wave explodes under the influence of the additive noise. The phenomenon appears respectively in Figure 1 (a and b) for the particular case where $\lambda_1 = \lambda_3$ and $\mu_1 = \mu_3$ and in Figure 2 (left side of a and b) for the more general case where $\lambda_1 \ne \lambda_3$ and $\mu_1 \ne \mu_3$.

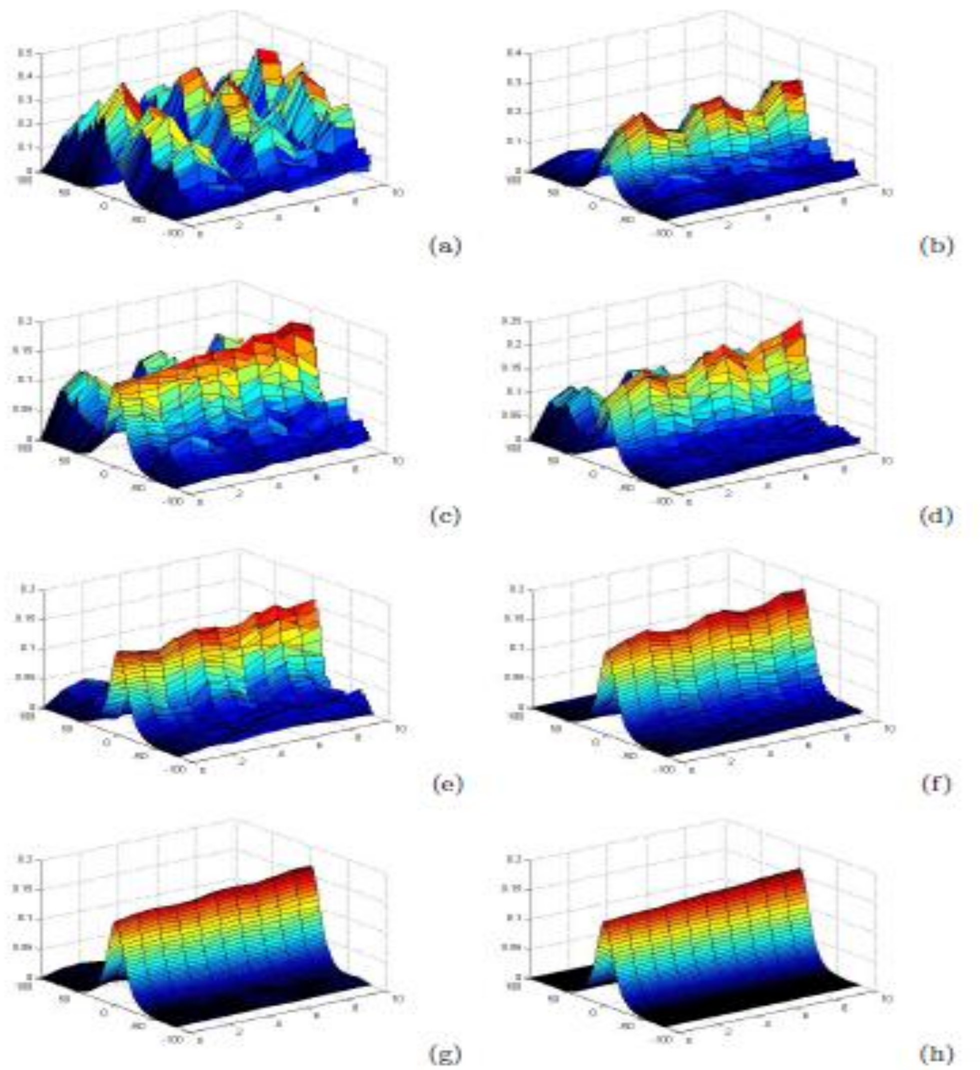
In the next, we will look at the general case. Being interested to the right side of Figures 2 and 3, we can see that the infinite norm of u presents many observable peaks. As we decrease the value of ε, the amplitude of the noise decreases consequently and its influence on the the deterministic solution disappears progressively. That is why the blow-up phenomenon appears less and less, and so are the peaks. More precisely, we can remark that the soliton wave starts to appear with small perturbations of the deterministic equation, corresponding to weak values of the parameter ε. As examples, we can cite $\varepsilon = 0.00007$, $\varepsilon = 0.00003$ and $\varepsilon = 0.00001$ respectively in Figures 3-g, 3-h and 3-i.

Finally, taking $\varepsilon = 0$ in Figure 3-j, we can say that we did proceed to a simulation of the solution in the deterministic case and the infinite norm's figure tends to a linear shape. We can say that we did start with a blow-up phenomenon to convert it into a soliton wave, also we did straightening the infinite norm of the solution.

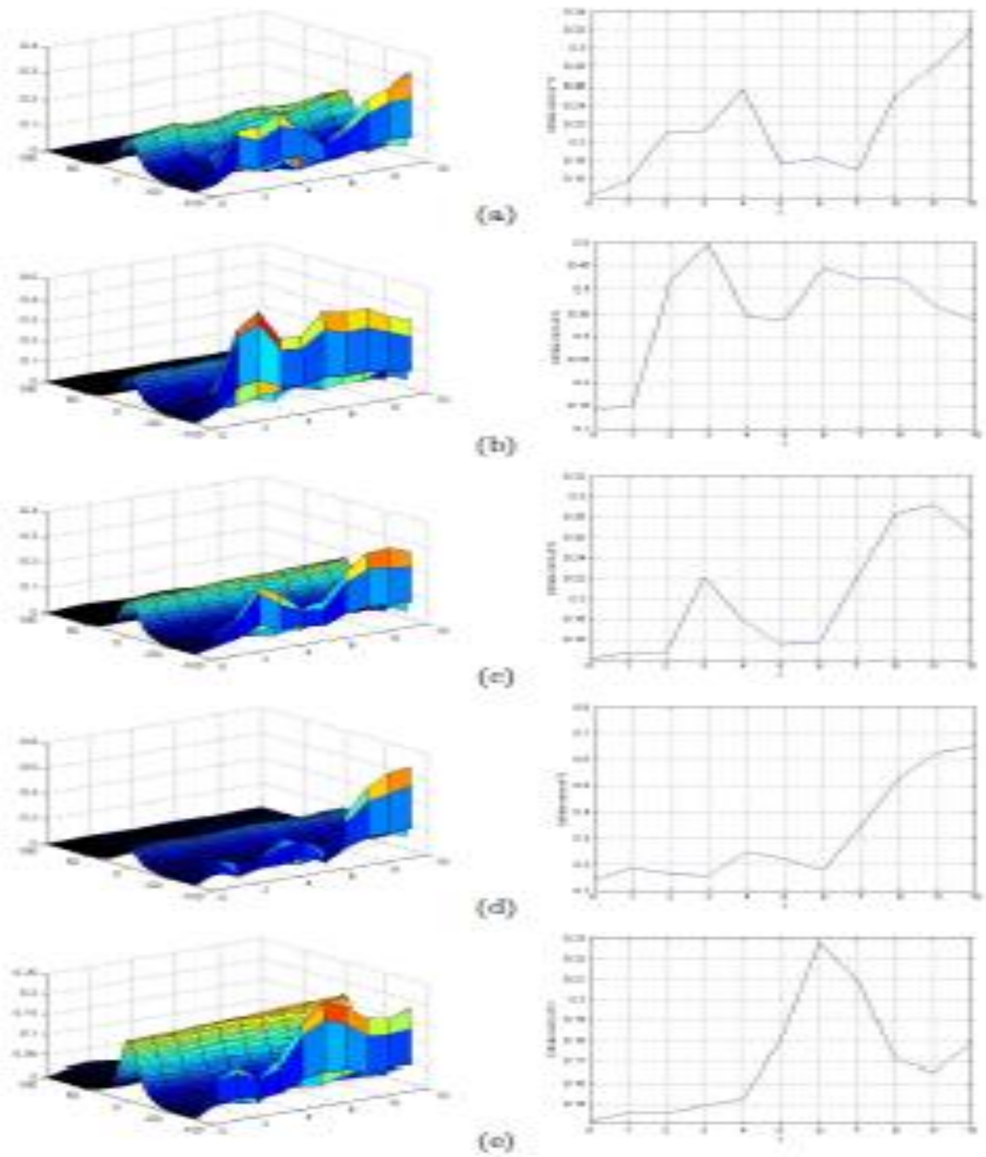
Looking at the effect of noise on one trajectory, we show, in Figure 4, the profiles of the solution with additive noise at a final computation time for several values of ε. We see that in any case the profile has the same shape as the solitary wave and has been diffused and damped. But, we see in Figure 4 (g and h) that the noise effect is really strong and the wave has been completely destroyed.

IV. Conclusion

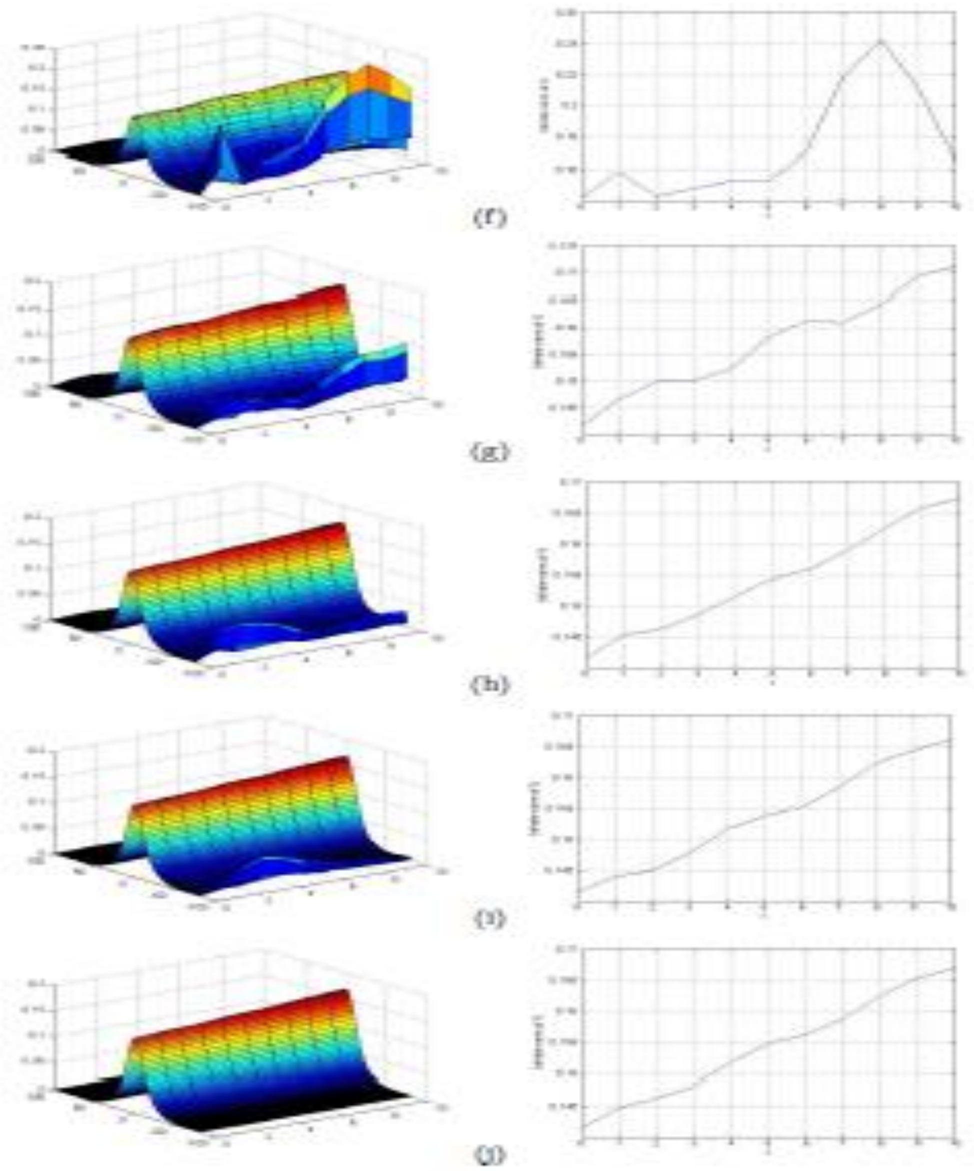
It is noted that the stochastic nonlinear equation (1.1) can be considered as a white noise random perturbation of the deterministic equation defined by $(\varepsilon = 0)$. Such a perturbation occurs when the size of the noise, described by the real-value parameter ε is positive. We proved, in this work, that as $\varepsilon \to 0$, the solution of the perturbed case converges to the unique trajectory of the deterministic equation. Then, we can say that the stochastic model would be more realistic, and we can observe a similar evolution phenomena about the solution as that given by the deterministic case.



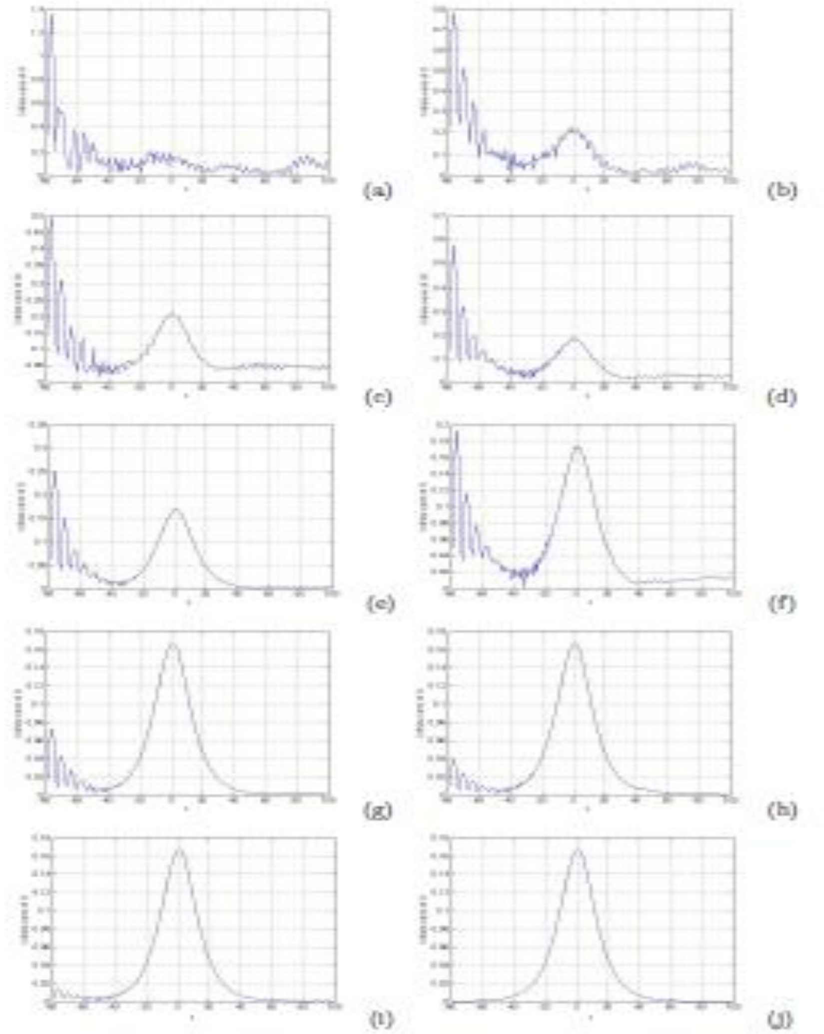
Plots in $(t,\ x)$-plane of $|u|$ for one trajectory in the case where $\lambda_1 = \lambda_3 = \lambda = \frac{1}{5}$ and $\mu_1 = \mu_3 = \mu = \frac{1}{3}$: 1.3cm (a) $\varepsilon = 0.005$, 0.5cm (b) $\varepsilon = 0.002$, 0.5cm (c) $\varepsilon = 0.001$, 0.5cm (d) $\varepsilon = 0.0009$, 1.9cm (e) $\varepsilon = 0.0007$, 0.87cm (f) $\varepsilon = 0.0003$, 0.68cm (g) $\varepsilon = 0.0001$, 0.65cm (h) $\varepsilon = 0$.



Plots in $(t,\ x)$-plane of $|u|$ (left side) and of its infinite norm (right side) for one trajectory in the general case: 2.3cm (a) $\varepsilon = 0.003$, 0.5cm (b) $\varepsilon = 0.001$, 0.7cm (c) $\varepsilon = 0.0009$, 0.7cm (d) $\varepsilon = 0.0003$, 0.8cm (e) $\varepsilon = 0.0001$.



Plots in $(t,\ x)$-plane of $|u|$ (left side) and of its infinite norm (right side) for one trajectory in the general case: 2.3cm (f) $\varepsilon = 0.00009$, 0.5cm (g) $\varepsilon = 0.00007$, 0.8cm (h) $\varepsilon = 0.00003$, 0.8cm (i) $\varepsilon = 0.00001$, 0.8cm (j) $\varepsilon = 0$.



Plots in $(t,\ x)$-plane of $|u|$ for one trajectory 1.3cm (a) $\varepsilon = 0.002$, 0.5cm (b) $\varepsilon = 0.0009$, 0.5cm (c) $\varepsilon = 0.0004$, 0.5cm (d) $\varepsilon = 0.0002$, 1.9cm (e) $\varepsilon = 0.00009$, 0.56cm (f) $\varepsilon = 0.00006$, 0.6cm (g) $\varepsilon = 0.00004$, 0.35cm (h) $\varepsilon = 0.00003$, 0.5cm (i) $\varepsilon = 0.00001$, 0.8cm (j) $\varepsilon = 0$.

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