IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 17, Issue 5 Ser. II (Sep. – Oct. 2021), PP 42-46
www.iosrjournals.org

Fourier Transformed: Insufficiencies In $\boldsymbol{L^1}(\mathbb{R})$ AND IN $\boldsymbol{L^2}(\mathbb{R})$

António Camunga Tchikoko, Msc.
ORCID: 0000-0002-9532-4133
Higher Institute of Educational Sciences of Huambo (Angola)

Abstract
For a function to admit Fourier transform it has to be absolutely integrable, that is, in other words this function has to be of classL^1. The present work presents a part of the Fourier transform theory that allows the analysis of the behavior of certain functions in the spaces L^1 andL^2. The main objective is to present some shortcomings of the spaces L^1 and L^2 when operating the functions regarding the normal product and the convolution product within the Fourier transform.

Date of Submission: 28-09-2021 Date of Acceptance: 12-10-2021

I. Introduction

The Fourier transform is the mathematical tool that shows us how to deconstruct the waveform into its sinusoidal components. This has a multitude of applications, aids in understanding the universe, and just makes life so much easier for the practicing engineer or scientist.

For a function to admit a Fourier transform it must be absolutely integrable. For this reason it is important to know the absolutely integrable class or function spaces, how to operate these functions within the class, since our main objective is to present some shortcomings of the $L^1(□)$ and $L^2(□)$ spaces when operating the functions as the normal product and the product of convolution within the Fourier transform.

Development
Fourier Transform
Definition:
If the integral

$$\int_{-\infty}^{+\infty} f(t)e^{ixt}\,dt$$

Exists for all $x \in □$, then we define the Fourier transform of $f(t)$ by:

$$F(x) = V[f(t)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)e^{ixt}\,dt.$$

If the integral

$$\int_{-\infty}^{+\infty} f(t)e^{ixt}\,dt$$

Exists for all $x \in □$, then we define the inverse Fourier transform of $f(t)$ by:

$$f(t) = V^{-1}[F(x)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(x)e^{-ixt}\,dx.$$

Fuction class $L(□)$
The function class $L(□)$ is the class formed by the functions $f: □ \to □$ or $□$.

$$L^1(□) = \left\{ f: □ \to □ \text{ ou } □ : \int_{-\infty}^{+\infty} |f(x)|dx < \infty \right\}$$

$L^1(□)$ set of functions absolutely integrable

$$L^2(\square) = \left\{ f: \square \to \square \text{ ou } \square : \int_{-\infty}^{+\infty} |f(x)|^2 dx < \infty \right\}$$

$L^2(\square)$ sets of functions with integrable square

Examples

I) Given the fuction

$f: (0,1) \to \square$

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{x}} \; se \; x \in (0,1) \\ 0 \; in \; a \; not\square er \; case \end{cases}$$

Demonstrates that $f \in L^1(0,1)$

Soluction:

$$f \in L^1(0,1) \Longrightarrow \int_0^1 |f(x)| dx < \infty$$

$$\int_0^1 |f(x)| dx = \int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{\varepsilon \to 0} \int_\varepsilon^1 \frac{1}{\sqrt{x}} dx = \lim_{\varepsilon \to 0} \left(2 - 2\sqrt{\varepsilon}\right) = 2 < \infty$$

Therefore $f \in L^1(0,1)$.

ii) be given $\alpha > 0$ consider the function

$f: (0,1) \to \square$

$$f(x) = \begin{cases} \dfrac{1}{x^\alpha} \; se \; x \in (0,1) \\ 0 \; in \; ot\square er \; case \end{cases}$$

Demonstrates that $f \in L^2(0,1)$ if and only if $\alpha < \frac{1}{2}$

$$f \in L^2(0,1) \Longrightarrow \int_0^1 |f(x)|^2 dx < \infty$$

Soluction

1. For $\alpha = \frac{1}{2}$ then

$$\int_0^1 |f(x)|^2 dx = \int_0^1 \left(\frac{1}{x^\alpha}\right)^2 dx = \int_0^1 \frac{1}{x^{2\alpha}} dx = \int_0^1 \frac{1}{x} dx =$$

$$= \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{x} dx = \lim_{\varepsilon \to 0^+} (\ln(1) - \ln(\varepsilon)) = +\infty$$

2. $\alpha > 0 \;\; \alpha \neq \frac{1}{2}$

$$\int_0^1 |f(x)|^2 dx = \int_0^1 \left(\frac{1}{x^\alpha}\right)^2 dx = \int_0^1 \frac{1}{x^{2\alpha}} dx = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{x^{2\alpha}} dx = \lim_{\varepsilon \to 0^+} \left[\frac{1}{-2\alpha + 1} - \frac{\varepsilon^{-2\alpha+1}}{-2\alpha + 1}\right]$$

$$= \frac{1}{-2\alpha + 1} \lim_{\varepsilon \to 0^+} [1 - \varepsilon^{-2\alpha+1}]$$

This limit is finite if $-2\alpha + 1 > 0 \Longrightarrow \alpha < \frac{1}{2}$

Sufficient condition for the existence of the Fourier transform

Theorem: The sufficient condition for the existence of the Fourier transform of the function $f(t)$ is that the integral of the absolute value of the function $f(t)$ must be finite, that is,

$$\int_{-\infty}^{+\infty} |f(t)| dt < \infty$$

Proof:

$e^{-ixt} = cosxt - isenxt \Longrightarrow \left|e^{-ixt}\right| = \sqrt{cos^2xt + sen^2xt} = 1$

$\left|f(t)e^{-ixt}\right| = |f(t)| \Longrightarrow$

$\int_{-\infty}^{\infty} |f(t)| \, dt = \int_{-\infty}^{\infty} \left|f(t)e^{-ixt}\right| dx$ it ie finite.

Some transform formulas

1-$V[f(\lambda t)] == \frac{1}{|\lambda|}F\left(\frac{t}{\lambda}\right)$

2-$V^{-1}[F.G] = F * G$

3-$V^2[f(t)] = f(-t)$

4-$V[f^m(x)] = (-ix)^m F(x)$

5-$V[t^m f(t)] = \frac{F^m(t)}{i^m}$

6-$V\left[\frac{\partial^q}{\partial x^q}f(x,y)\right] = (ix)^m F(x,y)$

7-$V\left[\frac{\partial^q}{\partial y^q}f(x,y)\right] = \frac{\partial^q}{\partial y^q}F(x,y)$

8-$V[f(t \pm ai)] = e^{\pm ax}F(x)$

Proof of formulas 3 and 8

3-$V^2[f(t)] = f(-t)$

Proof: we know that

$V[f(t)] = F(x)$

Applying the transform to the previous expression

$$V^2[f(t)] = VF(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x)e^{ixt}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x)e^{-ix(-t)}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x)e^{-i(-t)x}dt = f(-t)$$

8- $V[f(t \pm ai)] = e^{\pm ax}F(x)$

Proof:

$$V[f(t \pm ai)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t \pm ai)e^{ixt}dt$$

$$\begin{cases} z = t \pm ai \\ t = z \mp ai \\ dt = dz \end{cases}$$

$$V[f(t \pm ai)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t \pm ai)e^{ixt}dz = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)e^{ix(z \mp ai)}dz$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)e^{ixz}.e^{\mp i^2 ax}dz = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)e^{ixz}.e^{\pm ax}dz = e^{\pm ax}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)e^{ixz}dz\right] = e^{\pm ax}F(x)$$

Fourier transform in $L^2(\mathbb{R})$

$$L^2(\square) = \left\{f: \square \to \square \text{ ou } \square : \int_{-\infty}^{+\infty} |f(x)|^2 dx < \infty\right\}$$

If $f \in L^2(\square)$ the transform of f is given as follows:

$V[f(x)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{ixt}\,dt$, it is observed that this transform is the same that is defined in $L^1(\square)$ which raises the following question.

$L^2(\square)$ is the subsect of $L^1(\square)$? ($L^2(\square) \subset L^1(\square)$?)

Let's look at the following example: Let's consider the function defined as follows

$$f(x) = \begin{cases} \frac{1}{x}\ se\ x \in [1, \infty[\\ 0\ anot\square er\ case \end{cases}$$

Let's analyze if f simultaneously belongs to

$L^2(\square)$ e $L^1(\square)$

$$\int_{-\infty}^{\infty}\left(\frac{1}{x}\right)^2 dx = \int_{1}^{\infty}\frac{1}{x^2}dx = 1$$

As the integral is finite this fuction belongsto $L^2(\square)$.

Now let´s analyze if f belongs simultaneaously to $L^1(\square)$

$\int_{-\infty}^{\infty}\left|\frac{1}{x}\right|dx = \int_{1}^{\infty}\frac{1}{x}dx = ln|x|_1^{\infty} = +\infty$ the fuction not belong to $L^1(\square)$

Conclusion: $L^2(\square)$ is not subsect of $L^1(\square)$.

Convolution product

If $f_1, f_2 \in L^1(\square)$ have transform in $L^1(\square)$, the convoluction product $(*)$ is given by:

$$f_1(t) * f_2(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(t).f_2(x-t)dt$$

Norm in $L^1(\mathbb{R})$

The norm inthis class is defined as follows:

$$\|f\| = \int_{-\infty}^{+\infty} |f(x)|dx$$

To prove that it is in fact a norm, check the axioms of the norm.

$$\|f\| = 0 \Leftrightarrow f = 0$$

$\|f\| = 0 \Longrightarrow \int_{-\infty}^{+\infty} |f(t)|dt = 0 \Longrightarrow f = 0,$ pois $f \in L^1(\square)$

$$\|\lambda f\| = |\lambda|\|f\|$$

$$\|\lambda f\| = \int_{-\infty}^{+\infty} |\lambda f(t)|dt = \int_{-\infty}^{+\infty} |\lambda||f(t)|dt = |\lambda| \int_{-\infty}^{+\infty} |f(t)|dt$$

$$\|f+g\| \leq \|f\| + \|g\|$$

$\|f+g\| = \int_{-\infty}^{+\infty} |f(t)+g(t)|dt \leq \int_{-\infty}^{+\infty} [|f(t)| + |g(t)|]dt \leq \int_{-\infty}^{+\infty} |f(t)|dt + \int_{-\infty}^{+\infty} |g(t)|dt$ by triangular inequality

Some Convolution Properties

Se $f_1(t) * f_2(t) \in L^1(\square)$

$$\|f_1 * f_2\| \leq \frac{1}{2\pi} \|f_1\|.\|f_2\|$$

Proof

$\|f_1 * f_2\| = \int_{-\infty}^{\infty} |f_1 * f_2|dx$ by definition of the norm in $L^1(\square)$

$$\|f_1 * f_2\| = \int_{-\infty}^{\infty} |f_1 * f_2|dt = \int_{-\infty}^{\infty} \left| \frac{1}{2\square} \int_{-\infty}^{\infty} f_1(t).f_2(x-t)dx \right| dt$$

$$\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f_1(t) f_2(x-t)|\, dt dx \quad \text{(a)}$$

Applying Fubini's theorem that says the following:

Let $A = [a,b] \times [c,d]$ a rectangle of $\square^2$, and let $f: A \to \square$ an integrable function, such that the functions $f_x: [a,b] \to \square$ defined by $f_x(y) = f(x,y)$ are integrable into $[c,d]$, for all $x \in [a,b]$. Then the fuction $x \mapsto \int_a^b f(x,y)dy$ is integrable into $[a,b]$, and $\int_A f = \int_a^b \left(\int_c^d f(x,y)dy \right) dx$. We have that:

$$\text{(a)} \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |f_1(t)|dt \int_{-\infty}^{\infty} |f_2(x-t)|dx = \frac{1}{2\pi} \|f_1\|.\|f_2\|$$

2- If $f_1(t) * f_2(t) \in L^1(\square)$

$V(f_1 * f_2) = V(f_1).V(f_2) = F_1.F_2$

Proof:

$$V(f_1 * f_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(f_1 * f_2)(t) e^{ixt} dt$$

Considering

$$(f_1 * f_2)(t) = (f_1 * f_2)(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(t-\tau) f_2(\tau) d\tau$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(t-\tau) f_2(\tau) d\tau \right) e^{ixt} dt$$

$$\begin{cases} u = t - \tau \to t = u + \tau \\ dt = du \end{cases}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(u) f_2(\tau) d\tau \right) e^{ix(u+\tau)} du$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(u) f_2(\tau) d\tau \right) e^{ixu} e^{ix\tau} du$$

Applying Fubini

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(u) e^{ixu} du \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} f_2(\tau) \right) e^{ix\tau} d\tau$$

$V(f_1 * f_2) = V(f_1).V(f_2)$

Convolution Algebra

$L^1(\mathbb{R})$ with the operations (+) of functions and (*) product of convolution and (.) product of a scalar, get the structure of constructive algebra, but without the unit element.

The convolution product complies with the properties:

1- Associative

$(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)$

2- $\alpha f_1 * f_2 = f_1 * \alpha f_2$

3- Comutative

$f_1 * f_2 = f_2 * f_1$

Note: It is not a algebra with unit product, as $f * 1 \neq f$

Convoluton in $L^2(\mathbb{R})$

Be given $f, g \in L^2(\mathbb{R})$

A convoltion of $f * g$ in $L^2(\mathbb{R})$ is defined as follows:

$$f * g = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x - t) g(t) dt$$

Transform of $f * g$ em $L^2(\mathbb{R})$

If $f * g \in L^2(\mathbb{R}), V[f * g] = F.G$

Properties

If $F, G \in L^2(\mathbb{R}) \Rightarrow F.G \in L^1(\mathbb{R})$

If $f, g \in L^2, V(f) = F, V(g) = G. V^{-1}(F.G) = f * g \in L^2$

Shortcomings in In $L^1(\mathbb{R})$

1-If $F, G \in L^1$ we cannot guarantee that $F.G \in L^1(\mathbb{R})$, even if $F \in L^1$ and $G \in L^2$ there are no guarantees that$F.G \in L^1(\mathbb{R})$.

Example: let's consider functions

$$F = G = \begin{cases} \frac{1}{\sqrt{x}} \; se \; x \in (0,1) \\ 0, other \; case \end{cases}$$

This fuction is the $L^1(\mathbb{R})$, that is, $F, G \in L^1(\mathbb{R})$, that is,

$$\int_{-\infty}^{\infty} F dx = \int_{-\infty}^{\infty} G \, dx = \int_0^1 \frac{1}{\sqrt{x}} dx = 2$$

Now we'll see what happens to the product(.)

$$\int_{-\infty}^{\infty} F.G \, dx = \int_0^1 \frac{1}{x} dx = ln|x|_0^1 = \infty$$

2-Does not always comply with the formula $V^{-1}(F.G) = f * g$.

3-$F.G \in L^1$implies that $F * G \in L^1$.

Shortcomings in In □^{2}(□)

1-If$F, G \in L^2$ does not necessarily imply that $F.G \in L^2$, we can only ensure that $F.G \in L^1$

2-If $F, G \in L^2$does not imply that $F * G \in L^2$

II. Conclusions:

Studying the Fourier transforms through a bibliographical review, and identifying certain insufficiencies concerning the behavior of certain functions in the spaces mentioned in the body of the work, is not an easy study.

This study allowed us to identify some limitations that, from the author's point of view, are crucial because in this way it allows operating the functions safely in the indicated classes and not jumping to conclusions, such as thinking that if two functions belong to the L^1 space, the product or the convolution is also de L^1.

Bibliography:

[1]. Bronson, R., Gabriel Costa (SA). Differential Equations. Third edition. Mexico: McGraw Hill Education.
[2]. Churchil, R. And James Ward, B.(1987). Fourier Series and Boundary value problems. 5th ed. United states: McGraw Hill
[3]. Ramos, E.(2008). Mathematical Analysis IV for Science and Engineering student. second edition. Lima-Peru: New editions
[4]. Tchikoko, A. (2020). Lessons from Equations in Partial Derivatives. Sevilla: Punto Rojo Libros, SL

António Camunga Tchikoko, Msc. "Fourier Transformed: Insufficiencies In $L^1(\mathbb{R})$ AND IN $L^2(\mathbb{R})$." *IOSR Journal of Mathematics (IOSR-JM)*, 17(5), (2021): pp. 42-46.